Sampling, Fourier Transform (FT), Discrete-time Fourier Transform (DTFT), & Discrete Fourier Transform (DFT)

The Fourier Transform (FT), Discrete-time Fourier Transform (DTFT), and Discrete Fourier Transform (DFT), are all interconnected via the sampling process. Our intention here is to show these interconnections.

Sampling: Let us consider a continuous-time signal $x(t)$ and its sampled signal $x_s(t)$,

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT), \tag{1}$$

where T is the sampling period. Let

$$\begin{aligned} f_s &= \frac{1}{T} = \text{ Sampling frequency in Hz} \\ \omega_s &= 2\pi f_s = \frac{2\pi}{T} \text{ Sampling frequency in Rad/Sec.} \end{aligned}$$

Since the impulse train $\sum_{n=-\infty}^{\infty} \delta(t-nT)$ is periodic with a period T, we can first construct its Fourier series as $\frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j k\omega_s t}$, and then rewrite $x_s(t)$ as

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = x(t)\frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j k\omega_s t} = \frac{1}{T} \sum_{k=-\infty}^{\infty} x(t)e^{j k\omega_s t}. \tag{2}$$

We can now take the Fourier Transform of the above equation to get

$$X_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s). \tag{3}$$

The above equation relates the Fourier Transform of the sampled signal to that of the original signal. As seen clearly, $X_s(\omega)$ is obtained by replicating $X(\omega)$ infinite number of times in frequency domain. This is often quoted as **Sampling in time domain replicates the spectrum in frequency domain**. Indeed this replication process leads us to derive **Sampling Theorem and Interpolation Formula**.

We can also rewrite $x_s(t)$ as

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT). \tag{4}$$

We can also take the Fourier Transform of the above equation to get

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega T n}. \tag{5}$$

Thus there are two equivalent expressions for $X_s(\omega)$, one as given by (3), and another as given by (5). As said earlier, equation (3) is the underlying basic expression that leads to sampling theorem. On the other hand, equation (5) is the underlying basic expression that leads to **Discrete-time Fourier Transform (DTFT)**.

Discrete-time Fourier Transform (DTFT): Let us next define a sequence of numbers representing $x_s(t)$ and thus $x(t)$,

$$x[0] = x(0),\ x[1] = x(T),\ x[2] = x(2T),\ \cdots,\ x[n] = x(nT),\ \cdots\cdots. \tag{6}$$

The Discrete-time Fourier Transform is defined by

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\,\Omega\,n}, \tag{7}$$

where Ω is known as discrete-time frequency. We need to make an important remark here.
Remark: Although $X(\Omega)$ is obtained from a discrete signal or a sequence of numbers, $X(\Omega)$ itself is a function of continuous variable Ω.

In view of (6), we see easily that $X_s(\omega)$ as given by (5) and $X(\Omega)$ as given by (7) are exactly the same if

$$\Omega = \omega\,T = \frac{\omega}{f_s} = 2\pi\frac{f}{f_s}. \tag{8}$$

The above equation relates the continuous-time frequency ω to the discrete-time frequency Ω and viceversa. In particular, if continuous-time frequency ω equals the sampling frequency ω_s, the discrete-time frequency Ω equals 2π. Next, it is easy to see from (7) that $X(\Omega)$ is periodic in Ω with a period equal to 2π. This also follows from (3) when we observe that $X_s(\omega)$ is periodic in ω with a period equal to ω_s; this implies that $X(\Omega)$ is periodic in Ω with a period equal to 2π.
Next, we need to make another important remark.
Remark: Since $X(\Omega)$ is periodic in Ω with a period equal to 2π, we need to evaluate $X(\Omega)$ only for $0 \le \Omega < 2\pi$ or equivalently for $-\pi \le \Omega < \pi$. Also, knowing $X(\Omega)$ for $0 \le \Omega < 2\pi$ or equivalently for $-\pi \le \Omega < \pi$, we can get back $x[n]$ easily as

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(\Omega)e^{j\,\Omega\,n}\,d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\,\Omega\,n}\,d\Omega\,.$$

The above is the inverse DTFT formula or IDTFT. To verify it follow the steps (**Do it**),
(1) substitute for $X(\Omega)$ from (7),
(2) interchange the order of integration and summation, and finally
(3) simplify the resulting integration and summation.

Although there are interconnections between DTFT and FT of a sampled signal, DTFT and IDTFT as summarized below can be thought of as a new transform pair for discrete-time signals,

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\,\Omega\,n} && \text{DTFT} \\ x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\,\Omega\,n}\,d\Omega\,. && \text{IDTFT} \end{aligned}$$

Discrete Fourier Transform (DFT): Suppose we have a finite sequence of numbers $x[0],\ x[1],\ x[2],\ \cdots,\ x[N-1]$. We can define a DTFT for this sequence. As we discussed earlier, DTFT is periodic in Ω and we need to evaluate it only for $0 \leq \Omega < 2\pi$. Now let us discretize the DTFT and evaluate it only at

$$\Omega = 0,\ \Omega = \frac{2\pi}{N},\ \Omega = 2\frac{2\pi}{N},\ \cdots\ \Omega = (N-1)\frac{2\pi}{N}.$$

Then we get a finite sequence of numbers representing DTFT . This finite sequence of numbers is known as **Discrete Fourier Transform (DFT)**. Although there are inter-connections between DTFT and DFT of a finite sequence, DFT and inverse DFT as given below can be thought of as a new transform pair for discrete-time signals,

$$X_k = \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi k\,n}{N}} \qquad \text{for } k = 0,\ 1,\ \cdots, N-1$$
(Discrete Fourier Transform)

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{\frac{j2\pi k\,n}{N}} \qquad \text{for } n = 0,\ 1,\ \cdots, N-1\,.$$
(Inverse Discrete Fourier Transform)