332:525 – Optimum Signal Processing
Computer Experiment 5 – Due March 24, 2011

1. The file `GNPC96.dat` contains the data for the U.S. Real Gross National Product (in billions of chained 2005 dollars, measured quarterly and seasonally adjusted.) In this experiment, you will test whether an ARIMA$(p,1,0)$ model is appropriate for these data.

   Extract the data from 1980 onwards and take their log. This results in $N = 119$ observations, $y_n = \ln(\text{GNP}_n)$, $n = 0, 1, \dots, N-1$. Because of the upward trend of the data, define the length-$(N-1)$ differenced signal $z_n = y_n - y_{n-1}$, for $n = 1, 2, \dots, N-1$. Then, subtract its sample mean, say μ, to get the signal $x_n = z_n - \mu$, for $n = 1, 2, \dots, N-1$.

   a. Calculate and plot the first $M = 24$ sample autocorrelation lags $R(k)$, $0 \le k \le M$, of the signal x_n. Send these into the Levinson algorithm to determine and plot the corresponding reflection coefficients γ_p, for $p = 1, 2 \dots, M$. Add on that graph the 95% confidence bands, that is, the horizontal lines at $\pm 1.96/\sqrt{N}$. Based on this plot, determine a reasonable value for the order p of an autoregressive model of fitting the x_n data.

   b. Check the chosen value of p against also the FPE, AIC, and MDL criteria, that is, by plotting them versus $p = 1, 2, \dots, M$, and identifying their minimum:

   $$\text{FPE}_p = E_p \frac{N+p+1}{N-p-1}, \quad \text{AIC}_p = N \ln E_p + 2(p+1), \quad \text{MDL}_p = N \ln E_p + (p+1)\ln N$$

   c. For the chosen value of p, use the Yule-Walker method to determine the linear prediction error filter $\mathbf{a}_p$ of order p, then, calculate the one-step-ahead prediction $\hat{x}_{n/n-1}$, add the mean μ to get the prediction $\hat{z}_{n/n-1}$, and undo the differencing operation to compute the prediction $\hat{y}_{n/n-1}$ of y_n. There is a small subtlety here that has to do with the initial value of y_n. On the same graph plot y_n and its prediction.

   d. Repeat part (c) for a couple of other values of p, say $p = 1$ and $p = 10$.

   e. Calculate the DTFT $|X(\omega)|^2$ of the data x_n over $0 \le \omega \le \pi$. For the chosen order p, calculate the corresponding AR power spectrum but this time use the Burg method to determine the order-p prediction filter. Plot the DTFT and AR spectra on the same graph, but for convenience in comparing them, normalize both spectra to their maximum values. Investigate if higher values of p can model these spectra better, for example, try the orders $p = 10$ and $p = 15$.

2. *Wiener Filter Design.* It is desired to design a Wiener filter to enhance a sinusoidal signal buried in noise. The noisy sinusoidal signal is given by

   $$x_n = s_n + v_n, \qquad \text{where } s_n = \sin(\omega_0 n)$$

   with $\omega_0 = 0.075\pi$. The noise component v_n is related to the secondary signal y_n by

   $$v_n = y_n + y_{n-1} + y_{n-2} + y_{n-3} + y_{n-4} + y_{n-5} + y_{n-6}$$

   a. Generate $N = 200$ samples of the signal y_n by assuming that it is an AR(4) process with reflection coefficients:

   $$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} = \{0.5, -0.5, 0.5, -0.5\}$$

The variance σ_ϵ^2 of the driving white noise of the model must be chosen in such a way as to make the variance σ_v^2 of the noise component v_n approximately $\sigma_v^2 = 0.5$, such that the two terms s_n and v_n of x_n have approximately equal strengths, that is, 0 dB signal-to-noise ratio.

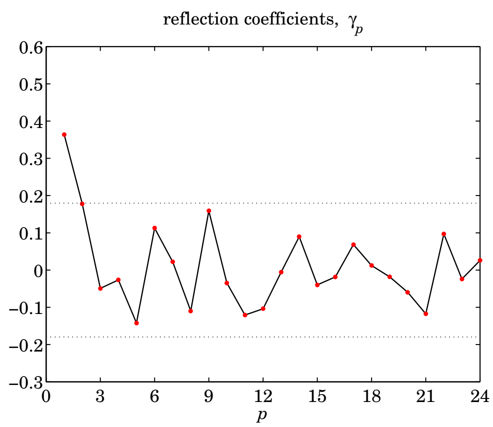
(This can be done by writing $v_n = \mathbf{c}^T \mathbf{y}(n)$ and therefore, $\sigma_v^2 = \mathbf{c}^T R \mathbf{c}$, where $\mathbf{c}$ is a 7-dimensional column vector of ones, and R is the order-6 autocorrelation matrix, you can then write $\sigma_v^2 = \sigma_y^2 \mathbf{c}^T R_{\text{norm}} \mathbf{c}$, where R_{norm} has all its entries normalized by $R(0) = \sigma_y^2$. You can easily determine R_{norm} by doing a maximum entropy extension to order six, starting with the four reflection coefficients and setting $\gamma_5 = \gamma_6 = 0$.)

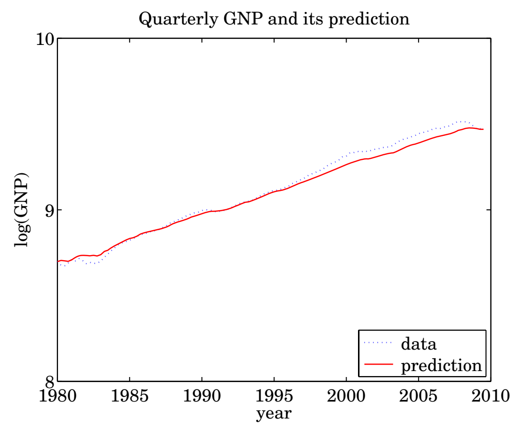
In generating y_n make sure that the transients introduced by the filter have died out. Then, generate the corresponding N samples of the signal x_n. On the same graph, plot x_n together with the desired signal s_n. On a separate graph (but using the same vertical scales as the previous one) plot the reference signal y_n versus n.

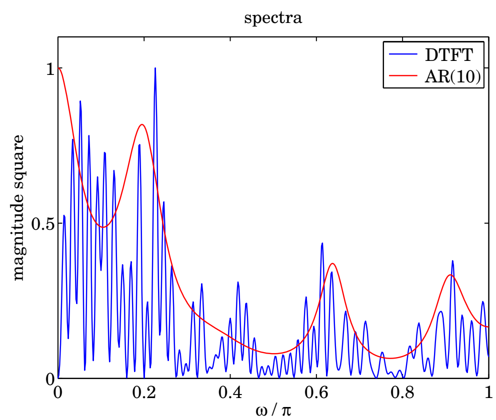
b. For $M = 4$, design a Wiener filter of order-M based on the generated signal blocks $\{x_n, y_n\}$, $n = 0, 1, \ldots, N-1$, and realize it in both the direct and lattice forms.

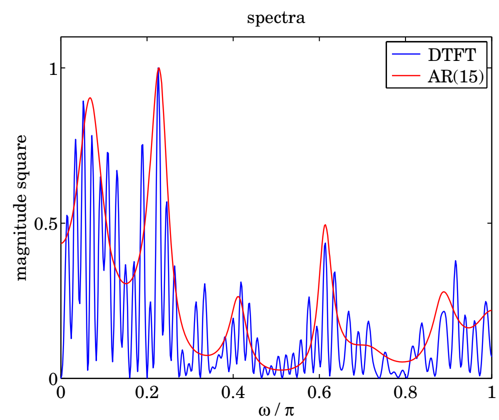
c. Using the *lattice* form, filter the signals x_n, y_n through the designed filter and generate the outputs $\hat{x}_n, e_n$. Explain why e_n should be an estimate of the desired signal s_n. On the same graph, plot e_n and s_n using the same vertical scales as in part (a).

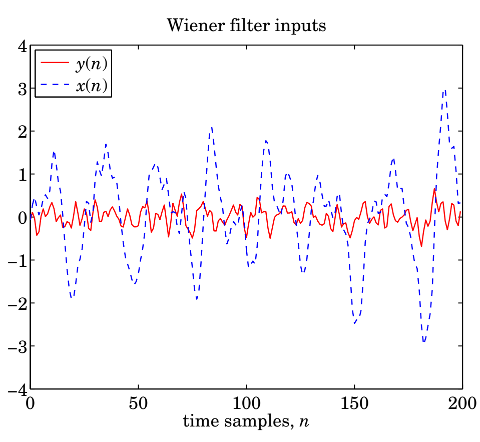
d. Repeat parts (b) and (c) for filter orders $M = 5, 6, 7, 8$. Discuss the improvement obtained with increasing order. What is the smallest M that would, at least theoretically, result in $e_n = s_n$?

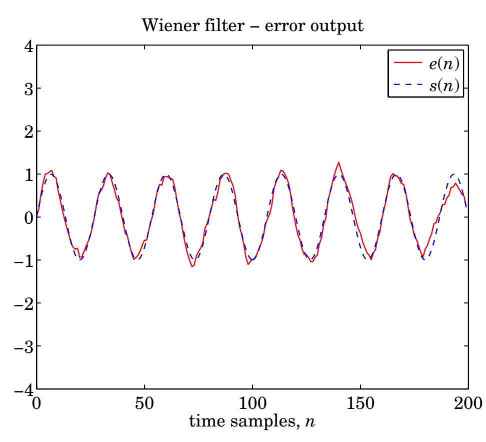

reflection coefficients, γ_p
p


Quarterly GNP and its prediction
log(GNP)
year
data
prediction


spectra
magnitude square
ω / π
DTFT
AR(10)


spectra
magnitude square
ω / π
DTFT
AR(15)


Wiener filter inputs
time samples, n
$y(n)$
$x(n)$


Wiener filter – error output
time samples, n
$e(n)$
$s(n)$