



School of Engineering  
Department of Electrical and Computer Engineering

332:224 Principles of Electrical Engineering II Laboratory

Experiment 3

## *Steady State Frequency Response Using Bode Plots*

### 1 Introduction

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- Objectives**
- To study the steady state frequency response of stable transfer functions of certain simple circuits using Bode plots

#### **Overview**

This experiment treats the subject of frequency response by the use of Bode plots. The basic equation and logic of Bode plots are introduced in section 2 for transfer functions with real and complex-conjugate poles and zeros. The asymptotes for the magnitude and phase plots are introduced, as is the method for correcting around the corner frequency to obtain realistic plots.

Four circuits with transfer characteristics that exhibit illustrative examples of the pole and zero combinations covered in section 2 are realized in section 4. Their theoretical Bode plots are derived in section 3 and data for plotting their actual ones are extracted in section 4 by varying the frequency and measuring output/input (magnitude of gain) and phase.

## 2 Theory

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### 2.1 Introduction to Bode plots<sup>1</sup>

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Bode plots are commonly used to display the steady state frequency response of a stable system. Let the transfer function of a stable system be  $H(s)$ . Also, let  $M(\omega)$  and  $\theta(\omega)$  be respectively the magnitude and the phase angle of  $H(j\omega)$ . In Bode plots, the magnitude characteristic  $M(\omega)$  and the phase angle characteristic  $\theta(\omega)$  of the frequency response are plotted vs.  $\log_{10}(\omega)$ . Also, in the magnitude plot, instead of plotting  $M(\omega)$ , vs.  $\omega$  one plots  $20 \log_{10}M(\omega)$  which is called the *decibel*<sup>2</sup> (abbreviated as dB) value of  $M(\omega)$ . By utilizing the logarithm, one accommodates a wide range of data on a compact graph. Moreover, as will be seen shortly from equations (3) and (4), the logarithmic operation permits the decomposition of the transfer function into simpler parts in a natural way. By analyzing each of the simpler parts individually, and then combining them appropriately, one can plot relatively easily the overall frequency response of a composite transfer function. Note here that the DC frequency  $\omega=0$  cannot be represented in the logarithmic scale since  $\log_{10}(0)=-\infty$ .

### 2.2 Transfer function standard form

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Consider a rational transfer function  $H(s)$  in which all the poles and zeros are shown in a factored form as

$$H(s) = \frac{Ks^r (s + z_1)(s + z_2)\dots(s + z_m)}{(s + p_1)(s + p_2)\dots(s + p_n)} \quad (1)$$

where  $m$ ,  $n$ , and  $r$  are some integers. It is convenient to rewrite  $H(j\omega)$  in the form known as a standard form for Bode plots,

$$H(j\omega) = \frac{K_o (j\omega)^r (1 + j\omega / z_1)(1 + j\omega / z_2)\dots(1 + j\omega / z_m)}{(1 + j\omega / p_1)(1 + j\omega / p_2)\dots(1 + j\omega / p_n)} \quad (2A)$$

where

$$K_o = \frac{Kz_1z_2\dots z_m}{p_1p_2\dots p_n} \quad (2B)$$

If any of the poles or zeros are complex, each of the complex conjugate pairs of such poles or zeros of  $H(s)$  can be combined to get a quadratic form of the type  $s^2 + (2\zeta/\omega_0)s + \omega_0^2$ , and then, by factoring out  $\omega_0^2$  and letting  $s = j\omega$ , it can be reduced to a standard form as

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<sup>1</sup> The subject is treated in detail in Appendix E of the text.

<sup>2</sup> The decibel is treated in detail in Appendix D of the text.

$$1 + (2\zeta/\omega_0)j\omega - \omega^2/\omega_0^2$$

### 2.3 Real Poles and Zeros

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First assume that all the poles and zeros are real. Then, one can easily determine the magnitude and phase functions  $20 \log_{10}M(\omega)$  and  $\theta(\omega)$  as

$$\begin{aligned} 20 \log_{10}M(\omega) = & 20 \log_{10}(|K_0|) + 20 r \log_{10}(j\omega) \\ & + 20 \log_{10}(|1+j\omega/z_1|) + 20 \log_{10}(|1+j\omega/z_2|) + \dots \\ & - 20 \log_{10}(|1+j\omega/p_1|) - 20 \log_{10}(|1+j\omega/p_2|) - \dots \end{aligned} \quad (3)$$

$$\begin{aligned} \theta(\omega) = & \angle K_0 + r 90^\circ + \tan^{-1}(\omega/z_1) + \tan^{-1}(\omega/z_2) + \dots \\ & - \tan^{-1}(\omega/p_1) - \tan^{-1}(\omega/p_2) - \dots \end{aligned} \quad (4)$$

The phase of  $K_0$  is zero if  $K_0 > 0$  and  $180^\circ$  if  $K_0 < 0$ .

The above equations reveal that the magnitude and phase functions  $20 \log_{10}M(\omega)$  and  $\theta(\omega)$  are obtained by simply adding the contributions due to several individual factors. In what follows, we examine individually each of such factors. Once this is done, Bode plots of composite transfer functions can easily be determined. For clarity, we denote  $20 \log_{10}M(\omega)$  by  $M_{dB}(\omega)$ .

Note that 
$$20 \log_{10} \left| 1 + \frac{j\omega}{p_1} \right| = 20 \log_{10} \sqrt{1 + (\omega/p_1)^2}$$

If  $\omega \ll p_1$  then  $\omega/p_1 \ll 1$  and 
$$20 \log_{10} \left| 1 + \frac{j\omega}{p_1} \right| = 20 \log_{10} \sqrt{1 + (\omega/p_1)^2} \approx 0$$

If  $\omega \gg p_1$  then  $\omega/p_1 \gg 1$  and

$$20 \log_{10} \left| 1 + \frac{j\omega}{p_1} \right| = 20 \log_{10} \sqrt{1 + (\omega/p_1)^2} \approx 20 \log_{10} \left( \frac{\omega}{p_1} \right)$$

The latter represents a line with slope of 20dB per decade or 6db per octave<sup>3</sup> since:

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<sup>3</sup> A decade is a ten-fold increase in frequency, while an octave is a two-fold increase in frequency.

$$\text{If } \omega=10 p_1 \text{ then } 20 \log_{10} \left( \frac{\omega}{p_1} \right) = 20 \log_{10} 10 = 20 \text{dB}$$

$$\text{If } \omega=2 p_1 \text{ then } 20 \log_{10} \left( \frac{\omega}{p_1} \right) = 20 \log_{10} 2 \approx 6 \text{dB}$$

Exactly the same is true for zeros. The only difference is that the contribution of poles is subtracted (negative slope  $-20\text{dB/dec}$ ) and that of zeros is added ( $+20\text{dB/dec}$ ).

These asymptotes of  $M_{\text{dB}}(\omega)$  meet at  $\omega = p_1$ . Thus, the frequency  $p_1$  is called the *corner frequency*. At this corner frequency, the actual value of  $M_{\text{dB}}(\omega)$  can be computed easily as 3 dB rather than the 0 dB predicted by either asymptote. At half of the corner frequency, i.e., at  $\omega = 0.5p_1$ , one can compute  $M_{\text{dB}}(\omega)$  as 1 dB rather than the 0 dB given by the low frequency asymptote. On the other hand, at twice the corner frequency, i.e., at  $\omega = 2 p_1$ , one can compute  $M_{\text{dB}}(\omega)$  as 7 dB rather than the 6 dB given by the high frequency asymptote. Using these actual values as a guide, an exact plot of  $M_{\text{dB}}(\omega)$  can easily be drawn<sup>4</sup>.

In terms of the angle, one can choose three cases;

$$\text{If } \omega \ll p_1 \text{ then } \omega/p_1 \ll 1 \text{ and } \tan^{-1} \left( \frac{\omega}{p_1} \right) \approx 0$$

$$\text{If } \omega = p_1 \text{ then } \omega/p_1 = 1 \text{ and } \tan^{-1} \left( \frac{\omega}{p_1} \right) = \tan^{-1} 1 = 45^\circ$$

$$\text{If } \omega \gg p_1 \text{ then } \omega/p_1 \gg 1 \text{ and } \tan^{-1} \left( \frac{\omega}{p_1} \right) \approx \tan^{-1} \infty = 90^\circ$$

This means that the *maximum contribution of any one zero or pole is  $90^\circ$*  (+ for zeros, - for poles). If one approximated “*much smaller than  $p_1$* ” by  $p_1/10$ <sup>5</sup> and as “*much larger than  $p_1$* ” by  $10 p_1$ ,<sup>6</sup> then the angle plot has a slope of  $-45^\circ/\text{dec}$ <sup>7</sup> and is flat elsewhere.

<sup>4</sup> Figure E-6 in the text shows both the low and high frequency asymptotes of  $M_{\text{dB}}(\omega)$  as well as the actual characteristic.

<sup>5</sup>  $\tan^{-1}(0.1) = 5.3^\circ \approx 0$

<sup>6</sup>  $\tan^{-1}(10) = 84.7^\circ \approx 90^\circ$

<sup>7</sup>  $+45^\circ/\text{dec}$  for a zero

### 2.3.1 Constant gain $K_0$ :

If  $H(s)=K_0$ , then  $M_{dB}(\omega) = 20 \log_{10}|K_0|$

which is a constant with respect to  $\log_{10} \omega$ . If  $K_0$  is positive, then the phase angle of  $H(j\omega)$  is zero for all  $\omega$ ; if  $K_0$  is negative, then the phase angle of  $H(j\omega)$  is  $180^\circ$  for all  $\omega$ .

### 2.3.2 Pole or zero at the origin:

For an ideal differentiator, the transfer function is  $s$ , while for an ideal integrator, the transfer function is  $1/s$ . Thus, the ideal differentiator has a zero at  $s=0$ , and an ideal integrator has a pole at  $s=0$ .

For an ideal differentiator,  $H(j\omega)=j\omega$ , and  $M_{dB}(\omega) = 20 \log_{10} \omega$ .

Thus, the plot of  $M_{dB}(\omega)$  as a function of  $\log_{10} \omega$  is a straight line with a slope of 20 dB per decade or approximately 6 dB per octave. The straight line passes through 0 dB at  $\omega=1$  since  $20 \log_{10} \omega$  equals 0 at  $\omega=1$ . The phase angle of  $H(j\omega)=j\omega$  is  $90^\circ$  for all  $\omega$ .

For an ideal integrator, the transfer function is  $1/s$  which is the reciprocal of that of the differentiator. The Bode plots of  $1/s$  are the negative replicas of those of  $s$ . That is, the magnitude plot is a straight line with a slope of -20 dB per decade and passes through 0 dB at  $\omega=1$ , while the phase angle is  $-90^\circ$  for all  $\omega$ .

### 2.3.3 Simple Real Pole or Zero:

Then the discussion of section 2.3 above is relevant and the asymptotes are at zero db and  $\pm 20$ dB/decade for zero and pole respectively.

### 2.3.4 Complex Conjugate Pairs of Poles or Zeros:

We consider next a transfer function having a pair of complex conjugate poles which together give rise to a quadratic factor, i.e.,

$$H(s) = \left[ 1 + \frac{2\zeta}{\omega_o} s + \frac{s^2}{\omega_o^2} \right]^{-1}$$

where  $\zeta$  and  $\omega_0$  are respectively called as the damping coefficient and the natural frequency of the two complex conjugate poles. For this case,

$$M_{dB}(\omega) = -20 \log_{10} \left| 1 - \frac{\omega^2}{\omega_o^2} + 2\zeta \frac{j\omega}{\omega_o} \right|$$

and

$$\theta(\omega) = -\tan^{-1} \left( \frac{2\zeta \frac{\omega}{\omega_o}}{1 - \frac{\omega^2}{\omega_o^2}} \right).$$

Two asymptotes of  $M_{dB}(\omega)$ , one for low frequencies and another for high frequencies, can easily be ascertained. For  $\omega/\omega_o \ll 1$ , we have

$$M_{dB}(\omega) \approx -20 \log_{10} 1 = 0.$$

Similarly, for  $\omega/\omega_o \gg 1$ , we have  $M_{dB}(\omega) \approx -40 \log_{10} (\omega/\omega_o)$ .

That is, for high enough frequencies such that  $\omega \gg \omega_o$ , the magnitude characteristic  $M_{dB}(\omega)$  is a straight line with a slope of -40 dB per decade and intersects the frequency axis at  $\omega = \omega_o$ . Both the low and high frequency asymptotes of  $M_{dB}(\omega)$  meet at the corner frequency  $\omega = \omega_o$ . The exact magnitude characteristic of  $M_{dB}(\omega)$  depends on the damping coefficient  $\zeta$ . As shown in Figure 1, for small values of  $\zeta$ , there is significant peaking in the neighborhood of the corner (or natural) frequency  $\omega_o$ . For  $\zeta < 1/\sqrt{2}$  and for certain representative frequencies, one can compute exactly  $M_{dB}(\omega)$  as outlined below:

1. At the corner frequency  $\omega_o$ ,  $M_{dB}(\omega)$  has a value of  $-20 \log_{10}[2\zeta]$ .
2. At  $\omega = 0.5 \omega_o$ ,  $M_{dB}(\omega)$  has a value of  $-10 \log_{10}[\zeta^2 + 0.5625]$ .
3. It can be shown that the peak of  $M_{dB}(\omega)$  occurs at  $\omega = \omega_o \sqrt{1 - 2\zeta^2}$  and that the peak value is  $-10 \log_{10}[4\zeta^2(1 - \zeta^2)]$ .
4.  $M_{dB}(\omega)$  crosses the 0 dB axis at the frequency  $\omega = \omega_o \sqrt{2(1 - 2\zeta^2)}$

For  $\zeta > 1/\sqrt{2}$  the exact characteristic of  $M_{dB}(\omega)$  lies entirely below its asymptotic approximation.

For the angle, as shown in Figure 1, the exact plot of  $\theta(\omega)$  depends on the damping factor  $\zeta$ . Without taking into account the value of  $\zeta$ , one can coarsely<sup>8</sup> approximate

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<sup>8</sup> There exist in the literature certain better mid frequency approximations which take into account the value of  $\zeta$ .

the phase angle plot by three asymptotes, similar to the ones derived for the simple real pole, only now the total contribution of the pair will be  $180^\circ$  rather than  $90^\circ$  of the simple pole or zero:

Low frequency approximation: For  $\omega < 0.1 \omega_0$ ,  $\theta(\omega)$  can be approximated by  $0$ .

High frequency approximation: For  $\omega > 10 \omega_0$ ,  $\theta(\omega)$  can be approximated by  $-180^\circ$ .

Mid frequency approximation: For  $0.1 \omega_0 < \omega < 10 \omega_0$ ,  $\theta(\omega)$  can be approximated by a straight line having a slope of  $-90^\circ$  per decade and passing through  $-90^\circ$  at  $\omega = \omega_0$ .

Consider next a transfer function with a pair of complex conjugate zeros,

$$H(s) = 1 + (2\zeta/\omega_0)s + s^2/\omega_0^2.$$

For this case, the magnitude and phase plots are negative replicas (inverted versions) of the corresponding plots derived for the case of complex conjugate poles. In particular, the high frequency asymptote of the magnitude characteristic has a slope of  $+40$  dB / decade, and the mid frequency asymptote of the phase characteristic has a slope of  $+90^\circ$ /decade.

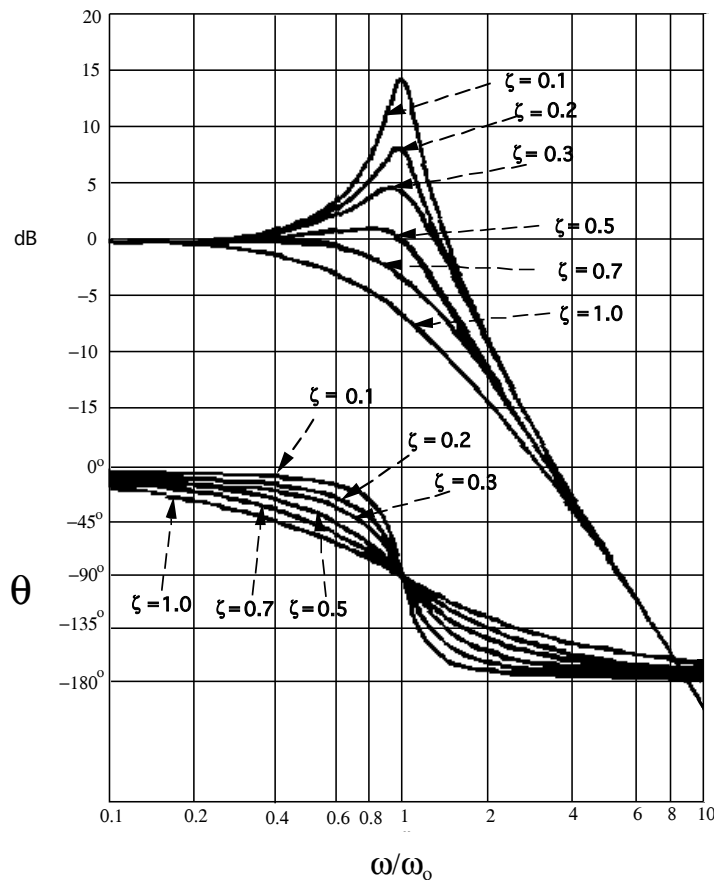


Fig. 1 Log-Magnitude and Phase-Angle curves for a pair of complex conjugate poles

### 3 Prelab Exercises

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- 3.1** Derive the transfer functions  $H(s)$  for the five circuits in Section 4. Note that the transfer function of circuit #2 is not the product of two simple transfer functions of type #1 circuits. This is the result of the so called *loading effect*. Circuits #4 and #5 differ in the sense that the transfer function of circuit #4 has a quadratic factor with complex roots in its denominator, whereas the transfer function of circuit #5 has a quadratic factor with real roots in its denominator.
- 3.2** Put each of the transfer functions derived in 3.1 in the appropriate standard form of the transfer functions in section 2. Derive or state any necessary arguments that are essential or necessary for plotting the theoretical Bode diagrams for each transfer function. Also, plot the theoretical Bode diagrams for each transfer function. Make sure that you use a semi-log graph paper for all your plots. Make sure that both plots, the straight line approximation and the corrected plot, are shown on the same graph paper for every circuit.



## 4 Experiments

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Suggested Equipment:

Tektronix FG 501A 2MHz Function Generator<sup>9</sup>  
Tektronix PS 503 Power Supply  
Tektronix DC 504A Counter-Timer  
HP 54600A or Agilent 54622A Oscilloscope  
741 Op-Amp  
Protoboard

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<sup>9</sup> NOTE: The oscillator is designed to work for a very wide range of frequencies but may not be stable at very low frequencies, say in the order of 100 Hz or 200Hz. To start with it is a good idea to have the circuit working at some mid-range frequency, say in the order of 1K Hz or 2K Hz, and then change the frequency slowly as needed.











Vary the frequency of the input, read the output on the DVM, and show your instructor that the circuit actually behaves like a band reject filter.

Find the following:

- (1) The maximum value of the output.
- (2) The 3 dB frequencies and the corresponding value of the output.
- (3) The stop band.
- (4) The phase angle at some particular frequencies.

## **5 Report**

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- 5.1** Tabulate all experimental data obtained in Section 4.
- 5.2** Plot the Bode diagrams (magnitude and phase) for the five circuits in Section 4 using the experimental data. Make sure that you use a semi-log graph paper for all your plots. For each circuit, compare the experimental Bode diagrams (plots) with the theoretical Bode diagrams (plots) previously completed in pre-lab exercise 3.2.
- 5.3** Prepare a summary.