7.8 Stability of Discrete-Time Linear Systems

We distinguish between internal system stability (stability of the system zero-input response) and bounded-input bounded-output (BIBO) stability (stability of the system zero-state response). As in the continuous-time domain, discrete-time internal system stability depends on the location of the system eigenvalues and system BIBO stability is determined by the nature of the system impulse response.

7.8.1 Internal Stability of Discrete-Time Linear Systems

Definition 7.3: A discrete-time input free system is *stable* if its zero-input response is bounded in time, that is

$$|y_{zi}[k]| < M = \text{const} < \infty, \qquad \forall k$$

The linear discrete-time system is *asymptotically stable* if in addition of being stable its zero-input response tends to zero as time increases, that is

$$\lim_{k\rightarrow\infty}\{y_{zi}[k]\}\rightarrow 0$$

Internal system stability is related to the system eigenvalues. The discrete-time linear system eigenvalues are the solutions of corresponding system characteristic equation. Consider first the case when the characteristic values are distinct. The homogeneous solution of the corresponding difference equation, which for a zero-input system represents the zero-input response is given by

$$y_{zi}[k] = C_1(\rho_1)^k + C_2(\rho_2)^k + \cdots + C_n(\rho_n)^k,\ \rho_1 \neq \rho_2 \neq \cdots \neq \rho_n,\ k \geq 0$$

Let us observe that ρ^k will decay to zero as k increases if $|\rho| < 1$. The term ρ^k will tend to infinity as time increases if $|\rho| > 1$. For $|\rho| = 1$, which represents the unit circle in the complex plane, ρ^k remains on the unit circle since $\left(1e^{j\varphi}\right)^k = 1^k e^{jk\varphi} = e^{jk\varphi}$. It can be concluded that asymptotic stability requires $|\rho_i| < 1, \forall i$. For marginal stability, it is needed that $|\rho_i| \leq 1, \forall i$. In the case when *only one* $|\rho_i| > 1$, the system is unstable. These simple observations can be stated in the form of the following discrete-time stability theorem valid for the case of distinct system eigenvalues.

Theorem 7.7 *A discrete-time linear time invariant system with distinct eigenvalues is asymptotically stable if* $|\rho_i| < 1, \forall i$. *It is stable (marginally stable) for* $|\rho_i| \leq 1, \forall i$, *and it is unstable if there exists an eigenvalue such that* $|\rho_i| > 1$.

Theorem 7.7 can be interpreted as: the discrete-time linear system with distinct eigenvalues is asymptotically stable if all its eigenvalues are located in the unit circle of the complex plane; the system is marginally stable if all eigenvalues are either inside or on the unit circle; and that the system is unstable if only one of its eigenvalues is outside the unit circle, see Figure 7.6.

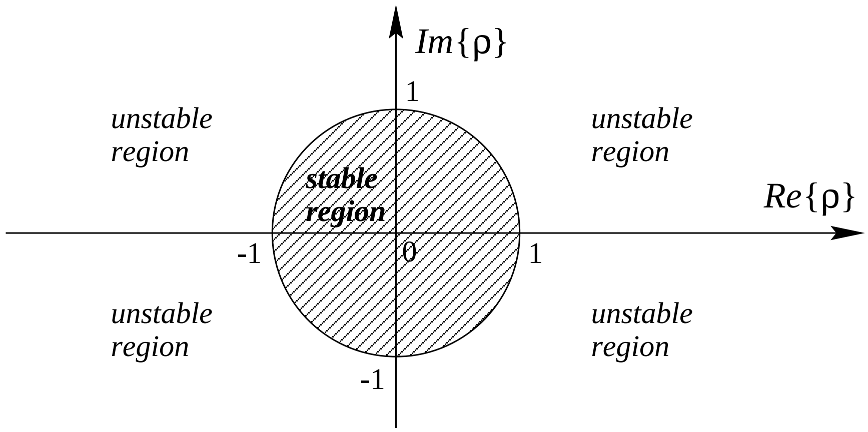

Figure 7.6: Unit circle eigenvalue stability criterion for discrete-time linear systems

In the case of a multiple eigenvalue, the zero-input system response has terms of the form

$$k^i(\rho)^k \rightarrow \begin{cases} 0, & |\rho| < 1 \\ \infty, & |\rho| > 1 \end{cases}, \quad k \rightarrow \infty, \quad i = 1, 2, \ldots, r-1$$

where r indicates the multiplicity of the given eigenvalue. It follows that in the case of multiple eigenvalues discrete-time system asymptotic stability requires

$|\rho_i| < 1, \forall i$ and that if there exists a multiple eigenvalue with $|\rho_i| \geq 1$ the system is unstable. This can be formulated in the following theorem.

Theorem 7.8 *A discrete-time linear time invariant system with multiple eigenvalues is asymptotically stable if $|\rho_i| < 1, \forall i$. The system is marginally stable if all multiple eigenvalues satisfy $|\rho_i| < 1$ and for all distinct eigenvalues the following hold $|\rho_i| \leq 1$. The system is unstable if there exists a multiple eigenvalue such that $|\rho_i| \geq 1$.*

Remark 7.3: The statement of Theorem7.8 is superficial. It claims that all multiple eigenvalues of a discrete-time linear system that lie on the unit circle are unstable. This is true in most cases. However, it is possible that multiple eigenvalues on the unit circle be marginally stable. A deeper study of this phenomenon requires discrete-time linear system representation in the state space form (to be introduced in Chapter 8) and the use of the corresponding Jordan canonical form.

This observation indicates that the system representation in the state space form is more general than the system representation using difference equations.

Example 7.33: The linear discrete-time system in Example 7.8 has the eigenvalues given by $\rho_1 = 1/2, \ \rho_2 = -1/3$. Since both eigenvalues are inside the unit circle the system is asymptotically stable. The same is true for the discrete-time linear system from Example 7.9 that has a triple eigenvalue within the unit circle, $\rho_1 = \rho_2 = \rho_3 = -0.5$. The system in Example 7.10 has four eigenvalues inside the unit circle, $\rho_1 = 0.5, \ \rho_2 = \rho_3 = \rho_4 = -0.25,$ and a double eigenvalue on the unit circle, $\rho_5 = \rho_6 = -1$. According to Theorem 7.8, this system is unstable due to a double eigenvalue at -1.

7.8.2 Algebraic Stability Tests for Discrete Systems

In this section we study the stability of time invariant linear discrete-time systems and present two algebraic methods: Jury's test and the bilinear transformation method.

7.8.2.1 Jury's Stability Test

Consider a polynomial represented in the z-domain by

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

The algebraic test to be presented is based on Table 7.2, the simplified Jury's table, which can be easily obtained by playing simple algebra with the polynomial's coefficients.

a_n	a_{n-1}	a_{n-2}	$\cdots$	a_2	a_1	a_0
b_{n-1}	b_{n-2}	$\cdots$	$\cdots$	b_1	b_0	
c_{n-2}	c_{n-3}	$\cdots$	$\cdots$	c_0		
$\cdots$	$\cdots$	$\cdots$	$\cdots$			
$\cdots$	$\cdots$	$\cdots$				
u_1	u_0					
w_0						

Table 7.2: Simplified Jury's table

$$b_{n-1} = a_n - k_0 a_0, \; b_{n-2} = a_{n-1} - k_0 a_1, \quad \ldots, \quad b_0 = a_1 - k_0 a_{n-1}$$

$$k_0 = \frac{a_0}{a_n}$$

$$c_{n-2} = b_{n-1} - k_1 b_0, \; c_{n-3} = b_{n-2} - k_1 b_1, \quad \ldots, \quad c_0 = b_1 - k_1 b_{n-2}$$

$$k_1 = \frac{b_0}{b_{n-1}}$$

Theorem 7.9 *Assume that* $a_n > 0$. *Then, the polynomial under consideration is asymptotically stable if and only if all coefficients in the first column of Table 4.2 are positive. In addition, the number of negative coefficients in the first column indicates the number of poles outside the unit circle.*

Example 7.34: The polynomial under consideration is given by

$$z^3 + 0.5z^2 + 0.3z + 0.1$$

The simplified Jury table for this example has the form

1	0.5	0.3	0.1
0.99	0.47	0.25	
0.93	0.35		
0.80			

with coefficients $k_0 = 0.1, \; k_1 = 0.25, \; k_2 = 0.24$.

From this table and Theorem 7.9, we conclude that the considered polynomial is stable. Using MATLAB, we find the eigenvalues as $p_1 = -0.3893,\ p_{2,3} = -0.0554 \pm j0.53038$, i.e. all of them are inside the unit circle.

7.8.2.2 Bilinear Transformation

It important to point out that the Routh–Hurwitz method can be used for studying the stability of discrete-time linear systems as well. That is, the very well-known bilinear transformation defined by

$$z = \frac{s+1}{s-1}$$

maps the unit circle in the z-domain into the left complex plane in the s-domain.

For the given discrete-time characteristic equation $\Delta(z) = 0$, by using the bilinear transformation, we get another characteristic equation in the s-domain, $\Delta_d(s) = 0$, so that the Routh–Hurwitz criterion can be directly applied to $\Delta_d(s)$.

The stability conclusion reached for the polynomial $\Delta_d(s)$ is valid also for the polynomial $\Delta(z)$. The following example demonstrates this procedure.

Example 7.35: Consider the discrete-time characteristic equation

$$\Delta(z) = z^3 + z^2 + z + 2 = 0$$

By using the bilinear transformation, this is mapped into

$$\left(\frac{s+1}{s-1}\right)^3 + \left(\frac{s+1}{s-1}\right)^2 + \left(\frac{s+1}{s-1}\right) + 2 = 0$$

which implies

$$\Delta_d(s) = 5s^3 - 3s^2 + 7s - 1 = 0$$

Using the knowledge from the previous section, we immediately conclude that this polynomial is unstable (it has coefficients of opposite signs). The same instability conclusion is valid for the polynomial $\Delta(z)$. If we form the Routh table we obtain

s^3	5	7	0
s^2	–3	–1	0
s^1	16/3	0	
s^0	–1	0	

The first column of this table indicates the existence of three s-domain unstable roots (three sign changes), which means also that in the z-domain there are three roots outside of the unit circle. This can be confirmed by MATLAB, which produces the following roots: $p_1 = -1.3532,\ p_{2,3} = 0.1766 \pm j1.2028$. Jury's table:

1	1	1	2
–3	–1	–1	
–8/3	–2/3		
–15/6			

Using Theorem 7.9, we conclude that all three roots are outside the unit circle.

7.8.3 Discrete-Time System BIBO Stability

Bounded-input bounded-output stability requires that a bounded input signal $|f[k]| < M_f = \mathbf{const} < \infty, \ \forall k$, produces a bounded signal on the system output, that is $|y_{zs}[k]| < M_y = \mathbf{const} < \infty, \ \forall k$. The zero-state response of a linear discrete-time system is given by the convolution formula, that is

$$y^{i}_{zs}[k] = \sum_{m=0}^{k} f[k-m]h[m] = \mathcal{Z}^{-1}\{F(z)H(z)\}$$

where $h[k]$ is the discrete-time system impulse response and $H(z)$ is the discrete-time system transfer function, $h[k] = \mathcal{Z}^{-1}\{H(z)\}$.

It follows that

$$|y_{zs}[k]| = \left| \sum_{m=0}^{k} f[k-m]h[m] \right| \leq \sum_{m=0}^{k} |f[k-m]h[m]|$$

$$\leq \sum_{m=0}^{k} |f[k-m]||h[m]| \leq M_f \sum_{m=0}^{k} |h[m]|$$

Since the above condition has to hold to any k, we conclude that

$$\sum_{m=0}^{\infty} |h(m)| \leq \text{const} < \infty \quad \Rightarrow \quad |y_{zs}[k]| \leq M_y = \text{const} < \infty, \ \ \forall k$$

The last formula produces the following bounded-input bounded-output stability theorem for discrete-time linear systems.

Theorem 7.10 *A discrete-time linear system is bounded-input bounded-output stable if and only if its impulse response is absolutely summable.*

We know from Chapter 5 that the system poles (at which the discrete-time system transfer function is equal to infinity, $H(p_i) = \infty$) are defined by

$$H(z) = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$$
$$= \frac{b_m (z - z_1)(z - z_2) \cdots (z - z_m)}{(z - p_1)(z - p_2) \cdots (z - p_n)}$$

It follows from this expression that the discrete-time system poles are equal to the system eigenvalues except for those eigenvalues that disappear from the system transfer function due to cancellations of common factors. Since the discrete-time impulse response is the inverse $\mathcal{Z}$ transform of the discrete-time system transfer function, we conclude that the discrete-time system impulse response is a linear combination of the discrete-time exponential terms p_i^k. The discrete-time system impulse response will be absolutely summable if all these exponential terms decay to zero as time increases, that is, if all the system poles are located in the unit circle.

Theorem 7.11 *A discrete-time linear system is bounded-input bounded-output stable if and only if all its poles are in the unit circle of the complex plane.*

Note that an asymptotically stable discrete-time system has all eigenvalues in the unit circle, which implies that all system poles are also in the unit circle of the complex plane. Hence, *an asymptotically stable discrete-time system is at the same time bounded-input bounded-output stable*. However, it is possible that all system poles are located in the unit circle, but not all system eigenvalues are in the unit circle (those eigenvalues cancelled in the system transfer function are either on the unit circle or outside the unit circle). It can be concluded that *bounded-input bounded-output stability does not imply in general asymptotic system stability*.

Note that only in the case when no cancellation of common factors in the discrete-time system transfer function takes place (when the number of the discrete-time system poles is identical to the number of the system eigenvalues) is discrete-time system asymptotic stability equivalent to discrete-time system BIBO stability.

Example 7.36: Let the discrete-time system transfer function be given by

$$H(z) = \frac{(z-2)(z+1.5)}{(z+0.3)(z-0.4)(z+1.5)} = \frac{z-2}{(z+0.3)(z-0.4)}$$

It can be observed that the eigenvalues of this discrete-time system are given by $\rho_1 = -0.3,\ \rho_2 = 0.4,\ \rho_3 = -1.5$. The discrete-time system poles are $p_1 = -0.3,\ p_2 = 0.4$. It follows that this system is BIBO stable since all its poles are in the unit circle, but the system is not asymptotically stable since it has the eigenvalue outside the unit circle, $\rho_3 = -1.5$.

The discrete-time system transfer function given by

$$H(z) = \frac{z+3}{z(z+1)(z-1)}$$

indicates that $\rho_1 = p_1 = 0$, $\rho_2 = p_2 = -1$, $\rho_3 = p_3 = 1$. The corresponding system is marginally stable since the eigenvalues are distinct and $|p_i| \leq 1$, $i = 1, 2, 3$. However, the system is not BIBO stable since the system poles $p_2 = -1$, $p_3 = 1$ are on the unit circle so that the corresponding impulse response is not absolutely summable. Note that $h[k]$ contains a term $(-1)^k u[k]$ contributed by the pole $p_2 = -1$ and a unit step function contributed by the pole $p_3 = 1$, which implies that the corresponding sum diverges, that is, that sum is equal to ∞.