Why Binary Number System?

- Because computers work with binary values
  0 & 1, LOW & HIGH, TRUE & FALSE
- Recall the basic building blocks
  -- AND, OR, NOT logic gates

- \[ Z = X \cdot Y \]
- \[ Z = X + Y \]
- \[ Z = X' \]

- We need to learn to work with the Binary Number System
Number Systems

A positional number system has a **radix** (or base of the number) any integer \( r \geq 2 \)

\[
d_{p-1} \ldots d_k \ldots d_1 d_0 \cdot d_{-1} \ldots d_{-j} \ldots d_{-n}
\]

Most Significant Digit  \( r^k \)  radix point  \( r^{-j} \)  Least Significant Digit

\[
D = \sum_{i = -n}^{p-1} d_i \cdot r^i \quad 0 \leq d_i \leq (r-1)
\]

Example: 25.375 radix 10

\[
2 \cdot 10^1 + 5 \cdot 10^0 + 3 \cdot 10^{-1} + 7 \cdot 10^{-2} + 5 \cdot 10^{-3}
\]

The integer and the fractional part are processed separately.

Powers of 2: \( 2^n \)

It will be convenient to remember these powers:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 2^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
</tr>
<tr>
<td>7</td>
<td>128</td>
</tr>
<tr>
<td>8</td>
<td>256</td>
</tr>
<tr>
<td>9</td>
<td>512</td>
</tr>
<tr>
<td>10</td>
<td>1024</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 2^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>0.5</td>
</tr>
<tr>
<td>(-2)</td>
<td>0.25</td>
</tr>
<tr>
<td>(-3)</td>
<td>0.125</td>
</tr>
</tbody>
</table>
Integer Part

\[ \sum_{i=0}^{p-1} d_i \cdot r^i = ((\cdots (d_{p-1} \cdot r + d_{p-2}) \cdot r + \cdots + d_2) \cdot r + d_1) \cdot r + d_0 \]

*Divide* by \( r \), the remainder is \( d_0, d_1, d_2, \ldots \) from the least significant to the most significant digit.

Will use \( r = 2 \), binary conversion: \( d_i = \{0, 1\} \)

Example: \( 25_{10} = \; ?_2 \)

\[
\begin{align*}
25:2 &= 12R1 & d_0 &= 1 \\
12:2 &= 6R0 & d_1 &= 0 \\
6:2 &= 3R0 & d_2 &= 0 \\
3:2 &= 1R1 & d_3 &= 1 \\
1:2 &= 0R1 & d_4 &= 1
\end{align*}
\]

\( \Rightarrow 25_{10} = 11001_2 \)  

Verify the result: \( 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^0 = 16 + 8 + 1 = 25 \)
Fractional Part

\[ \sum_{i=-n}^{-1} d_i \cdot r^i = r^{-1}(d_{-1} + r^{-1} \cdot (d_{-2} + \ldots )) \]

Same like before, but now we multiply with the radix.

Example: \[0.375_{10} = ?_2\]

\[
\begin{align*}
0.375 \times 2 &= 0.750 < 1 \quad \Rightarrow \quad d_{-1} = 0 \\
0.750 \times 2 &= 1.500 > 1 \quad \Rightarrow \quad d_{-2} = 1 \\
0.500 \times 2 &= 1.000 \quad \Rightarrow \quad d_{-3} = 1
\end{align*}
\]

Verify the result: \[1 \cdot 2^{-2} + 1 \cdot 2^{-3} = 0.25 + 0.125 = 0.375\]
Important Radices

Radices important to computer engineers are: \( r = 2, 8, 16 \)

<table>
<thead>
<tr>
<th>Binary</th>
<th>Decimal</th>
<th>Octal</th>
<th>Hexadecimal</th>
<th>Octal</th>
<th>Hexadecimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>000</td>
<td>0</td>
<td>000</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>001</td>
<td>1</td>
<td>001</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>2</td>
<td>010</td>
<td>2</td>
<td>0010</td>
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<tr>
<td>11</td>
<td>3</td>
<td>3</td>
<td>011</td>
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<td>0011</td>
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<tr>
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<td>0100</td>
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<tr>
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<td>5</td>
<td>0101</td>
</tr>
<tr>
<td>110</td>
<td>6</td>
<td>6</td>
<td>110</td>
<td>6</td>
<td>0110</td>
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<td>111</td>
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<td>111</td>
<td>7</td>
<td>0111</td>
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<td>1001</td>
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<td>1001</td>
</tr>
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<td>1010</td>
<td>10</td>
<td>12</td>
<td>A</td>
<td>10</td>
<td>1010</td>
</tr>
<tr>
<td>1011</td>
<td>11</td>
<td>13</td>
<td>B</td>
<td>11</td>
<td>1011</td>
</tr>
<tr>
<td>1100</td>
<td>12</td>
<td>14</td>
<td>C</td>
<td>12</td>
<td>1100</td>
</tr>
<tr>
<td>1101</td>
<td>13</td>
<td>15</td>
<td>D</td>
<td>13</td>
<td>1101</td>
</tr>
<tr>
<td>1110</td>
<td>14</td>
<td>16</td>
<td>E</td>
<td>14</td>
<td>1110</td>
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<tr>
<td>1111</td>
<td>15</td>
<td>17</td>
<td>F</td>
<td>15</td>
<td>1111</td>
</tr>
</tbody>
</table>

Example: \( 11100001.011_{2} = 01110001001.011_{2} = 341.3_{8} \)

\[
341.3_{8} = 3 \cdot 8^2 + 4 \cdot 8^1 + 1 \cdot 8^0 + 3 \cdot 8^{-1} = 225.375_{10}
\]

Fourth digit was added to the fractional part

\( 11100001.011_{2} = 11100001.011_{2} = E1.6_{16} \)

\( E1.6_{16} = 14 \cdot 16^1 + 1 \cdot 16^0 + 6 \cdot 16^{-1} = 225.375_{10} \)

Binary Addition

\[
s = \text{sum} \quad c_{\text{in}} = \text{carry in} \quad c_{\text{out}} = \text{carry out}
\]

\[
X, Y, c_{\text{in}} \rightarrow s, c_{\text{out}}
\]

Decimal:

\[
1_{10} + 1_{10} = 2_{10}
\]

Binary:

\[
1_{2} + 1_{2} \downarrow = 10_{2}
\]

\[
1_{2} + 1_{2} = 10_{2}
\]
Binary Addition

<table>
<thead>
<tr>
<th>$c_{in}$</th>
<th>$X$</th>
<th>$Y$</th>
<th>$c_{out}$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0</td>
<td>0</td>
<td>1</td>
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<td>1</td>
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<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

$c_{in} = \text{carry in}$  
$c_{out} = \text{carry out}$

$X, Y, C_{in} \rightarrow s, C_{out}$

Example addition:

<table>
<thead>
<tr>
<th>X</th>
<th>1 1 0 1 1 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>+ $Y$</td>
<td>1 0 1 1 1</td>
</tr>
<tr>
<td>+ carries</td>
<td>1 1 1 1 1</td>
</tr>
<tr>
<td>sum =</td>
<td>________</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>1 1 0 1 1 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>+ $Y$</td>
<td>1 0 1 1 1</td>
</tr>
<tr>
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<td>1 1 1 1 1</td>
</tr>
<tr>
<td>sum =</td>
<td>________</td>
</tr>
</tbody>
</table>

Subtraction

Decimal:

$25_{10}$

$- 7_{10}$

$= 18$

$25_{10}$ borrow $10_{10}$ from the next leftward digit

$25 \rightarrow$ minuend

$7 \rightarrow$ subtrahend

$18 \rightarrow$ difference

Binary:

$100_{2}$

$- 1_{2}$

$= 011_{2}$

$100_{2}$ borrow $10_2 (=2_{10})$ from the next leftward digit

$100 \rightarrow$ minuend

$1 \rightarrow$ subtrahend

$011 \rightarrow$ difference

$10_2 = 1_2$

because $-1_2 = 1_2$
Binary Subtraction

<table>
<thead>
<tr>
<th>b&lt;sub&gt;in&lt;/sub&gt;</th>
<th>x</th>
<th>y</th>
<th>b&lt;sub&gt;out&lt;/sub&gt;</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

b<sub>in</sub> = borrow in
b<sub>out</sub> = borrow out

X, Y, B<sub>in</sub> → s, B<sub>out</sub>

X – Y – B<sub>in</sub> = d

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Example Subtraction (1)

<table>
<thead>
<tr>
<th>minuend</th>
<th>X</th>
<th>229</th>
</tr>
</thead>
<tbody>
<tr>
<td>subtrahend</td>
<td>Y</td>
<td>46</td>
</tr>
<tr>
<td>difference</td>
<td>X – Y</td>
<td>183</td>
</tr>
</tbody>
</table>

Must borrow 1, yielding the new subtraction 102 – 12 = 12

After the first borrow, the new subtraction for the column is (1 – 1) – 1, so we must borrow again.

The borrow ripples through leftwards until there is a non-zero digit from which to borrow.
Example Subtraction (2)

<table>
<thead>
<tr>
<th>minuend</th>
<th>X</th>
<th>77</th>
</tr>
</thead>
<tbody>
<tr>
<td>subtrahend</td>
<td>Y</td>
<td>– 23</td>
</tr>
<tr>
<td>difference</td>
<td>X – Y</td>
<td>54</td>
</tr>
</tbody>
</table>

Verify the result:

\[
\begin{array}{c}
\text{X} \quad 77 \\
\text{Y} \quad -23 \\
\hline
\text{X} - \text{Y} \quad 54
\end{array}
\]

Signed-Magnitude Representation

Use the MSB for the sign:

- For positive numbers: \( d_{n-1} = 0 \)
- For negative numbers: \( d_{n-1} = 1 \)

- \( d_{n-1} \) is the sign bit
- \( d_{n-2} \ldots \) are the magnitude bits

- \( n \) bits represent \( 2^n \) numbers

- Largest positive number: 011...1
  \[
  \sum_{i=0}^{n-2} 1 \cdot 2^i = 2^{n-1} - 1
  \]
- Smallest negative number: 111...1
  \[-(2^{n-1} - 1)\]
- Two representations for zero: 000...0 and 100...0
Signed-Magnitude Arithmetic

Arithmetic operations must process the sign separately.
For example, subtraction: $A - B$

1. Compare the magnitudes $A \geq B$
2. Subtract smaller magnitude from larger magnitude
3. If $B > A$, then change the sign of the result

… too complicated … will NOT use it for computations.

Instead, we use two’s complement representation …

Radix-Complement Representation

Assumptions:
- fixed number of digits, $n$
- $D = d_{n-1} \ldots d_k \ldots d_1 d_0$, radix $r$

radix-complement representation of $D$:

$$[D]_r = r^n - D$$

The involution property:

$$[[D]_r]_r = r^n - (r^n - D) = D$$

How to compute it?
(would like to avoid subtraction)
Radix-Complement Computation

\[ [D]_r = r^n - D \]

- Rewrite \( r^n = (r^n - 1) + 1 \)
- Then, \[ [D]_r = r^n - D = ((r^n - 1) - D) + 1 \]

- Observe that \((r^n - 1)\) has the form \(mm \ldots mm\) where \(m = r - 1\)
  - For example, for \(r = 10\) and \(n = 4\), \((r^n - 1) = 9999\)
  - For \(r = 2\) and \(n = 5\), \((r^n - 1) = 11111\)

- Define the complement of a digit \(d\) to be \(d' = r - 1 - d\)
  - For example, for \(r = 10\), the complements of 3, 5, and 8 are
    \(3'_{10} = 10 - 1 - 3 = 6\)
    \(5'_{10} = 10 - 1 - 5 = 4\)
    \(8'_{10} = 10 - 1 - 8 = 1\)

- Then, the complement of \(D\) is obtained by complementing individual digits of \(D\) and adding 1

2’s-Complement Representation

n-bit 2’s-complement representation of \(D\):
\[ [D]_2 = 2^n - D_2 \]

Compute two’s complement as:
\[ [D]_2 = (2^n - 1 - D_2) + 1 \]

\[
\begin{align*}
2^n - 1: & \quad 1 \ 1 \ \ldots \ 1 \quad \leftarrow n \text{ bits} \\
- \ D: & \quad -d_{n-1}d_{n-2} \ldots d_0 \\
\text{d'}_{n-1} \text{d'}_{n-2} \ldots \text{d'}_0 & \quad + 1 \\
\hline
[D]_2 & \quad 1 - d_i = d'_i, \quad 1 - 0 = 1 \\
& \quad 1 - 1 = 0
\end{align*}
\]
2's-Complement Computation

1. Complement the digits
2. Add 1 to the Least Significant Bit
3. Discard carry out from Most Significant Bit

\[ [D]_2 = (2^n - 1 - D_2) + 1 \]

Two's Complement Number System

![Diagram showing two's complement number system with binary representations of numbers from -8 to 7.](image)
### 2’s-Complement Representation (2)

Range of n-bit 2’s complement: \(-2^{n-1} \leq A \leq 2^{n-1} - 1\)

Example: \(n = 5\)
represent \(-13_{10}\) in 2’s complement:

\[
\begin{align*}
13_{10} &= 01101_2 \rightarrow 10010 \\
& \quad \downarrow \\
10011 &= -13_{10}
\end{align*}
\]

What decimal number is represented in 5-bit 2’s complement:
11010

?  

---

### 2’s-Complement Representation (2)

Range of n-bit 2’s complement: \(-2^{n-1} \leq A \leq 2^{n-1} - 1\)

Example: \(n = 5\)
represent \(-13_{10}\) in 2’s complement:

\[
\begin{align*}
13_{10} &= 01101_2 \rightarrow 10010 \\
& \quad \downarrow \\
10011 &= -13_{10}
\end{align*}
\]

What decimal number is represented in 5-bit 2’s complement:

\[
\begin{align*}
\text{negative number for magnitude: } & \quad 00101 \\
& \quad \text{complement the digits and add 1} \\
& \quad 00110 + 1 \\
00111 & \quad \text{that is } 6_{10}
\end{align*}
\]

so the number is: \(-6_{10}\)
<table>
<thead>
<tr>
<th>Representation</th>
<th>1011001₂ = ?₁₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>Answer:</td>
<td>depends on the representation!</td>
</tr>
<tr>
<td>Unsigned:</td>
<td>1011001₂ = 89₁₀</td>
</tr>
<tr>
<td>Signed-magnitude:</td>
<td>1011001₂ = – 25₁₀</td>
</tr>
<tr>
<td>Two’s complement:</td>
<td>1011001₂ = – 39₁₀</td>
</tr>
</tbody>
</table>