4.3 Laplace Transform in Linear System Analysis

The main goal in analysis of any dynamic system is to find its response to a given input. The system response in general has two components: zero-state response due to external forcing signals and zero-input response due to system initial conditions. The Laplace transform will produce both the zero-input and zero-state components of the system response. We will also present procedures for obtaining the system impulse, step, and ramp responses. Note that using the Fourier transform, we have been able to find only the zero-state system response.

It is important to point out that the Laplace transform is very convenient for dealing with the system input signals that have jump discontinuities (and delta impulses). Note that in general the linear system differentiates input signals. The delta impulse inputs can come from the system differentiation of input signals that have jump discontinuities.

Recall from Chapter 2 the definition of the generalized derivative, which indicates that at the point of a jump discontinuity, the generalized derivative generates the impulse delta signal. Furthermore, for the same reason, a signal that is continuous and differentiable for all $t > 0$, but has a jump discontinuity at $t = 0$, for example $e^{-t}u(t)$, will generate an impulse delta signal (after being differentiated) at $t = 0$. The same is true for the signal $u(t)\sin(t)$ (after being differentiated twice), which is continuous but not differentiable at $t = 0$. Note that

$$\frac{D}{Dt}\{u(t)\sin(t)\} = u(t)\cos(t), \quad \frac{D^2}{Dt^2}\{u(t)\sin(t)\} = \delta(t) - u(t)\sin(t)$$

Using the Laplace transform as a method for solving differential equations that represent dynamics of linear time invariant systems can be done in a straight forward manner despite delta impulses generated by the system differentiation of input signals.

4.3.1 System Transfer Function and Impulse Response

Let us take the Laplace transform of both sides of a linear differential equation that describes the dynamical behavior of an nth order linear system

$$\mathcal{L}\left\{\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + a_{n-2}\right\}$$

$$= \mathcal{L}\left\{b_m\frac{d^m f(t)}{dt^m} + b_{m-1}\frac{d^{m-1}f(t)}{dt^{m-1}} + \cdots + b_1\frac{df(t)}{dt} + b_0 f(t)\right\}$$

Using the time derivative property of the Laplace transform we have

$$\left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right)Y(s) - I(s)$$

$$= \left(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0\right)F(s)$$

where $I(s)$ contains terms coming from the system initial conditions

Recall that

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\} = sY(s) - y(0^-)$$

$$\mathcal{L}\left\{\frac{d^2y(t)}{dt^2}\right\} = s^2Y(s) - sy(0^-) - y^{(1)}(0^-)$$

$$\cdots$$

$$\mathcal{L}\left\{\frac{d^ny(t)}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0^-) - s^{n-2}y^{(1)}(0^-) - \cdots - y^{(n-1)}(0^-)$$

Note that we assume that the input signal $f(t)$ represents a causal signal for which $f(t) = 0,\ t < 0$. Thus, *we have to set* $f^{(i)}(0^-) = 0, i = 0, 1, 2, ..., m$. Hence, $I(s)$ is a function of the coefficients a_i only and the system initial conditions.

It can be easily shown that in general

$$
\begin{aligned}
I(s) = &\Big(a_1 y(0^-) + a_2 y^{(1)}(0^-) + \cdots + a_{n-1} y^{(n-2)}(0^-) + y^{(n-1)}(0^-)\Big) \\
&+ s\Big(a_2 y(0^-) + a_3 y^{(1)}(0^-) + \cdots + a_{n-1} y^{(n-3)}(0^-) + y^{(n-2)}(0^-)\Big) \\
&+ s^2\Big(a_3 y(0^-) + a_4 y^{(1)}(0^-) + \cdots + a_{n-1} y^{(n-4)}(0^-) + y^{(n-3)}(0^-)\Big) \\
&\cdots \\
&+ s^{n-2}\Big(a_{n-1} y(0^-) + y^{(1)}(0^-)\Big) + s^{n-1} y(0^-)
\end{aligned}
$$

Note that in practice we do not need to use this formula.

System Response

The input signal is applied to the system at $t = 0$, and we are interested in finding the complete system response—the response due to both system initial conditions and input signals. From the original derivations we have

$$Y(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} F(s) + \frac{I(s)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

which produces the solution $Y(s)$ in the frequency domain of the original differential equation. To get the time domain solution, we must use the inverse Laplace transform, that is $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

If the initial conditions are set to zero, then $I(s) = 0$. The quantity

$$H(s) = \frac{Y(s)}{F(s)}|_{I(s)=0} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

defines ***the system transfer function***. The transfer function can also be written as

$$H(s) = K\frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}, \quad K = b_m$$

where $z_i, i = 1, 2, ..., m,$ are the *transfer function zeros* (note that $H(z_i) = 0$) and $p_j, j = 1, 2, ..., n,$ are the *transfer function poles* at which $H(p_j) = \infty$. Very often, we call them the *system* zeros and poles. K is called the *static gain*. We have assumed that $z_i \neq p_j$ for all i, j so that we have n poles and m zeros. However, in the case when there are common factors in the transfer function numerator and denominator they have to be cancelled out before the system poles and zeros are identified.

Example 4.17: Let the system transfer function be given by

$$H(s) = \frac{(s+1)(s+3)}{s(s+2)(s+3)(s+4)} = \frac{s+1}{s(s+2)(s+4)}$$

After the cancellation of common factors, the system zeros and poles are identified as $z_1 = -1$ and $p_{1,2,3} = \{0, -2, -4\}$. Hence, -3 is neither the system zero nor the system pole.

Example 4.18: Consider the electrical circuit from Figure 1.10. Denote the input voltage by $f(t)$, that is $f(t) = e_i(t)$, and denote the output voltage by $y(t)$, that is $y(t) = e_0(t)$. The corresponding differential equation is given by

$$\frac{d^2y(t)}{dt^2} + \left(\frac{L + R_1R_2C}{R_2LC}\right)\frac{dy(t)}{dt} + \left(\frac{R_1 + R_2}{R_2LC}\right)y(t) = \frac{1}{LC}f(t)$$

To simplify notation, we introduce

$$a_1 = \frac{L + R_1R_2C}{R_2LC}, \qquad a_0 = \frac{R_1 + R_2}{R_2LC}, \qquad b_0 = \frac{1}{LC}$$

Assuming that the system initial conditions are zero, the Laplace transform produces

$$s^2Y(s) + a_1sY(s) + a_0Y(s) = b_0F(s)$$

so that the system transfer function is given by

$$H(s) = \frac{Y(s)}{F(s)}|_{I.C.=0} = \frac{b_0}{s^2 + a_1s + a_0}$$

The quantity

$$H(0) = K\frac{(-z_1)(-z_2)\cdots(-z_m)}{(-p_1)(-p_2)\cdots(-p_n)} = \frac{b_0}{a_0}$$

is called the *system DC gain* (system gain at zero frequency). It is also called the *system steady state gain* since it shows how much the system multiplies a constant input signal at steady state. It follows from the final value theorem of the Laplace transform that for $f(t) = au(t) \leftrightarrow F(s) = a/s$, we have

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0}\left\{sH(s)\frac{a}{s}\right\} = H(0)a$$

Note that the final value theorem is applicable under the assumption that the function $sY(s) = aH(s)$ has no poles on the imaginary axis and in the right-hand half of the complex plane, that is, when all poles of $H(s)$ are strictly in the left-hand half of the complex plane. In such a case, the system output can reach its steady state value that can be found by using this simple formula.

Example 4.19: The steady state response to a constant input $f(t) = 5u(t)$ of a system whose transfer function is given by

$$H(s) = \frac{2s^3 + 3s^2 + s + 2}{s^4 + 3s^3 + 5s^2 + s + 1} \quad \Rightarrow \quad H(0) = 2$$

exists since all poles of $H(s)$ are in the left-hand half of the complex plane (the pole location can be checked by MATLAB). The steady state system output value is

$$y_{ss}(t) = 5H(0) = 10$$

Since for the impulse delta signal the Laplace transform is given by $F(s) = 1$, we conclude from $Y(s) = H(s)F(s)$ that under zero initial conditions, the system response to the impulse delta signal is equal to $\mathcal{L}^{-1}\{H(s)\}$. In the time domain, the ***system impulse response*** is defined by

$$h(t) \triangleq \mathcal{L}^{-1}\{H(s)\}$$

For the system impulse response, the system initial conditions must be set to zero.

Example 4.20: The system impulse response for

$$y^{(2)}(t) + 3y^{(1)}(t) + 2y(t) = f^{(1)}(t) + 3f(t)$$

is obtained as follows

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{s+3}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+3}{(s+1)(s+2)}\right\}$$
$$= \frac{2}{s+1} - \frac{1}{s+2} = \left(2e^{-t} - e^{-2t}\right)u(t)$$

The system impulse response is plotted using MATLAB and presented in Figure 4.2.

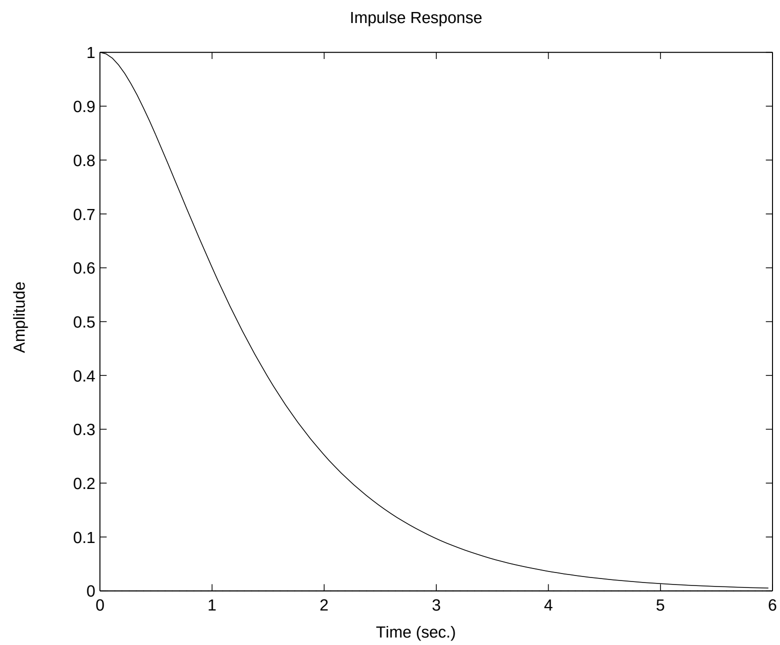

Figure 4.2: The system impulse response

Figure 4.2 represents a typical impulse response of a real physical linear system. The impulse delta signal at $t = 0$ brings the energy into the system (system excitation), which basically sets up system initial conditions to nonzero values at $t = 0^+$. As time passes, the system energy dissipates and the system response tends to zero. Note that in this example we have $h(0^-) = 0$, $h^{(1)}(0^-) = 0$.

It is easy to conclude from the expression obtained for $h(t)$ that at $t = 0^+$ we have $h(0^+) = 1,\ h^{(1)}(0^+) = 0$. In Chapter 7, where we will present the method for finding the system impulse in the time domain, we will address this phenomenon of instantaneous system signal changes at the initial time in more detail.

4.3.2 System Zero-State Response

The system response (system output) due to the given input $f(t)$ and zero system initial conditions is obtained in the frequency domain using the formula

$$Y_{zs}(s) = H(s)F(s)$$

Applying the convolution property of the Laplace transform, we obtain

$$y_{zs}(t) = h(t) * f(t) = \mathcal{L}^{-1}\{H(s)F(s)\}$$

This formula states that the system output under system zero initial conditions is equal to the convolution of the system input and the system impulse response.

We have established the most fundamental results of linear time invariant dynamic system theory. From these results, we can notice that in order to find the system response to any input signal, one must first find the system response due to the impulse delta signal, and then convolve the obtained system impulse response with the given system input signal. Note that the *system impulse response is obtained (and defined) for the system at rest* (zero initial conditions). It should be also emphasized that for the given time invariant system, the impulse response has to be found only once. Hence, any linear time invariant system is uniquely characterized by its impulse response (or by its transfer function in the frequency domain). In general, it is easy to find the system impulse response by using $h(t) = \mathcal{L}^{-1}\{H(s)\}$. As a matter of fact, this formula represents the most efficient way for finding the impulse response of continuous-time linear time invariant systems.

Example 4.21: The zero-state response of the system defined by

$$y^{(3)}(t) + 3y^{(2)}(t) + 2y^{(1)}(t) = f^{(1)}(t) + 3f(t), \quad t \geq 0$$

due to the input signal $f(t) = e^{-5t}u(t)$ can be obtained as follows. We first find the system transfer function and the Laplace transform of the input signals, that is

$$H(s) = \frac{s+3}{s^3 + 3s^2 + 2s}, \qquad F(s) = \frac{1}{s+5}$$

Then, we have

$$y_{zs}(t) = \mathcal{L}^{-1}\{H(s)F(s)\} = \mathcal{L}^{-1}\left\{\frac{(s+3)}{s(s+1)(s+2)(s+5)}\right\} =$$

$$= \mathcal{L}^{-1}\left\{\frac{3/10}{s} - \frac{1/2}{s+1} + \frac{1/6}{s+2} + \frac{1/30}{s+5}\right\}$$

$$= \left(\frac{3}{10} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-2t} + \frac{1}{30}e^{-5t}\right)u(t)$$

This response can also be obtained by using MATLAB, see Figure 4.3.

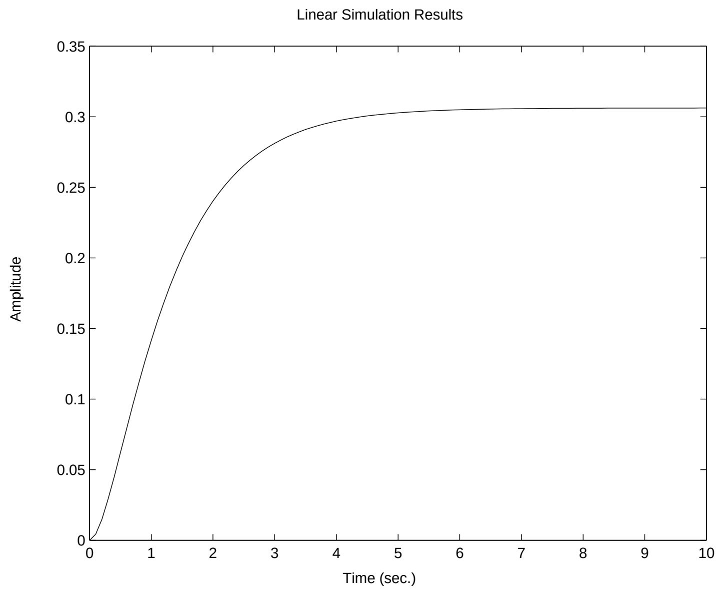

Figure 4.3: The zero-state response for Example 4.21

It should be pointed out that in Example 4.21 the initial conditions originally were given by $y_{zs}(0^-) = y_{zs}^{(1)}(0^-) = y_{zs}^{(2)}(0^-) = 0$. However, if we check the expression obtained for $y_{zs}(t)$, we will find that $y_{zs}(0^+) = y_{zs}^{(1)}(0^+) = 0$ and $y_{zs}^{(2)}(0^+) = 1$.

We know that only an impulse delta signal at zero is able to change the system initial conditions instantaneously. In this example, the impulse delta function is generated by the system's differentiation of the input signal. Namely, we have

$$f^{(1)}(t) + 3f(t) = \frac{df(t)}{dt} + 3f(t) = \frac{d}{dt}\{e^{-5t}u(t)\} + 3e^{-5t}u(t)$$

$$= -5e^{-5t}u(t) + e^{-5t}\delta(t) + 3e^{-5t}u(t) = -2e^{-5t}u(t) + \delta(t)$$

Note that when no differentiation of the input signal takes place ($m = 0$), the instantaneous change of the initial conditions could happen only in the case when the input signal is the impulse delta signal. Hence, *we can conclude that the system input signal differentiation, in general, can produce an instantaneous change in the system initial conditions*.

It is also interesting to observe that

$$\mathcal{L}\{-2e^{-5t}u(t)+\delta(t)\} = \frac{-2}{s+5}+1 = \frac{-2+s+5}{s+5} = \frac{s+3}{s+5}$$

which is identical to

$$\mathcal{L}\left\{f^{(1)}(t)+3f(t)\right\} = sF(s)-f\left(0^-\right)+3F(s) = (s+3)F(s) = \frac{s+3}{s+5}$$

It can be concluded from the previous example and the follow up discussion that the **Laplace transform has a built-in mechanism that takes into the account the delta impulses generated by the system differentiation of the input signals**. Note that the delta impulses are generated by taking the first and higher order derivatives of the input signals satisfying $f\left(0^-\right) \neq f\left(0^+\right)$. Also, in the case when $f\left(0^-\right) = f\left(0^+\right)$, but the input signal is not differentiable in the ordinary sense at $t = 0$, for example $\sin(t)u(t)$, the second and higher order generalized derivatives of this signal will produce the delta impulses, see Problem 4.21.

Another interesting phenomenon can be deduced from Example 4.21 by observing the expression obtained for the system *zero-state* response. It can be seen that the input signal $f(t) = e^{-5t}u(t)$ produces on the system output the component proportional to $e^{-5t}u(t)$ (which is expected) and components that correspond to system modes (poles) $p_1 = 0$, $p_2 = -1$, $p_2 = -2$. Hence, despite the fact that all initial conditions are zero, *the system input excites all system modes so that, in general, all of them appear on the system output.*

4.3.3 Unit Step and Ramp Responses

Finding the system response due to a unit step is a common problem in engineering. The unit step response can be related to the system impulse response by a very simple formula, assuming that in this case the *system initial conditions are also set to zero*. Namely, for $f(t) = u(t)$, we have

$$h(t) * u(t) = \int_{0^-}^{t} u(t-\tau)h(\tau)d\tau = \int_{0^-}^{t} h(\tau)d\tau \triangleq y_{step}(t)$$

where $y_{step}(t)$ denotes the system step response under system zero initial conditions (system at rest). Note that we have taken the lower integration limit at $t = 0^-$ in order to be able to completely include the delta impulse signal $\delta(\tau)$ within the integration limits since in the case when $n = m$ the system impulse response contains the delta impulse signal at the origin. From the above formula we have

$$h(t) = \frac{dy_{step}(t)}{dt}$$

This corresponds in the frequency domain to

$$Y_{step}(s) = \frac{1}{s}H(s)$$

Note that by the definition of the system step response, the zero system initial conditions must be used. Very often this formula represents an easier way to find the system step response than the corresponding time-convolution formula.

Similarly, we can get the system unit *ramp response* subject to zero-initial conditions. In this case $f(t) = r(t)$ so that

$$y_{ramp}(t) = h(t) * r(t) = \int_{0^-}^{t} (t-\tau)u(t-\tau)h(\tau)d\tau$$

$$= \int_{0^-}^{t} (t-\tau)h(\tau)d\tau$$

Integrating by parts and using the result for $y_{step}(t)$, we have

$$\int_{0^-}^{t} (t-\tau)h(\tau)d\tau = (t-\tau)y_{step}(\tau)|_{\tau=0^-}^{\tau=t} - \int_{0^-}^{t} y_{step}(\tau)(-d\tau)$$

$$= \int_{0^-}^{t} y_{step}(\tau)d\tau$$

that is

$$y_{ramp}(t) = h(t) * r(t) = \int_{0^-}^{t} y_{step}(\tau)d\tau$$

By taking the derivative of $y_{ramp}(t)$, we obtain

$$y_{step}(t) = \frac{dy_{ramp}(t)}{dt}$$

The relationship between the impulse response and the ramp response is

$$h(t) = \frac{dy_{step}(t)}{dt} = \frac{d^2 y_{ramp}(t)}{dt^2}$$

It follows also from the above discussion that

$$y_{ramp}(t) = \int_{0^-}^{t} y_{step}(\tau)d\tau = \int_{0^-}^{t}\int_{0^-}^{\tau} h(\sigma)d\sigma d\tau$$

Note that in the frequency domain this result corresponds to

$$Y_{ramp}(s) = \frac{1}{s^2}H(s)$$

Example 4.22: The step response of the system at rest considered in Example 4.21 can be obtained by integrating the corresponding impulse response. The impulse response is obtained from

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{s+3}{s^3+3s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{s+3}{s(s+1)(s+2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1.5}{s} - \frac{2}{s+1} + \frac{0.5}{s+2}\right\} = \left(1.5 - 2e^{-t} + 0.5e^{-2t}\right)u(t)$$

The step response is given by

$$y_{step}(t) = \int_0^t h(\tau)d\tau = \int_0^t \left(1.5 - 2e^{-\tau} + 0.5e^{-2\tau}\right)d\tau$$

$$= \left(1.5t + 2e^{-t} - 2 - 0.25e^{-2t} + 0.25\right)u(t)$$

$$= \left(-1.75 + 1.5t + 2e^{-t} - 0.25e^{-2t}\right)u(t)$$

The ramp response, obtained for zero initial conditions, is

$$y_{ramp}(t) = \int_0^t y_{step}(\tau)d\tau = \int_0^t \left(-1.75 + 1.5\tau + 2e^{-\tau} - 0.25e^{-2\tau}\right)d\tau$$

$$= 1.75 - 1.75t + 0.75t^2 - 2e^{-t} + 0.25e^{-2t}, \quad t \geq 0$$

4.3.4 Complete System Response

Let us define the *system characteristic polynomial* by

$$\Delta(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

The complete system response in the frequency domain is given by

$$Y(s) = H(s)F(s) + \frac{I(s)}{\Delta(s)} = Y_{zs}(s) + Y_{zi}(s)$$

Hence, the complete system response is obtained as the sum of the zero-state and zero-input responses. By taking the Laplace inverse of the last equation, we obtain the complete system response in the time domain

$$y(t) = \mathcal{L}^{-1}\{H(s)F(s)\} + \mathcal{L}^{-1}\left\{\frac{I(s)}{\Delta(s)}\right\} = y_{zs}(t) + y_{zi}(t)$$

In some textbooks, the zero-state response is also called the *forced system response* and the zero-input response is called the *natural system response*.

Note that there is no need that we treat in a separate section the problem of finding the system zero-input response by using the Laplace transform. It can be simply found by using the results presented in this section as

$$y_{zi}(t) = \mathcal{L}^{-1}(Y_{zi}(s)) = \mathcal{L}^{-1}\left(\frac{I(s)}{\Delta(s)}\right)$$

Example 4.23: The complete response of the system

$$y^{(2)}(t) + 6y^{(1)}(t) + 9y(t) = f(t), \quad f(t) = e^{-2t}u(t)$$

$$y(0^-) = -1, \;\; y^{(1)}(0^-) = 2$$

can be obtained as follows. Applying the Laplace transform, we have

$$\left(s^2Y(s) - sy(0^-) - y^{(1)}(0^-)\right) + 6\big(sY(s) - y(0^-)\big) + 9Y(s) = \frac{1}{s+2}$$

which implies

$$Y(s) = \frac{1}{(s+3)^2}\frac{1}{(s+2)} - \frac{s+4}{(s+3)^2} = H(s)F(s) + \frac{I(s)}{\Delta(s)}$$

Note that the system transfer function and the characteristic polynomial are given by

$$H(s) = \frac{1}{s^2 + 6s + 9}, \quad \Delta(s) = s^2 + 6s + 9$$

Using the Laplace inverse we obtain the zero-state response

$$y_{zs}(t) = \mathcal{L}^{-1}\{Y_{zs}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+3} - \frac{1}{(s+3)^2}\right\}$$
$$= \left(e^{-2t} - e^{-3t} - te^{-3t}\right)u(t)$$

and the zero-input response

$$y_{zi}(t) = \mathcal{L}^{-1}\{Y_{zi}(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{s+3} - \frac{1}{(s+3)^2}\right\}$$
$$= \left(-e^{-3t} - te^{-3t}\right)u(t)$$

The complete system response in now given by

$$y(t) = y_{zs}(t) + y_{zi}(t) = \left(e^{-2t} - 2e^{-3t} - 2te^{-3t}\right)u(t)$$

In the case when the input signal has a non standard analytical form, for example

$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

the input signal should be represented in terms of elementary signals considered in Chapter 2 (in this case, $f(t) = r(t) - r(t-1) - u(t-1)$) and the linearity and time invariance principles should be used. Note that the linearity principle is established for linear systems at rest, that is for zero initial conditions. Hence, this way we obtain $y_{zs}(t)$ component of the system response. The system response due to initial conditions, $y_{zi}(t)$, has to be found independently. Let $y_{ramp}(t)$ represent the system zero-state response due to $r(t)$, and let $y_{step}(t)$ represent the system zero-state response due to $u(t)$, then the system zero-state response due to $f(t)$ can be obtained by using linearity and time invariance as

$$y_{zs}(t) = y_{ramp}(t) - y_{ramp}(t-1) - y_{step}(t-1)$$

Example 4.24: Find the complete response of the system defined by

$$y^{(2)}(t) + 3y^{(1)}(t) + 2y(t) = f(t), \qquad y(0^-) = 1,\ y^{(1)}(0^-) = 1$$

due to the forcing function given by $f(t) = r(t) - r(t-1) - u(t-1)$. Since $y(t) = y_{zi}(t) + y_{zs}(t)$ we can find independently $y_{zi}(t)$ and $y_{zs}(t)$.

The zero-input response satisfies

$$y_{zi}^{(2)}(t) + 3y_{zi}^{(1)}(t) + 2y_{zi}(t) = 0, \qquad y_{zi}(0^-) = 1,\ y_{zi}^{(1)}(0^-) = 1$$

By applying the Laplace transform we have

$$Y_{zi}(s) = \frac{s+4}{s^2+3s+2} = \frac{I(s)}{\Delta(s)}$$

The time domain zero-input response is obtained as follows

$$y_{zi}(t) = \mathcal{L}^{-1}\{Y_{zi}(s)\} = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1} - \frac{2}{s+2}\right\}$$
$$= \left(3e^{-t} - 2e^{-2t}\right)u(t)$$

The impulse response is

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\}$$
$$= \left(e^{-t} - e^{-2t}\right)u(t)$$

The step response is given by

$$y_{step}(t) = \int_0^t h(\tau)d\tau = \int_0^t \left(e^{-\tau} - e^{-2\tau}\right)d\tau = \left(\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right)u(t)$$

The ramp response is obtained as follows

$$y_{ramp}(t) = \int_0^t y_{step}(\tau)d\tau = \int_0^t \left(\frac{1}{2} - e^{-\tau} + \frac{1}{2}e^{-2\tau}\right)d\tau$$
$$= \left(-\frac{3}{4} + \frac{1}{2}t + e^{-t} - \frac{1}{4}e^{-2t}\right)u(t)$$

The zero-state response is given by

$$y_{zs}(t) = y_{ramp}(t) - y_{step}(t-1) - y_{ramp}(t-1)$$

$$= \left(-\frac{3}{4} + \frac{1}{2}t + e^{-t} - \frac{1}{4}e^{-2t}\right)u(t) - \left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right)u(t-1)$$

$$-\left(-\frac{3}{4} + \frac{1}{2}(t-1) + e^{-(t-1)} - \frac{1}{4}e^{-2(t-1)}\right)u(t-1)$$

which simplifies to

$$y_{zs}(t) = \left(-\frac{3}{4} + \frac{1}{2}t + e^{-t} - \frac{1}{4}e^{-2t}\right)u(t)$$

$$+\left(\frac{3}{4} - \frac{1}{2}t - \frac{1}{4}e^{-2(t-1)}\right)u(t-1)$$

The complete system response is obtained as the sum of the zero-input and zero-state responses

$$y(t) = y_{zi}(t) + y_{zs}(t) = \left(-\frac{3}{4} + \frac{1}{2}t + 4e^{-t} - \frac{9}{4}e^{-2t}\right)u(t)$$
$$+\left(\frac{3}{4} - \frac{1}{2}t - \frac{1}{4}e^{-2(t-1)}\right)u(t-1)$$

Note that in this particular example, the system response is a continuous function at $t = 1$ despite the fact that the system input signal has a jump at that point. This can be easily checked by observing that the coefficient that multiplies $u(t-1)$ is equal to zero at $t = 1$, that is $3/4 - 1/2 - 1/4 = 0$. Also, the first derivative of $y(t)$ is continuous at $t = 1$ since

$$\frac{d}{dt}\left\{\left(\frac{3}{4} - \frac{1}{2}t - \frac{1}{4}e^{-2(t-1)}\right)u(t-1)\right\}_{|t=1} =$$

$$= \left(-\frac{1}{2} + \frac{1}{2}e^{-2(t-1)}\right)_{|t=1} u(t-1)$$

$$+ \left(\frac{3}{4} - \frac{1}{2}t - \frac{1}{4}e^{-2(t-1)}\right)_{|t=1} \delta(t-1)$$

$$= 0u(t-1) + 0\delta(t-1) = 0$$

The second derivative of $y(t)$ has a jump discontinuity at $t = 1$ equal to -1, which is identical to a jump in the input signal at the same time instant. This observation can be easily confirmed by finding the second derivative of $y(t)$ and evaluating it at $t = 1$.

Sinusoidal Response via the Laplace Transform

Let the system input be sinusoidal

$$f(t) = A\cos(\omega_0 t) \leftrightarrow F(s) = \frac{As}{s^2 + \omega_0^2}$$

Then, the system zero-state response is given by

$$Y_{zs}(s) = H(s)F(s) = H(s)\frac{As}{s^2+\omega_0^2} = \frac{N(s)}{\Delta(s)} + \frac{c}{s - j\omega_0} + \frac{c^*}{s + j\omega_0}$$

where

$$c = \lim_{s \to j\omega_0} \{(s - j\omega_0)H(s)F(s)\} = H(s)\frac{As}{(s + j\omega_0)}|_{s=j\omega_0} = \frac{A}{2}H(j\omega_0)$$

Using the inverse Laplace transform, we obtain

$$y_{zs}(t) = \mathcal{L}^{-1}\left\{\frac{N(s)}{\Delta(s)}\right\} + A|H(j\omega_0)|\cos(\omega_0 t + \measuredangle H(j\omega))$$

The first component comes from the system natural modes excited by the input signal $A\cos(\omega_0 t)$.