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332:521 – Digital Signals and Filters
Exam 1 – October 11, 2000

1. Determine all possible inverse z-transforms and corresponding regions of convergence of the following z-transform:

$$X(z)=\frac{5}{(1-0.5z^{-1})(1+2z^{-1})}$$

2. The Fibonacci sequence and a variant of it are:

$$\mathbf{h} = [1,1,2,3,5,8,13,21,\dots]$$

$$\mathbf{h} = [2,1,3,4,7,11,18,29,\dots]$$

where, starting with the third entry, each entry is obtained by adding the previous two.

   (a) If we think of these sequences as the impulse response sequences $h(n)$, $n=0,1,2\dots$ of a causal filter, determine the difference equation satisfied by $h(n)$ in the two cases.

   (b) Determine the corresponding transfer functions $H(z)$ and express them as ratios of quadratic polynomials in z^{-1}.

3. Consider the following signal $x(n)$ defined for $-\infty < n < \infty$:

$$x(n)=4+3\cos\left(\frac{3\pi n}{8}\right)+2\sin\left(\frac{3\pi n}{4}\right)+(-1)^n$$

   (a) Using pole/zero placement, design an IIR filter of order-four numerator and denominator that completely removes the second and third terms of the signal $x(n)$. Determine the transfer function $H(z)$ of this filter as a cascade of second-order sections with real coefficients. The 40-dB time constant of this filter must be 100 samples.

   (b) Explain why the steady-state output from this filter will have the form:

$$y(n)=A+B(-1)^n$$

   Determine the numerical values of A and B.

4. Consider the two filters:

$$H_1(z)=0.1\,\frac{1+z^{-8}}{1-0.8z^{-8}}\,,\quad H_2(z)=0.9\,\frac{1-z^{-8}}{1-0.8z^{-8}}$$

Such filters (with z^{-8} replaced by a higher power) are used to separate luminance and chrominance components in color video signals.

   (a) Determine the poles and zeros of the two filters and place them on the z-plane.

   (b) Make a rough sketch of the magnitude response of each filter versus $0<\omega<2\pi$.

   (c) Show the so-called complementarity properties:

$$H_1(z)+H_2(z)=1$$

$$|H_1(\omega)|^2+|H_2(\omega)|^2=1$$

   (d) These filters can be used to cancel periodic interference signals. What are the frequencies of the periodic interference signals (in units of f_s) that get canceled by these filters?

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Exam 2 – November 30, 2000

1. The signal $\mathbf{x} = [4, 8, 16, 4, 8, 16, 4, 8]$ is sent to the input of the filter:

$$H(z) = \frac{1 + 2^4 z^{-4}}{1 + 2^{-4} z^{-4}}$$

(a) Show that the filter $H(z)$ is an allpass filter. What is its gain?

(b) Draw the canonical and transposed realization forms and write the corresponding sample processing algorithms. Use a circular buffer for the canonical case. Compare the computational cost of the two realizations.

(c) For the canonical realization implemented with a circular buffer, calculate the corresponding eight output samples. Make a table that shows the input, the circular buffer entries, and the output samples at each time instant.

2. (a) Calculate by hand the radix-2, decimation-in-time, FFT of the signal:

$$\mathbf{x} = \begin{bmatrix} 2 \\ -2 + \frac{\sqrt{2}}{2} \\ 3 \\ -2 + \frac{\sqrt{2}}{2} \\ 2 \\ -2 - \frac{\sqrt{2}}{2} \\ 1 \\ -2 - \frac{\sqrt{2}}{2} \end{bmatrix}$$

(b) Express this signal as a sum of sinusoidal or cosinusoidal signals.

3. (a) Design a 9-tap linear phase FIR filter that approximates the low-pass differentiator:

$$D(\omega) = \begin{cases} j\omega, & \text{if} \quad |\omega| \le 0.5\pi \\ 0, & \text{if} \quad 0.5\pi < |\omega| \le \pi \end{cases}$$

(b) Show that the noise-reduction ratio of the ideal infinitely-long low-pass differentiator is eight times smaller than that of the usual full-band differentiator.

(c) Show that up to a scale factor and up to initial transients, the designed 9-tap filter correctly differentiates the signals $x(n) = n$ and $x(n) = n^2$, that is, it produces $y(n) = A$ and $y(n) = 2An$, where A is the aforementioned scale factor. What is the value of A?

4. (a) Using the bilinear transformation and a lowpass Butterworth analog prototype filter, design a digital *highpass* filter operating at a rate of 8 kHz with the following specs:

Its passband begins at 3 kHz and the maximum allowed passband attenuation is 1 dB. Its stopband ends at 1 kHz and the minimum stopband attenuation is 40 dB.

Express the designed transfer function as the cascade of first or second order factors.

(b) Sketch the magnitude response of the designed filter over the range $0 \le f \le 16$ kHz.

Hints

$$\int xe^x \, dx = (x-1)e^x$$

$$G_i = \frac{\Omega_0^2}{1 - 2\Omega_0 \cos\theta_i + {\Omega_0}^2}$$

$$a_{i1} = -\frac{2(\Omega_0^2 - 1)}{1 - 2\Omega_0 \cos\theta_i + {\Omega_0}^2}\,, \qquad a_{i2} = \frac{1 + 2\Omega_0 \cos\theta_i + \Omega_0^2}{1 - 2\Omega_0 \cos\theta_i + {\Omega_0}^2}$$

332:521 – Digital Signals and Filters
Final Exam – December 18, 2000

1. Let $\mathbf{X} = A\mathbf{x}$ be the N-point DFT of a length-N signal $\mathbf{x}$ expressed in matrix form, where A is the $N\times N$ DFT matrix, and $\mathbf{x}, \mathbf{X}$ are column vectors.

   (a) Show the following matrix relationship for any N:

   $$AA^* = NI\,, \qquad \text{where } I = N\times N \text{ identity matrix}$$

   (b) Using the above result, prove the DFT version of Parseval's identity:

   $$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

   You may assume that $x(n)$ is complex-valued. [*Hint:* Write this equation in terms of the vectors $\mathbf{x}$ and $\mathbf{X}$.]

2. Using the bilinear transformation and a lowpass analog Butterworth prototype filter, design a *digital* third-order *highpass* IIR filter operating at a rate of 8 kHz that has a maximum attenuation of 1 dB over the passband range of [2,4] kHz.

   Determine its transfer function $H(z)$ and its 3-dB frequency in kHz.

3. It is desired to design a 4× oversampling digital FIR interpolation filter for a CD player. Assume the following specifications: audio sampling rate of 44.1 kHz, passband range $[0, 20]$ kHz, stopband range $[24.1, 88.2]$ kHz, and stopband attenuation of 80 dB.

   Using the Kaiser window design method, determine the *filter length* and the total *computational rate* in MAC/sec for the following cases:

   (a) Single-stage design implemented in its polyphase form.

   (b) Two-stage (2×2) design implemented in its polyphase form. What are the design specifications of the two stages?

   Draw a sketch of the magnitude responses of the designed filters versus frequency in the range $0 \le f \le 176.4$ kHz, and of the two individual filter responses in the two-stage design case. What are the computational savings of design (b) versus design (a)? Can a 20 MIPS DSP chip handle the computational rates?

4. A digital audio tape recorder uses a third-order analog antialiasing Butterworth prefilter, which precedes an L-times oversampled $\Delta\Sigma$-ADC, which is followed by a digital decimation filter and an L-fold downsampler that reduces the sampling rate down to 48 kHz.

   (a) The passband attenuation of the prefilter is required to be less than 0.1 dB. Assuming an ideal decimator filter, determine the passband frequency and the 3-dB frequency of the prefilter in kHz.

   (b) What is the minimum value of the oversampling ratio L that will allow the above prefilter to achieve a stopband attenuation of 70 dB? What is the stopband frequency of the prefilter in kHz?

   (c) If the oversampling ratio is $L = 16$, what would be the stopband attenuation of the prefilter in dB?

Hints

$$W_N = e^{-2\pi j/N}\,, \qquad A_{ij} = W_N^{ij}$$

$$G_i = \frac{\Omega_0^2}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2}\,, \qquad \theta_i = \frac{\pi}{2N}\,(N - 1 + 2i)$$

$$a_{i1} = -\frac{2(\Omega_0^2 - 1)}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2}\,, \qquad a_{i2} = \frac{1 + 2\Omega_0\cos\theta_i + \Omega_0^2}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2}$$

$$D = \frac{A - 7.95}{14.36}$$

$$A(f) = 10\log_{10}\left[1 + \left(\frac{f}{f_0}\right)^{2N}\right]$$

332:521 – Digital Signals and Filters
Exam 1 – October 10, 2002

1. (a) Working with Laplace transforms, show that the frequency response of the staircase D/A reconstructor is given by:

$$H_{\text{D/A}}(f) = Te^{-j\pi f/f_s}\frac{\sin(\pi f/f_s)}{\pi f/f_s}, \qquad T = \frac{1}{f_s}$$

(b) After having been properly prefiltered by an antialiasing filter, an analog signal is sampled at a rate of 6 kHz. The digital signal is then filtered by a digital filter designed to act as an ideal lowpass filter with cutoff frequency of 1 kHz. The filtered digital signal is then fed into a staircase D/A reconstructor and then into a lowpass anti-image postfilter.

The overall reconstructor is required to suppress the spectral images caused by sampling by more than $A = 40$ dB. Determine the *least stringent* specifications for the analog postfilter that will satisfy this requirement.

(c) Using the "6N dB/octave" rule, determine the order N of the postfilter that will meet the required specifications.

2. Using partial fractions, determine the *stable* inverse z-transform of the following z-transform:

$$X(z) = \frac{0.75}{(1-0.5z^{-1})(1-0.5z)}$$

3. Consider the filter:

$$H(z) = \frac{z^{-1}+z^{-2}-2z^{-3}}{1+0.125\,z^{-3}}$$

(a) Draw the direct, canonical, and transposed realization forms. Write down the sample processing algorithm of the transposed form.

(b) Determine the poles and zeros of the transfer function and place them on the z-plane. Draw a rough sketch of the magnitude response $|H(\omega)|$ of this filter.

(c) Factor this transfer function into second-order sections with real coefficients (2nd order means 1st or 2nd order). Then, realize this filter in its cascade form, where each cascade factor is realized in its canonical form. Write down the corresponding sample processing algorithm using linear buffers. [*Hint:* $1-x^3 = (1-x)(1+x+x^2)$]

4. The finite signal $\mathbf{x} = [8,6,4,2,4,6,8]$ is sent to the input of the filter of the previous problem.

(a) Write down the sample proceesing algorithm of its canonical form using circular buffers.

(b) Iterating the above algorithm, compute the corresponding 7 output samples.

Make a table that shows the input, buffer contents $[w_0, w_1, w_2, w_3]$, internal states $[s_0, s_1, s_2, s_3]$, and output, such that the nth row of the table will represent the values at the nth time instant.

332:521 – Digital Signals and Filters
Exam 2 – November 14, 2002

1. The following signal is sampled at a rate of 40 kHz:

$$x(t) = \cos(9\pi t) + 0.0001\cos(10\pi t) + \cos(11\pi t)\,, \quad t \text{ is in msec}$$

In computing the DTFT of this signal using a Kaiser window, how many samples should be collected? It is required that the middle component, if present, should stand 20 dB above the window sidelobes of the other components.

2. Consider the following signal and filter:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

(a) Form the complex signal $\mathbf{z} = \mathbf{x} + j\mathbf{h}$ and calculate its 4-point FFT, $\mathbf{Z} = \text{FFT}(\mathbf{z})$, using the decimation-in-time radix-2 FFT algorithm.

(b) Extract the 4-point FFTs $\mathbf{X} = \text{FFT}(\mathbf{x})$ and $\mathbf{H} = \text{FFT}(\mathbf{h})$ from the single FFT $\mathbf{Z}$ that was computed in part (a).

(c) Calculate the mod-4 circular convolution of $\mathbf{x}$ and $\mathbf{h}$ by using the formula: $\tilde{\mathbf{y}} = \text{IFFT}\big[\text{FFT}(\mathbf{x})\cdot\text{FFT}(\mathbf{h})\big]$.

(d) Recompute the above circular convolution by using the time-domain formula: $\tilde{\mathbf{y}} = \widetilde{\mathbf{h} * \mathbf{x}}$.

(e) Determine two other signals $\mathbf{x}_1$ and $\mathbf{x}_2$ of length 6 that have the same 4-point DFT as the signal $\mathbf{x}$.

3. Consider the following signal in which the *second* term is a desired signal and the remaining terms represent interference:

$$x(n) = \cos\big(\frac{\pi n}{8}\big) + \cos\big(\frac{\pi n}{2}\big) + \sin\big(\frac{3\pi n}{4}\big) + \cos(\pi n)$$

(a) Using the bilinear transformation method, design a peaking resonator filter $H(z)$ that will let the desired signal pass through unchanged (up to a possible phase-shift) and will attenuate the interfering components by at least 10 dB. The filter must be designed on the basis of its 10-dB width, which can be obtained from the figure below.

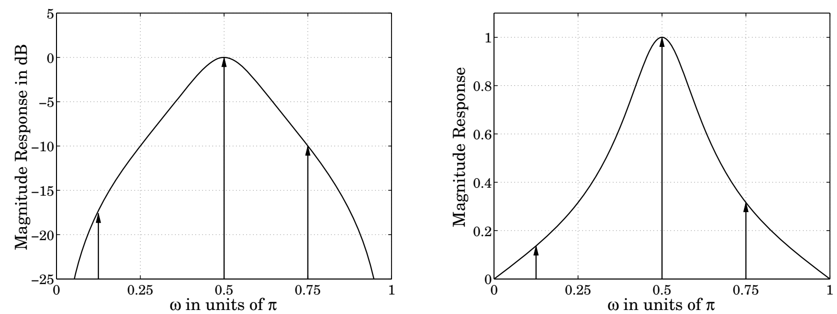
(b) Show that the equivalent analog filter, whose bilinear transformation generates the required digital filter $H(z)$, is given by:

$$H(s) = \frac{\frac{2}{3}s}{s^2 + \frac{2}{3}s + 1}$$

(c) Working with the above *analog* filter, show that the steady-state output of the *digital* filter will be:

$$y(n) = 0.137\cos\big(\frac{\pi n}{8} + 1.434\big) + \cos\big(\frac{\pi n}{2}\big) + 0.316\sin\big(\frac{3\pi n}{4} - 1.249\big)$$

Verify that the required filtering specifications are met.



Hints

$$c = \frac{6(R+12)}{155}$$

$$A_k = \text{DFT}(a_n)\,, \quad B_k = \text{DFT}(b_n)\,, \quad Z_k = \text{DFT}(a_n + jb_n) \quad \Rightarrow$$

$$A_k = \frac{1}{2}(Z_k + Z^*_{N-k})\,, \quad B_k = \frac{1}{2j}(Z_k - Z^*_{N-k})$$

$$H(z) = \left(\frac{\beta}{1+\beta}\right)\frac{1 - z^{-2}}{1 - 2\left(\frac{\cos\omega_0}{1+\beta}\right)z^{-1} + \left(\frac{1-\beta}{1+\beta}\right)z^{-2}}$$

$$\beta = \frac{G_B}{\sqrt{1 - G_B^2}}\tan\left(\frac{\Delta\omega}{2}\right)$$

$$0.019 + 0.136j = 0.137e^{1.434j}\,, \quad 0.1 - 0.3j = 0.316\,e^{-1.249j}$$

332:521 – Digital Signals and Filters
Final Exam - December 19, 2002

1. A digital audio tape recorder uses an order-N analog antialiasing Butterworth prefilter, which precedes a 16-times oversampled delta-sigma ADC, which is followed by a digital decimation filter and a 16-fold downsampler that reduces the sampling rate down to the standard rate of 48 kHz.

   The passband attenuation of the prefilter is required to be less than 0.1 dB and its stopband attenuation more than 70 dB.

   (a) The passband frequency of the prefilter is 24 kHz. What should its stopband frequency be in kHz that would result in the smallest filter order N while still suppressing the aliased components by more than 70 dB?

   (b) Assuming a near-perfect decimation filter, determine the order N of the analog prefilter and its 3-dB normalization frequency in kHz.

   (c) What is the actual stopband attenuation achieved by this prefilter?

2. A first-order noise-shaping ADC operating at the overasampled rate of $f_s' = Lf_s$ reshapes the quantization noise power within the f_s–Nyquist interval according to the high-pass filter:

$$|H_{\rm NS}(f)|^2 = \left|1 - e^{-2\pi jf/Lf_s}\right|^2 = 4\sin^2\left(\frac{\pi f}{Lf_s}\right)$$

   (a) Explain how the noise-shaping ADC achieves a savings in bits as compared to a non-oversampled system of equivalent quality.

   (b) Then, show that the savings in bits, $\Delta B = B - B'$, is given by the following exact formula:

$$\Delta B = -0.5\log_2\left[\frac{2}{\pi}\left(\frac{\pi}{L} - \sin\frac{\pi}{L}\right)\right]$$

   (do not make any approximations in computing the integral of $|H_{\rm NS}(f)|^2$.)

   (c) Finally, show that in the limit of large L, this reduces to:

$$\Delta B = 1.5\log_2 L - 0.5\log_2\left(\frac{\pi^2}{3}\right)$$

3. Consider the peaking analog filter

$$H(s) = \frac{\alpha s}{s^2 + \alpha s + \Omega_0^2}$$

   It was shown in class that the 3-dB width of this filter is $\Delta\Omega = \alpha$.

   (a) Show that the bandedge frequencies at the 10-dB attenuation level must satisfy the quartic equation:

$$\Omega^4 - (2\Omega_0^2 + 9\alpha^2)\Omega^2 + \Omega_0^4 = 0$$

   Show that the 10-dB width is given by $\Delta\Omega = 3\alpha$.

   (b) A peaking digital filter $H(z)$ operating at an 8 kHz sampling rate was designed using the bilinear transformation method. It turned out that the equivalent bilinearly transformed analog filter was:

$$H(s) = \frac{0.5\,s}{s^2 + 0.5\,s + 1}$$

   Determine the transfer function $H(z)$, realize it in its canonical form, and write its sample processing algorithm using circular buffers.

   (c) Determine the center frequency f_0 and 10-dB width Δf of the peaking digital filter in kHz.

4. (a) Calculate by hand the radix-2, decimation-in-time, FFT of the signal:

$$\mathbf{x} = \begin{bmatrix} 4 \\ -4+\sqrt{2} \\ 6 \\ -4+\sqrt{2} \\ 4 \\ -4-\sqrt{2} \\ 2 \\ -4-\sqrt{2} \end{bmatrix}$$

   (b) Using the inverse DFT formula, express this signal as a sum of real-valued sinusoids.

332:521 – Digital Signals and Filters
Exam 1 - October 9, 2003

1. For a complex-valued stable signal x_n, prove Parseval's identity:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(\omega)|^2\,d\omega=\sum_{n=-\infty}^{\infty}|x_n|^2$$

where $X(\omega)$ is the DTFT:

$$X(\omega)=\sum_{n=-\infty}^{\infty}x_n\,e^{-j\omega n}$$

Where was the assumption of stability used?

2. In this problem, assume that the quantities a, b are real-valued such that $|a|<1$ and $|b|<1$.

(a) Determine *all possible* inverse z-transforms of the following z-transform:

$$X(z)=\frac{1-ab}{(1-az^{-1})(1-bz)}$$

Identify the stable case and the causal case.

(b) Using the results of part (a), show the following integral:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\omega}{(1-2a\cos\omega+a^2)(1-2b\cos\omega+b^2)}=\frac{1+ab}{(1-ab)(1-a^2)(1-b^2)}$$

3. A filter has transfer function:

$$H(z)=\frac{N(z)}{D(z)}=\frac{2+z^{-2}+4z^{-4}}{1-0.5z^{-2}}$$

(a) Draw the *canonical* realization form of $H(z)$ and write the corresponding sample processing algorithm using a *circular* delay-line buffer and a circular pointer. (You may invoke the functions `tap` and `cdelay`, or you may use the sloppier notation $s_i=*(p+i)$, etc.)

(b) The following signal $\mathbf{x}=[1,2,3,4,5,6]$ is sent to the input of the above filter. Iterate the sample processing algorithm of part (a) and compute the output signal $y(n)$ for $0\le n\le 5$. In the process, fill the table of values of the circular buffer $\mathbf{w}$ and the states s_0, s_2, s_4:

x	w_0	w_1	w_2	w_3	w_4	s_0	s_2	s_4	y
1	*	*	*	*	*	*	*	*	*
2	*	*	*	*	*	*	*	*	*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
6	*	*	*	*	*	*	*	*	30

(To help you check your answer, the last output sample is $y=30$.)

(c) Draw the *transposed* realization of $H(z)$ and state its sample processing algorithm.

(d) Without any further calculations, write down the first six output samples of the filter:

$$\frac{1}{D(z)}=\frac{1}{1-0.5z^{-2}}$$

driven by the same input as in part (b). Explain your reasoning.

Hints

$$|A-Be^{-j\omega}|^2=A^2-2AB\cos\omega+B^2\,,\quad\text{for real } A,B,\omega$$

$$1+x+x^2+x^3+\cdots=\frac{1}{1-x}\,,\quad x+x^2+x^3+\cdots=\frac{x}{1-x}\,,\quad\text{for } |x|<1$$

$$\frac{1}{1-az^{-1}}\longleftrightarrow\begin{cases}a^n u(n), & \text{if } |z|>|a|\\ -a^n u(-n-1), & \text{if } |z|<|a|\end{cases}$$

332:521 – Digital Signals and Filters
Exam 2 - November 13, 2003

1. You have available the two separate filter blocks z^{-1} and $H(z)$ as shown:

$$\longrightarrow \boxed{z^{-1}} \longrightarrow \qquad \longrightarrow \boxed{H(z)} \longrightarrow$$

   (a) Show how to use the block $H(z)$ and *only one* block z^{-1} to implement the block diagram of the filter:

$$G(z) = \frac{z^{-1}}{1 - z^{-1}H(z)}$$

   (b) Suppose that the function $[y, \mathbf{w}] = h(x, \mathbf{w})$ implements the input-output sample processing algorithm of the block $H(z)$, where $\mathbf{w}$ is an appropriate set of internal states for $H(z)$. Using this function, write down the sample processing algorithm for the block diagram of the filter $G(z)$.

2. An 8-point signal $x(n)$, $n = 0, 1, \ldots, 7$, has the following 8-point FFT:

$$\mathbf{X} = \begin{bmatrix} 24 \\ 0 \\ -8j \\ 0 \\ 8 \\ 0 \\ 8j \\ 0 \end{bmatrix}$$

   (a) Express the signal $x(n)$ as a sum of *real-valued* sinusoidal signals.

   (b) Carry out by hand the inverse 8-point FFT of $\mathbf{X}$. Verify that the resulting time signal samples agree with those you found in part (a).

3. You wish to compute a 1024-point FFT, but your hardware can only accommodate 256-point FFTs.

   (a) Explain how you might use this hardware to compute the required 1024-point FFT in pieces. Discuss how you must partition the time data, what FFTs must be computed, and how they must be re-combined. Assume you have adequate hard-disk space available.

   (b) Determine the total number of complex multiplications required by your method and compare it to the cost of performing a single 1024-point FFT.

4. A signal $x(n)$ has the following z-transform:

$$X(z) = \frac{1 - z^{-8}}{1 - z^{-1}}$$

   (a) Working directly with $X(z)$ and without going to the time domain, calculate the 8-point DFT of the signal $x(n)$.

   (b) Calculate by hand the inverse 8-point FFT of the DFT of part (a) and obtain the time-domain samples $x(n)$, $n = 0, 1, \ldots, 7$. Verify that they are the same time samples as those obtained by the inverse z-transform of $X(z)$.

   (c) Make a rough sketch of the magnitude $|X(\omega)|$ of the DTFT of $x(n)$ for $0 \le \omega \le 2\pi$, and superimpose on the sketch the 8-point DFT values from part (a).

   (d) Consider the signal $x_1(n)$ whose z-transform is:

$$X_1(z) = (1 + z^{-1})(1 + z^{-2})(1 + z^{-4})(3 - 2z^{-1})$$

   Without calculating the 8-point DFT of $x_1(n)$, explain why the 8-point DFTs of the signals $x(n)$ and $x_1(n)$ are the *same*. (You may work in the z-domain or give a time-domain explanation.)

332:521 – Digital Signals and Filters
Final Exam - December 16, 2003

1. For two complex-valued stable signals x_n, y_n, prove the generalized Parseval's identity:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} Y^*(\omega)X(\omega)\,d\omega = \sum_{n=-\infty}^{\infty} y_n^* x_n$$

where $X(\omega), Y(\omega)$ are the DTFTs:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x_n\, e^{-j\omega n}\,, \quad Y(\omega) = \sum_{n=-\infty}^{\infty} y_n\, e^{-j\omega n}$$

Where was the assumption of stability used?

2. Consider two length-8, real-valued, signals a_n, b_n and form the complex valued signal $z_n = a_n + jb_n$ for $n = 0, 1, \dots, 7$. Suppose the 8-point DFT of z_n is given as follows:

$$Z_k = \begin{bmatrix} 0 \\ 0 \\ 8 \\ 0 \\ 0 \\ 8j \\ 0 \\ 0 \end{bmatrix}$$

(a) Extract from Z_k the 8-point DFTs, A_k, B_k, of the signals a_n, b_n.

(b) Applying the inverse DFT formula on A_k and B_k and without performing any DFT/FFT calculations, determine the real-valued signals a_n, b_n, expressed as sums of real-valued sinusoids.

3. It is desired to design a third-order digital lowpass Butterworth filter using the bilinear transformation method. The sampling rate is 8 kHz and the filter's 3-dB frequency is 2 kHz.

(a) Determine the transfer function $H(z)$.

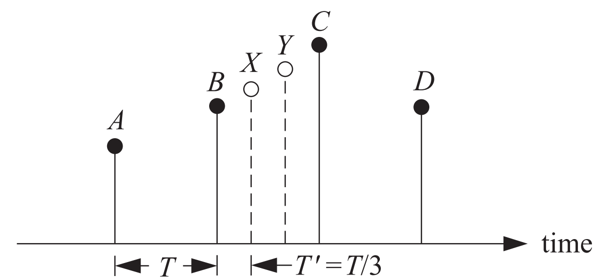
(b) Draw a rough sketch of the magnitude response of the filter over the frequency range $0 \le f \le 16$ kHz.

4. A 3-times interpolation filter calculates two missing samples between any two low-rate samples. For example, the two missing samples X, Y shown below may be calculated as linear combinations of four low-rate surrounding samples:

$$X = c_1 A + c_2 B + c_3 C + c_4 D$$
$$Y = d_1 A + d_2 B + d_3 C + d_4 D$$

(a) Determine the coefficients c_i, d_i if a sinc-interpolation filter is used that has length 13. (Do not just give the answer - you must work out all the design details.)

(b) Determine the coefficients c_i, d_i if a linear interpolator is used.



Hints

$$Z_k = A_k + jB_k\,, \quad Z_{N-k}^* = A_k - jB_k$$

$$G_i = \frac{\Omega_0^2}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2}\,, \quad \theta_i = \frac{\pi}{2N}(N - 1 + 2i)$$

$$a_{i1} = \frac{2\,(\Omega_0^2 - 1)}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2}\,, \quad a_{i2} = \frac{1 + 2\Omega_0\cos\theta_i + \Omega_0^2}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2}$$

$$G_0 = \frac{\Omega_0}{\Omega_0 + 1}, \qquad a_{01} = \frac{\Omega_0 - 1}{\Omega_0 + 1}$$

332:521 – Digital Signals and Filters
Exam 1 - October 14, 2004

1. The analog antialiasing prefilter used in a particular DSP application has a flat passband over the frequency interval of interest, $0 \le f \le f_{\max}$. Beyond $f_{\max}$, the prefilter attenuates at a rate of α dB per octave. The input analog signal to the prefilter has the following spectrum beyond $f_{\max}$ (given in dB and normalized to unity at $f_{\max}$):

$$A_{\text{in}}(f) = -10\log_{10}\left[\frac{1.1}{1+0.1(f/f_{\max})^4}\right], \quad f \ge f_{\max}$$

It is required to pick the value of the attenuation constant α of the prefilter in order that the aliased frequency components due to sampling that are aliased *within* the $f_{\max}$ interval of interest be suppressed by at least 50 dB.

(a) Determine α if the sampling rate is taken to be $f_s = 3f_{\max}$.

(b) Determine α if the sampling rate is taken to be $f_s = 5f_{\max}$.

(c) Determine the sampling rate f_s if $\alpha = 24$ dB/octave.

2. Consider the causal and stable filter with impulse response $h(n) = a^n u(n)$, where $|a| < 1$. The input-on behavior of this filter may be studied by applying to it a one-sided sinusoid that starts at $n = 0$ and continues till $n = \infty$. The input-off behavior may be studied by applying a sinusoid that has been on since $n = -\infty$ and turns off at $n = 0$.

(a) Using z-transforms and partial fraction expansions, show that the output signals are given as follows in the two cases:

$$x(n) = e^{j\omega_0 n}u(n) \xrightarrow{H} y(n) = H(\omega_0)e^{j\omega_0 n}u(n) + Ba^n u(n)$$

$$x(n) = e^{j\omega_0 n}u(-n-1) \xrightarrow{H} y(n) = H(\omega_0)e^{j\omega_0 n}u(-n-1) - Ba^n u(n)$$

What is the coefficient B?

(b) What are the ROCs of the output signals $y(n)$ in the two cases?

(c) Using the results of part (a), determine the output signal $y(n)$ when the input is the *double-sided* sinusoid $x(n) = e^{j\omega_0 n}$, $-\infty < n < \infty$.

3. Consider the signal:

$$x(n) = u(n) + (-1)^n u(n) + 2\sin\left(\frac{\pi n}{2}\right)u(n)$$

(a) Determine its z-transform and the corresponding ROC.

(b) It is desired to design a notch filter that filters out the *third* term of the above signal $x(n)$. Design such a notch filter (that is, determine its transfer function $H(z)$) assuming that the 3-dB width of the notch is $\Delta\omega = 0.02$ rads/sample and that the filter is normalized to unity-gain at DC.

(c) Determine the steady-state form of the output signal $y_{\text{steady}}(n)$ and discuss whether or not the filter notches out the last term of $x(n)$ and faithfully reproduces the first two terms.

4. Consider the period-4 causal periodic signal:

$$x(n) = \{\underbrace{1, 2, 3, 4}_{\text{one period}}, 1, 2, 3, 4, \dots\}$$

where the dots ... represent the repetition of the basic period $[1, 2, 3, 4]$.

(a) Determine the z-transform of $x(n)$ and its ROC.

(b) Show that $x(n)$ can be written as a linear combination of four causal complex sinusoids. What are the frequencies of these sinusoids in rads/sample? (It is not necessary to evaluate numerically the amplitudes of the sinusoids.)

(c) Consider the filter $H(z) = \dfrac{1+2z^{-4}}{1+0.5z^{-4}}$.

Show that this is an *allpass* filter, that is, show that $|H(\omega)| = G$, for all ω. What is the value of the constant gain G?

(d) The signal $x(n)$ is sent to the input of $H(z)$. Using the results of part (b), show that in the *steady-state* limit, the output signal $y(n)$ will also be a periodic sequence. What is that sequence? Approximately, how long would it take for the periodic output to start appearing?

330:521 – Digital Signals and Filters
Exam 2 — November 18, 2004

1. Consider the filter $H(z) = \dfrac{1 + 2z^{-2} - z^{-4}}{1 - 0.5z^{-4}}$.

   (a) Draw its canonical realization form and write the corresponding sample processing algorithm using a circular delay-line buffer.

   (b) Draw its transposed realization form and state its sample processing algorithm with the help of appropriate internal states.

   (c) For the canonical realization, iterate the circular-buffer sample processing algorithm of part (a) and calculate the output samples $y(n)$, $n = 0, 1, \dots 5$, for the following input $\mathbf{x} = [8, 6, 4, 2, 8, 6]$. Make a table, as shown below, that displays, at each time instant, the values of the circular buffer entries w_0, w_1, w_2, w_3, w_4, the states s_0, s_2, s_4, and the input and output samples x, y.

x	w_0	w_1	w_2	w_3	w_4	s_0	s_2	s_4	y
8	*	*	*	*	*	*	*	*	*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
6	*	*	*	*	*	9	*	*	*

   (To help you check your answer, the last value of s_0 is 9.)

2. Consider the causal periodic sequence of period four:

$$x(n) = [\underbrace{4, 3, 2, 1}_{\text{one period}}, 4, 3, 2, 1, 4, 3, 2, 1, \dots\,]$$

   (a) Show that this sequence may be expressed as a sum of four sinusoidal signals:

$$x(n) = A_0 e^{j\omega_0 n} + A_1 e^{j\omega_1 n} + A_2 e^{j\omega_2 n} + A_3 e^{j\omega_3 n}, \quad n \ge 0$$

   Determine the frequencies ω_i. Determine the coefficients A_i by calculating an appropriate 4-point DFT.

   (b) Suppose that the sequence $x(n)$ is sent to the input of the filter $H(z)$ of the previous problem. Show that, in the steady-state, the output signal will also be periodic and will have the form:

$$y_{\text{steady}}(n) = B_0 e^{j\omega_0 n} + B_1 e^{j\omega_1 n} + B_2 e^{j\omega_2 n} + B_3 e^{j\omega_3 n}, \quad n \ge 0$$

   Determine the numerical values of B_i. Explain your reasoning.

   (c) By performing an inverse 4-point DFT, determine the numerical values of the periodic sequence $y_{\text{steady}}(n)$.

3. Consider the 8-point signal:

$$\mathbf{x} = \begin{bmatrix} 4 \\ 3 - 2\sqrt{2} \\ 0 \\ -3 + 2\sqrt{2} \\ -4 \\ 3 + 2\sqrt{2} \\ 0 \\ -3 - 2\sqrt{2} \end{bmatrix}$$

   (a) Calculate its 8-point FFT by hand.

   (b) Express the time samples $x(n)$ as a linear combination of *real-valued* sinusoidal (or cosinusoidal) signals.

4. Consider the following analog signal consisting of three sinusoids:

$$x(t) = \cos(2\pi f_1 t) + 10^{-3}\cos(2\pi f_2 t) + \cos(2\pi f_3 t)$$

   where $f_1 = 1.5$ kHz and $f_3 = 3.2$ kHz. The middle term represents a weak sinusoid of unknown frequency f_2 whose presence we wish to detect by sampling $x(t)$ and computing its DTFT spectrum. The sampling rate is 10 kHz. We know from prior information that the frequency f_2 must lie somewhere in the interval $2 \le f_2 \le 3$ kHz.

   It is desired to assess and compare the use of the rectangular, Hamming, and Kaiser windows for this spectral analysis problem. For each of these three windows, determine the following:

   (a) Can this particular window be used? Why or why not?

   (b) If the answer to part (a) is yes, then how many time samples should be collected in order for the DTFT to exhibit the three sinusoidal spectral peaks of this signal?

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Final Exam — December 16, 2004

1. Consider the 8-point signal:

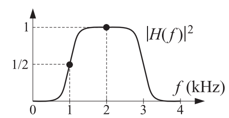
$$\mathbf{x} = [1, 0, -1, -\sqrt{2}, -1, 0, 1, \sqrt{2}]$$

   (a) Calculate its 8-point FFT by hand.

   (b) Based on the results of (a), express $x(n)$ as a sum of real-valued sinusoidal and/or cosinusoidal signals. Verify that your expression correctly generates the given time samples.

2. Using a *third-order* lowpass analog Butterworth prototype and the bandpass version of the bilinear transformation, design a *bandpass* digital filter that has the following specifications:

   – sampling rate of 8 kHz
   – center frequency of 2 kHz
   – left 3-dB frequency of 1 kHz



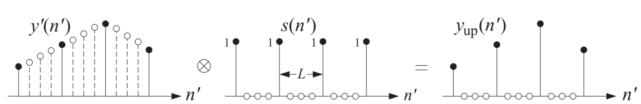
   (a) From these specifications, determine the values of the Butterworth parameter Ω_0 and bilinear transformation parameter c.

   (b) Determine the transfer function $H(z)$ of the designed digital filter.

   (c) The magnitude response of such a filter is shown above. Show that this response is given by:

$$|H(f)|^2 = \frac{1}{1 + \cot^6(\pi f/4)}\,, \quad f \text{ in units of kHz}$$

3. A fast-rate signal $y'(n')$ is downsampled by a factor of L resulting into the slow-rate signal $y_{\text{down}}(n) = y'(nL)$. Let $y_{\text{up}}(n')$ be the upsampled version of $y_{\text{down}}(n)$ consisting of the low-rate samples and $L-1$ zeros inserted between them, as shown in the figure below.



The process of going from the signal $y'(n')$ to $y_{\text{up}}(n')$ may be thought of as the multiplication of $y'(n')$ by a periodic train of unit pulses spaced at multiples of L with respect to the fast time scale, that is,

$$y_{\text{up}}(n') = s(n')y'(n')\,, \quad \text{where} \quad s(n') = \sum_{n=-\infty}^{\infty} \delta(n' - nL)$$

   (a) Because $s(n')$ is periodic in n' with period L, it can be expanded in a discrete Fourier series representation of the form:

$$s(n') = \frac{1}{L}\sum_{k=0}^{L-1} S(k)e^{2\pi jkn'/L}$$

   where $S(k)$ is the L-point DFT of one period of $s(n')$. Determine the DFT $S(k)$, for $k = 0, 1, \ldots, L-1$.

   (b) Using the result of part (a), show the basic spectrum replication property that was proven in class by different means:

$$Y_{\text{down}}(f) = Y_{\text{up}}(f) = \frac{1}{L}\sum_{k=0}^{L-1} Y'(f - kf_s)$$

   where f_s and $f_s' = Lf_s$ are the slow and fast sampling rates and the DTFT's are defined as usual by:

$$Y_{\text{down}}(f) = \sum_n y_{\text{down}}(n)e^{-2\pi jfn/f_s}\,, \quad Y_{\text{up}}(f) = \sum_{n'} y_{\text{up}}(n')e^{-2\pi jfn'/f_s'}$$

$$Y'(f) = \sum_{n'} y'(n')e^{-2\pi jfn'/f_s'}$$

4. A discrete-time model of a second-order noise-shaping delta-sigma quantizer is shown in the figure below.

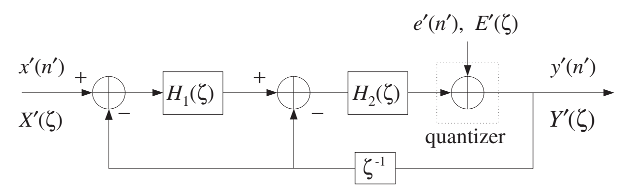
   (a) Write the I/O equation in the form:

$$Y'(\zeta) = H_x(\zeta)X'(\zeta) + H_{\text{NS}}(\zeta)E'(\zeta)$$

   and determine the signal and noise transfer functions $H_x(\zeta)$ and $H_{\text{NS}}(\zeta)$ in terms of the loop filters $H_1(\zeta)$ and $H_2(\zeta)$.

   (b) Then, determine $H_1(\zeta)$ and $H_2(\zeta)$ in order that the signal and noise transfer functions have the forms:

$$H_x(\zeta) = 1, \qquad H_{\text{NS}}(\zeta) = (1 - \zeta^{-1})^2$$



332:521 – Digital Signals and Filters
Exam 1 - October 13, 2005

1. Consider the following noisy "speech" signal, where t is in milliseconds:

$$x(t) = \underbrace{\sin(2\pi t) + \sin(4\pi t)}_{\text{speech part}} + \underbrace{\sin(14\pi t) + \sin(20\pi t)}_{\text{noise part}}$$

This signal is prefiltered by an analog antialiasing prefilter $H(f)$ and then sampled at the speech rate of 8 kHz, as shown below.

The resulting samples are immediately reconstructed by an *ideal reconstructor.* Consider the cases of the following three antialiasing prefilters:

(A) There is no prefilter, that is, $H(f) \equiv 1$.

(B) $H(f)$ is an *ideal* prefilter with cutoff of 4 kHz.

(C) $H(f)$ is an Nth order Butterworth filter with magnitude response:

$$|H(f)| = \frac{1}{\sqrt{1 + (f/4)^{2N}}}, \qquad (f \text{ is in kHz})$$

(a) For filters (A) and (B), determine the outputs $y(t)$ and $y_a(t)$ of the prefilter and reconstructor, and compare $y_a(t)$ with the *speech part* of $x(t)$ (you may ignore the filter's phase response.)

(b) For filter (C), determine the filter order N such that the components of the signal $x(t)$ that are aliased into the Nyquist interval be suppressed by more than 34 dB. For this value of N, determine the outputs $y(t)$ and $y_a(t)$.

2. Let $X(z)$ be the z-transform of a double-sided signal $x(n)$. Assume that its ROC is an annular region $R_1 < |z| < R_2$. Consider the following four transformations of $X(z)$:

(A) $\quad Y(z) = X(1/z)$

(B) $\quad Y(z) = X(z/a)\,, \quad a$ is a given constant

(C) $\quad Y(z) = -z\dfrac{dX(z)}{dz}$

(D) $\quad Y(z) = X(z^4)$

(a) For each case, determine the relationship between the inverse z-transform signal $y(n)$ and the original signal $x(n)$. Discuss also how the ROC is transformed.

(b) Using the differentiation property (C), show the z-transform pair:

$$(n+1)a^n u(n) \quad \Leftrightarrow \quad \frac{1}{(1 - az^{-1})^2}, \quad |z| > |a|$$

3. Consider the filter defined by the block diagram shown below.

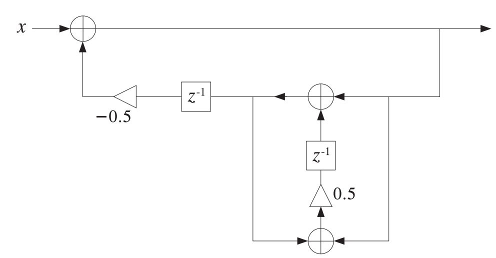
(a) Introduce appropriate internal states and write the *sample by sample processing* algorithm for computing each output sample y from each input sample x.

(b) Working in the z-domain, show that the transfer function from the input x to the output y of this block diagram is given by:

$$H(z) = \frac{1 - 0.5z^{-1}}{1 + 0.25z^{-2}}$$

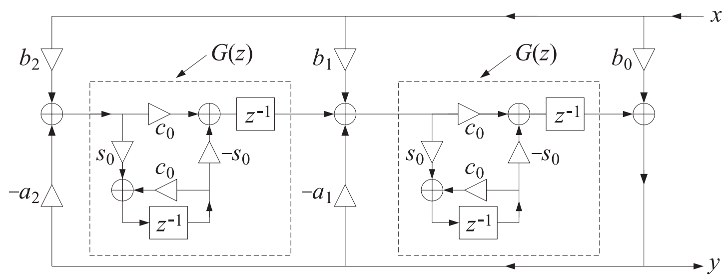
(c) Using *partial fraction expansions*, show that the corresponding causal impulse response is given by:

$$h(n) = (0.5)^n \left[\cos\left(\frac{\pi n}{2}\right) - \sin\left(\frac{\pi n}{2}\right)\right] u(n)$$



332:521 – Digital Signals and Filters
Exam 2 - November 17, 2005

1. A filter is defined by the following block diagram:



   (a) Introduce appropriate internal states and then write the sample processing algorithm for computing each output sample y from each input sample x.

   (b) Assuming that c_0 is real such that $|c_0| < 1$ and $s_0 = (1-c_0^2)^{1/2}$, show that the transfer function $G(z)$ of the sub-diagrams demarcated by the dashed lines is given by

$$G(z) = \frac{z^{-1}(c_0 - z^{-1})}{1 - c_0 z^{-1}}$$

   (c) Draw a realization of $G(z)$ that uses *only one* multiplier c_0.

   (d) Replacing the dashed boxes by $G(z)$, determine the transfer function $H(z)$ from x to y, expressed in terms of the coefficients $\{a_1, a_2, b_0, b_1, b_2\}$ and $G(z)$.

2. Consider the filter: $H(z) = \dfrac{1 + z^{-1}}{1 + z^{-1} - 0.5z^{-3}}$.

   Draw its canonical realization and write the corresponding sample processing algorithm using a circular-delay-line buffer. Then, for the input signal $\mathbf{x} = [8, 6, 4, 2, 1]$, iterate the sample processing algorithm, and compute the output signal $y(n)$ for $0 \le n \le 4$. In the process, fill in the table of values of the circular buffer $\mathbf{w}$ and the states s_0, s_1, s_2, s_3:

x	w_0	w_1	w_2	w_3	s_0	s_1	s_2	s_3	y
8	*	*	*	*	*	*	*	*	*
6	*	*	*	*	*	*	*	*	*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	*	*	*	*	*	*	*	*	*

3. Calculate the 8-point FFT of the following 8-point time signal:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2+\sqrt{2} \\ -1 \\ 2+\sqrt{2} \\ 1 \\ 2-\sqrt{2} \\ -1 \\ 2-\sqrt{2} \end{bmatrix}$$

   Using the calculated DFT, express the signal samples $x(n)$ as a sum of *real-valued* sinusoidal signals. Show all steps.

332:521 – Digital Signals and Filters
Final Exam - December 20, 2005

1. It is desired to design an FIR interpolation filter for the playback system of a digital audio system with an oversampling ratio of $L = 5$. The audio band extends to 20 kHz and the input to the interpolator is arriving at the low rate of $f_s = 40$ kHz.

   The output of the interpolation filter is reconstructed by a staircase DAC and then fed into a third-order Butterworth analog anti-image postfilter.

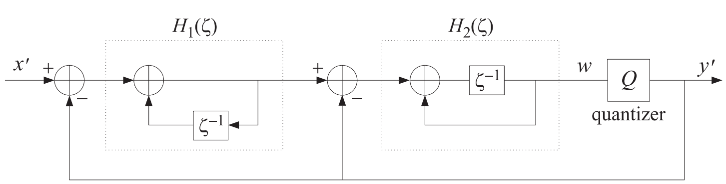
   (a) Using a Kaiser design, determine the length N of such an interpolation filter if it is to have a stopband attenuation of 70 dB and a transition width of 4 kHz (about the 20 kHz passband frequency.)

   (b) The combination of the staircase DAC and postfilter is required to suppress the spectral images at the output of the interpolator by more than 70 dB. How much of that 70 dB attenuation is going to be provided by the DAC and how much by the postfilter?

   (c) Determine the 3-dB frequency in kHz of the Butterworth anti-image postfilter. Then, determine the maximum attenuation in dB that the DAC and postfilter introduce within the 20-kHz audio band.

   (d) The FIR interpolation filter whose length N was determined above is required to equalize this maximum attenuation within the audio band arising from the DAC and postfilter. Write down the explicit design equation from which one may to calculate the interpolation filter's impulse response (it must include the Kaiser window factor). [*Hint:* $\alpha = 0.1102(A - 8.7)$.]

2. A discrete-time model of an oversampled noise-shaping quantizer is shown below, where ζ^{-1} represents the unit delay with respect to the fast time scale.



   (a) Assuming that the quantizer function is $Q(w) = \text{sign}(w)$, write down the sample processing algorithm for computing each quantized fast sample y' from each unquantized fast sample x'.

   (b) Replace this quantizer with an equivalent noise source, that is, set $Q(w_{n'}) = w_{n'} + e_{n'}$. Show that the input/output relationship of this block diagram can be written in the form:

$$Y'(\zeta) = H_x(\zeta)X'(\zeta) + H_{NS}(\zeta)E'(\zeta)$$

   and determine the explicit form of $H_x(\zeta)$ and $H_{NS}(\zeta)$.

3. The following digital filter operating at a rate of 8 kHz was the result of applying the (lowpass) bilinear transformation on an equivalent lowpass analog Butterworth filter:

$$H(z) = \frac{\frac{1}{6}(1 + z^{-1})^3}{1 + \frac{1}{3}z^{-2}}$$

   (a) Determine the order N and 3-dB frequency Ω_0 of the equivalent analog Butterworth filter.

   (b) Determine the equivalent analog transfer function $H_a(s)$ that gave rise to the above $H(z)$.

   (c) Determine the 3-dB frequency of the digital filter in kHz.

   (d) Suppose you want to design a Butterworth bandpass filter centered at 2 kHz and having the same passband and stopband attenuation specifications as the above $H(z)$. What simple change in $H(z)$ would generate such a bandpass filter?

332:521 – Digital Signals and Filters
Exam 1 – October 12, 2006

1. The maximum frequency of interest in a speech signal is 4 kHz. Beyond 4 kHz, the signal attenuates at a rate of 20 dB/octave.

   (a) Determine the required sampling rate f_s in kHz if no antialiasing filter is used and if the frequency components aliased back into the 4-kHz interval of interest must be suppressed by more than 20 dB.

   (b) Suppose now that an order-4 Butterworth filter is used as an antialiasing prefilter, having a magnitude response squared:

   $$|H(f)|^2 = \frac{1}{1 + a\left(\frac{f}{4}\right)^8}$$

   First, determine the constant a such that the attenuation caused by the filter *within* the 4-kHz interval remain less than 0.2 dB.
   Second, using the value of f_s from part (a), determine the total suppression in dB of the aliased components into the 4-kHz interval.

   (c) For the above signal and filter, set up a single equation that would determine f_s given that the total attenuation of the components aliased into the 4-kHz interval is required to be greater than A dB.
   Then, by trial-and-error determine f_s in kHz when $A = 45$ dB. How much of the 45 dB is due to the signal and how much to the filter?

2. Consider the two signals:

   $$x_1(n) = a^n u(n)\,, \quad x_2(n) = a^{-n} u(-n)$$

   where a is a real parameter satisfying $|a| < 1$.

   (a) Determine the z-transforms $X_1(z)$ and $X_2(z)$ and the corresponding ROC's.

   (b) Working in the time domain, calculate the convolution of $x_1(n)$ and $x_2(n)$, that is,

   $$y(n) = \sum_m x_1(m) x_2(n-m)$$

   where you must first determine the range of the output index n, and the proper range of summation over m. [*Hint:* $y(n)$ is a double-sided function of n, so do the cases $n \geq 0$ and $n < 0$ separately.]

   (c) Using the time-domain result of part (b), determine the z-transform of $y(n)$ and its ROC by adding up the series $\sum_n y(n) z^{-n}$. Verify that $Y(z) = X_1(z) X_2(z)$.

   (d) Discuss the stability properties of $x_1(n), x_2(n), y(n)$.

3. Consider the following filter and *causal* input signal:

   $$H(z) = \frac{-0.5 + z^{-1}}{1 - 0.5z^{-1}}$$

   $$x(n) = 1 + \cos\left(\frac{\pi n}{2}\right) + \cos(\pi n)\,, \quad n \geq 0$$

   (a) Show that the corresponding steady-state output will have the form:

   $$y_{\text{steady}}(n) = 1 - 0.8\cos\left(\frac{\pi n}{2}\right) + 0.6\sin\left(\frac{\pi n}{2}\right) - \cos(\pi n)\,, \quad n \geq 0$$

   (b) Defining the angles $\theta_1 = \arctan(0.6/0.8) - \pi$ and $\theta_2 = \pi$, show that the above output can be written as:

   $$y_{\text{steady}}(n) = 1 + \cos\left(\frac{\pi n}{2} + \theta_1\right) + \cos(\pi n + \theta_2)$$

   What do these angles θ_1, θ_2 have to do with $H(z)$?

   (c) Determine the causal impulse response of $H(z)$.

   (d) Show that this filter is an *allpass* filter, that is, $|H(\omega)| = 1$, for all values of ω (that is why the three input sinusoids pass through unchanged except for a phase shift.)

332:521 – Digital Signals and Filters
Exam 2 – November 16, 2006

1. Consider the filter: $H(z) = \dfrac{1 - 2z^{-1} + z^{-2}}{1 + z^{-1} - 0.5z^{-2}}$.

   Draw its canonical realization and write the corresponding sample processing algorithm using a circular-delay-line buffer. Then, for the input signal $\mathbf{x} = [1, 2, 4, 6, 8]$, iterate the sample processing algorithm, and compute the output signal $y(n)$ for $0 \le n \le 4$. In the process, fill in the table of values of the circular buffer $\mathbf{w}$ and the states s_0, s_1, s_2, s_3:

x	w_0	w_1	w_2	w_3	s_0	s_1	s_2	s_3	y
8	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
6	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

2. Consider the causal periodic sequence of period 4:

$$x(n) = [\underbrace{5, 2, 5, 4}, \underbrace{5, 2, 5, 4}, \underbrace{5, 2, 5, 4}, \dots]$$

   where the dots represent the repetition of the period $[5, 2, 5, 4]$. The signal $x(n)$ is next filtered through the filter

$$H(z) = \frac{3 - 1.5z^{-1}}{1 - 0.5z^{-2}}$$

   Using FFT methods, determine the steady-state output from this filter and show that it is also periodic with period 4. Write down this periodic output explicitly.

   How quickly does the filter output become periodic after sending $x(n)$ in?

3. Let $X(k), Y(k)$ be the N-point DFTs of the two length-N complex-valued signals $x(n), y(n)$. Prove the following generalized form of Parseval's identity:

$$\sum_{n=0}^{N-1} y^*(n)x(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y^*(k)X(k)$$

4. Calculate the 8-point FFT of the following time signal:

$$\mathbf{x} = \begin{bmatrix} 6 \\ 5+\sqrt{2} \\ 4 \\ 3+\sqrt{2} \\ 6 \\ 5-\sqrt{2} \\ 8 \\ 3-\sqrt{2} \end{bmatrix}$$

5. An 8-point real-valued signal has the following 8-point DFT:

$$\mathbf{X} = \begin{bmatrix} 40 \\ 0 \\ -4j \\ -8j \\ 8 \\ * \\ * \\ * \end{bmatrix}$$

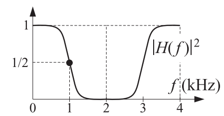
   (a) What are the values of the starred entries?

   (b) Express the time-domain signal $x(n)$ as a linear combination of real-valued sinusoidal and/or cosinusoidal signals.

330:521 – Digital Signals and Filters
Final Exam — December 19, 2006

1. Using a *third-order* lowpass analog Butterworth prototype and the bandstop version of the bilinear transformation, design a *bandstop* digital filter that has the following specifications:

   – sampling rate of 8 kHz
   – center frequency of 2 kHz
   – left 3-dB frequency of 1 kHz



   (a) From these specifications, determine the values of the Butterworth parameter Ω_0 and bilinear transformation parameter c_0.

   (b) Determine the transfer function $H(z)$ of the designed digital filter.

   (c) The magnitude response of such a filter is shown above. Show that this response is given analytically by:

$$|H(f)|^2 = \frac{1}{1+\tan^6(\pi f/4)}, \quad f \text{ in units of kHz}$$

2. Calculate the 8-point FFT of the following time signal:

$$\mathbf{x} = \begin{bmatrix} 4 \\ 3+\sqrt{2} \\ 2 \\ -3-\sqrt{2} \\ 0 \\ 3-\sqrt{2} \\ 2 \\ -3+\sqrt{2} \end{bmatrix}$$

3. A periodic signal $x(n)$ with period eight has the following 8-point DFT:

$$\mathbf{X} = \begin{bmatrix} 8 \\ 8 \\ -12j \\ 0 \\ 8 \\ 0 \\ 12j \\ 8 \end{bmatrix}$$

   (a) Express $x(n)$ as a sum of real-valued sinusoidal signals.

   (b) The periodic signal $x(n)$ is sent to the filter designed in Problem 1. Determine the steady output signal $y_{\text{steady}}(n)$ and express it as a sum of real-valued sinusoidal signals. [*Hint:* $(1+j)^3 = -2+2j$.]

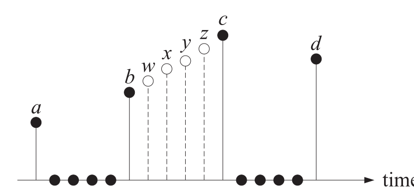
   (c) Explain the results of part (b) based on the frequency response shown in Problem 1.

4. A 5-times interpolation filter calculates four missing samples between any two low-rate samples, as shown below. For example, three possible ways of calculating the four missing samples $[w, x, y, z]$ as linear combinations of four low-rate surrounding samples $[a, b, c, d]$ are as follows:

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -0.10 & 0.23 & 0.94 & -0.16 \\ -0.19 & 0.50 & 0.76 & -0.22 \\ -0.22 & 0.76 & 0.50 & -0.19 \\ -0.16 & 0.94 & 0.23 & -0.10 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -0.01 & 0.16 & 0.91 & -0.06 \\ -0.03 & 0.41 & 0.69 & -0.06 \\ -0.06 & 0.69 & 0.41 & -0.03 \\ -0.06 & 0.91 & 0.16 & -0.01 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \\ 0.6 & 0.4 \\ 0.8 & 0.2 \end{bmatrix} \begin{bmatrix} c \\ b \end{bmatrix}$$



   (a) In each case, please explain where these expressions come from and how the matrix coefficients were obtained.

   (b) For each case, determine the length N of the interpolation filter, its passband and stopband attenuations in dB, and its transition width in units of the low rate f_s.

332:521 – Digital Signals and Filters
Exam 1 – October 11, 2007

1. In a particular DSP application the maximum frequency of interest is $f_{\max}$. Beyond $f_{\max}$, the input signal to be sampled attenuates at a rate of α dB/octave.

   To suppress aliased frequency components within the desired $f_{\max}$ range, an analog antialiasing prefilter was designed and found to have the following Butterworth magnitude-square response:

$$|H(f)|^2 = \frac{1}{1 + 0.1\left(\dfrac{f}{f_{\max}}\right)^8}$$

   It is known that within the $f_{\max}$ range of interest the aliased components due to sampling are suppressed by 50 dB. Moreover, it is known that 30% of that arises from the filter and 70% from the input signal.

   (a) Determine the sampling rate f_s in units of $f_{\max}$.

   (b) Determine the attenuation rate α in dB/octave of the input signal.

2. Consider the two signals:

$$h(n) = a^n u(n)\,, \quad x(n) = -b^n u(-n-1)$$

   (a) Working in the time domain, determine the convolution:

$$y(n) = \sum_m x(m)h(n-m)$$

   and express it in closed form for $n \geq 0$ and $n \leq -1$ by using the geometric series to perform the above summation. Under what conditions on a, b are your results valid?

   (b) Determine the z-transform of $y(n)$ and its ROC. Under what conditions on a, b does the z-transform exist? Under what conditions is $y(n)$ stable?

3. Consider the filter:

$$H(z) = \frac{2 - z^{-2} + z^{-3}}{1 + z^{-1} - 0.5z^{-3}}$$

   (a) Draw the transposed realization of this filter. Let v_1, v_2, v_3 denote the contents of the three delays that appear in this realization. With the help of these variables, state the sample processing algorithm for computing each output sample y from each input sample x.

   (b) For the following input signal $\mathbf{x} = [5, 4, 3, 2, 1]$, iterate the above sample processing algorithm to compute the corresponding output samples, and in the process fill in the entries of the following table:

x	v_1	v_2	v_3	y
5	$*$	$*$	$*$	$*$
4	$*$	$*$	$*$	$*$
3	$*$	$*$	$*$	$*$
2	$*$	0	$*$	$*$
1	$*$	$*$	$*$	$*$
	$*$	4.5	$*$	

   To help you check your answer, a couple of table entries are given.

   (c) Draw the canonical realization of this filter and state its sample processing algorithm using circular buffers.

   (d) Iterate the sample processing algorithm on the above input signal and in the process fill-in the entries of the following table, where w_i are the circular buffer entries, and s_i, the states.

x	w_0	w_1	w_2	w_3	s_0	s_1	s_2	s_3	y
5	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
4	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
3	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
2	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
1	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

332:521 – Digital Signals and Filters
Exam 2 – November 15, 2007

1. The following digital filter operating at a rate of 20 kHz was the result of applying the highpass bilinear transformation on an equivalent lowpass analog Butterworth filter:

$$H(z) = \frac{\frac{1}{6}(1 - z^{-1})^3}{1 + \frac{1}{3}z^{-2}}$$

   (a) Determine the order N and 3-dB frequency Ω_0 of the equivalent LP analog Butterworth filter.
   (b) Determine the equivalent LP analog transfer function $H_a(s)$ that gave rise to the above $H(z)$.
   (c) Determine the 3-dB frequency of the digital filter in kHz. Make a rough sketch of $|H(f)|^2$ for $0 \le f \le 10$ kHz.

2. Compute the FFT of the 8-point time signal:

$$\mathbf{x} = \begin{bmatrix} 2 \\ -1-\sqrt{2} \\ 2 \\ -1 \\ 0 \\ -1+\sqrt{2} \\ 0 \\ -1 \end{bmatrix}$$

3. The following signal is sampled at a rate of 8 kHz and 8 consecutive time samples are collected:

$$x(t) = \cos(6\pi t) + \cos(8\pi t) + \sin(10\pi t)\,, \quad \text{with } t \text{ in msec}$$

   Without performing any DFT/FFT calculations, determine the 8-point FFT of the 8 collected time samples.

4. Consider the LP analog shelving filter:

$$|H_a(\Omega)|^2 = \frac{4 + 2\Omega^2}{1 + 2\Omega^2}$$

   (a) What is the DC gain G of this filter in absolute units and in dB? What is the 3-dB frequency Ω_0 of this filter (defined by the condition $|H_a(\Omega)|^2 = G^2/2$)? Make a sketch of $|H_a(\Omega)|^2$ versus Ω indicating the 3-dB point.

   (b) What is the transfer function $H_a(s)$ of this filter? (Both zeros and poles must be in the left-hand plane.)
   (c) Using the BP bilinear transformation, the analog filter $H_a(s)$ is to be transformed into a digital equalizer filter operating at a rate of $f_s = 8$ kHz and having a center frequency at $f_0 = 2$ kHz. Show that the resulting filter is

$$H(z) = b\,\frac{1 + az^{-2}}{1 - az^{-2}}\,, \qquad b = \sqrt{2}\,, \quad a = \frac{1-\sqrt{2}}{1+\sqrt{2}}$$

   (d) What is the gain of this filter in dB at DC? At the peak? At Nyquist?
   (e) What is the 3-dB width in kHz of this filter measured from the peak? What are the corresponding left and right 3-dB bandedge frequencies in kHz? Sketch the magnitude response $|H(f)|^2$ versus $0 \le f \le 4$ kHz.
   (f) The above filter was a boosting filter. A cutting filter that cuts frequencies in the opposite way as the boosting filter is defined by inverting the gains:

$$|H_a(\Omega)|^2 = \frac{4^{-1} + 2^{-1}\Omega^2}{1 + 2^{-1}\Omega^2}$$

   Show that the same design procedure leads to the exact inverse of both the analog shelving filter and the digital equalizer, for example,

$$H(z) = b^{-1}\,\frac{1 - az^{-2}}{1 + az^{-2}}$$

Hints

$$G_i = \frac{\Omega_0^2}{1 - 2\Omega_0\cos\theta_i + \Omega_0{}^2}\,, \quad \theta_i = \frac{\pi}{2N}(N - 1 + 2i)$$

$$a_{i1} = -\frac{2(\Omega_0^2 - 1)}{1 - 2\Omega_0\cos\theta_i + \Omega_0{}^2}\,, \quad a_{i2} = \frac{1 + 2\Omega_0\cos\theta_i + \Omega_0^2}{1 - 2\Omega_0\cos\theta_i + \Omega_0{}^2}$$

$$G_0 = \frac{\Omega_0}{\Omega_0 + 1}\,, \qquad a_{01} = -\frac{\Omega_0 - 1}{\Omega_0 + 1}$$

$$\frac{\cos\omega_0 - \cos\omega_\pm}{\sin\omega_\pm} = \pm\Omega_0$$

$$\Omega_0 = \tan\left(\frac{\omega_+ - \omega_-}{2}\right)\,, \quad \cos\omega_\pm = \frac{\cos\omega_0 \mp \Omega_0\sqrt{\Omega_0^2 + \sin^2\omega_0}}{1 + \Omega_0^2}$$

332:521 – Digital Signals and Filters
Final Exam – December 19, 2007

1. Consider the 8-point signal:

$$x(n) = \cos(\pi n) + \cos\left(\frac{\pi n}{2}\right) + \sin\left(\frac{\pi n}{2}\right), \quad n = 0, 1, \dots, 7$$

Determine its 8-point DFT in two ways:

(a) By calculating it numerically doing an 8-point FFT by hand.

(b) By matching the appropriate terms in the inverse DFT formula.

2. It is desired to design a 10-times oversampling interpolator filter for the playback system of a CD player. The low rate sampling frequency is 44.1 kHz and the transition width of the filter is 4.41 kHz. The minimum stopband attenuation is required to be 60 dB.

(a) Using a Kaiser design, determine the length N of the filter. Calculate the number of low-rate CD samples that are used to compute each interpolated sample. [*Hint:* $D = (A - 7.95)/14.36$.]

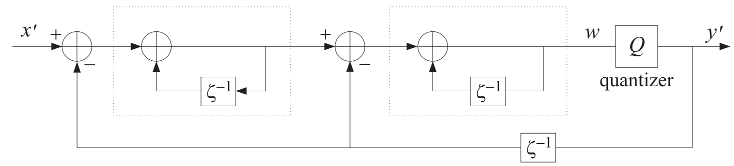
(b) Assuming the filter is implemented in its polyphase form, calculate the computational rate R of the filter in MACs/sec. Can a modern DSP chip handle this filter? If the DSP chip has a MAC instruction time of 20 nsec, then what would be the maximum filter length N of the interpolation filter that it can handle?

(c) Assume next that the filter is to be implemented as a 2×5 multistage design. Calculate the required lengths N_0 and N_1 of the 2-times and 5-times interpolating stages. Calculate the total computational rate R_{multi} in MACs/sec and compare it with the rate R of the single-stage design of the previous question (assume that each stage is implemented in its polyphase form.)

(d) Repeat part (c) for a 5×2 multistage design. Which of the three designs is the most efficient?

(e) For parts (c,d), make a sketch of the magnitude responses of the filters in the two stages over the frequency range $0 \le f \le 10 f_s$.

3. A discrete-time model of an oversampled noise-shaping quantizer is shown below, where ζ^{-1} represents the unit delay with respect to the fast time scale, and the dashed boxes represent discrete-time integrators:



(a) Assuming that the quantizer function is $Q(w) = \text{sign}(w)$, write down the sample processing algorithm for computing each quantized fast sample y' from each unquantized fast sample x'.

(b) Replace this quantizer with an equivalent noise source, that is, set $Q(w_{n'}) = w_{n'} + e_{n'}$. Show that the input/output relationship of this block diagram can be written in the form:

$$Y'(\zeta) = H_x(\zeta) X'(\zeta) + H_{NS}(\zeta) E'(\zeta)$$

and determine the explicit form of $H_x(\zeta)$ and $H_{NS}(\zeta)$.

332:521 – Digital Signals and Filters
Exam 1 – October 8, 2009

1. In recent audio applications, it is common to use the sampling rate $f_s = 2\times 48 = 96$ kHz. It is desired to use an Nth order Butterworth analog prefilter whose attenuation in dB is given as follows, where f_0 is a parameter:

$$A(f) = 10\log_{10}\left[1 + \left(\frac{f}{f_0}\right)^{2N}\right]$$

   (a) The highest audio frequency of interest is $f_{\max} = 20$ kHz. What should be the passband and stopband frequencies $f_{\text{pass}}, f_{\text{stop}}$ in kHz for this prefilter?

   (b) The audio signal to be sampled has a spectrum that attenuates at a rate of 15 dB/octave beyond the 20 kHz maximum frequency. It is desired to suppress the aliased components within the audio band by at least 50 dB.

   What should be the attenuation A_{stop} in dB that must be provided by the prefilter at its stopband frequency f_{stop}?

   (c) Let A_{pass} be the attenuation of the prefilter within the audio band. Show that the quantities $f_s, f_{\max}, A_{\text{stop}}, A_{\text{pass}}$ must be related by:

$$f_s = f_{\max}\left[1 + \left(\frac{10^{A_{\text{stop}}/10} - 1}{10^{A_{\text{pass}}/10} - 1}\right)^{\frac{1}{2N}}\right]$$

   How would you interpret the limit $N \to \infty$?

   (d) Assuming that $A_{\text{pass}} = 0.2$ dB, determine the filter parameters N, f_0 to meet the above aliasing requirements.

   For the derived values of N, f_0, determine the actual values for A_{stop}, A_{pass} achieved by the filter at $f_{\text{pass}}, f_{\text{stop}}$.

2. Consider the following z-transform whose ROC is $|z| > |a|$ with $|a| < 1$,

$$X(z) = \frac{az^{-1}}{(1 - az^{-1})^2} = \left[\frac{az^{-1}}{1 - az^{-1}}\right]\cdot\left[\frac{1}{1 - az^{-1}}\right]$$

   Determine the inverse z-transforms of the two factors in the brackets. Then, convolve them in the time domain to determine the inverse z-transform of $X(z)$.

   As a separate question, consider the signal $x(n) = a^n u(n) + b^n u(-n)$. For what relative values of the complex numbers a, b does the z-transform of $x(n)$ exist?

3. Consider the multi-notch filter discussed in class:

$$H(z) = G\frac{1 - z^{-D}}{1 - az^{-D}}, \quad G = \frac{1 + a}{2}, \quad 0 < a < 1$$

   (a) Using the geometric series expansion determine the impulse response $h(n)$ of this filter and sketch it over $0 \le n \le 4D$.

   (b) Show that the magnitude response squared of this filter is given by,

$$|H(\omega)|^2 = G^2\frac{2 - 2\cos(\omega D)}{1 - 2a\cos(\omega D) + a^2}$$

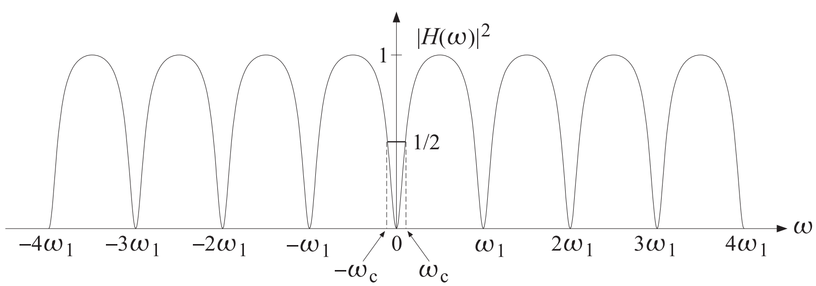
   It is depicted below for $a = 0.5$ and $D = 8$.

   (c) Show that the 3-dB width, defined in terms of the 3-dB frequency by $\Delta\omega = 2\omega_c$ and depicted below, is given by

$$\tan\left(\frac{D\Delta\omega}{4}\right) = \frac{1 - a}{1 + a}$$

   Hint: First solve for ω_c and then use the trig identity:

$$\tan^2\left(\frac{x}{2}\right) = \frac{1 - \cos x}{1 + \cos x}$$



332:521 – Digital Signals and Filters
Exam 2 – November 15, 2009

1. Consider the 8-point signal:

$$x(n) = 2\cos(\pi n) + \cos\left(\frac{3\pi n}{2}\right) + \sin\left(\frac{3\pi n}{2}\right), \quad n = 0, 1, \dots, 7$$
$$= [3,\ -3,\ 1,\ -1,\ 3,\ -3,\ 1,\ -1]$$

Determine its 8-point DFT in two ways:

(a) By calculating it numerically doing an 8-point FFT by hand.

(b) By matching the appropriate terms in the inverse DFT formula.

2. Consider the filter $H(z) = \dfrac{1 - z^{-2}}{1 - 0.5z^{-4}}$.

(a) Draw its canonical realization form and write the corresponding sample processing algorithm using a circular delay-line buffer.

(b) Draw its transposed realization form and state its sample processing algorithm with the help of appropriate internal states.

(c) For the canonical realization, iterate the circular-buffer sample processing algorithm of part (a) and calculate the output samples $y(n)$, $n = 0, 1, \dots, 5$, for the following input $\mathbf{x} = [3, -3, 1, -1, 3, -3]$. Make a table, as shown below, that displays, at each time instant, the values of the circular buffer entries w_0, w_1, w_2, w_3, w_4, the states s_0, s_2, s_4, and the input and output samples x, y.

x	w_0	w_1	w_2	w_3	w_4	s_0	s_2	s_4	y
3	*	*	*	*	*	*	*	*	*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
−3	*	*	*	*	*	*	*	*	−3.5

(To help you check your answer, the last value of y is -3.5.)

3. The input signal of Problem 3 corresponds to the first 6 samples of the 8-point signal given in Problem 1. Suppose that 8-point signal $\mathbf{x} = [3, -3, 1, -1, 3, -3, 1, -1]$ is sent periodically into the above filter, that is, its input is now the signal $[\mathbf{x}, \mathbf{x}, \mathbf{x}, \dots,]$. After the transients die out, the output will be periodic with period 8.

Using FFT methods, determine one period of this periodic output. For this calculation, you will need the values of the frequency response of the above filter at the 8 DFT frequencies.

To save you time, these are given below:

$$H(\omega_k) = \begin{bmatrix} 0 \\ 2(1+j)/3 \\ 4 \\ 2(1-j)/3 \\ 0 \\ 2(1+j)/3 \\ 4 \\ 2(1-j)/3 \end{bmatrix}$$

Roughly, how many periods does it take for the output to settle into the steady-state periodic output?

4. (a) Given a DTFT pair $x(n) \longleftrightarrow X(\omega)$, what is the DTFT of the complex conjugate signal $x^*(n)$ in terms of $X(\omega)$?

(b) Given a N-point DFT pair $x(n) \longleftrightarrow X(k)$, what is the N-point DFT of the complex conjugate signal $x^*(n)$ in terms of $X(k)$? Is this property consistent with that of part (a)?

(c) The following 8-point DFT pair can be easily verified:

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \\ 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} + j \begin{bmatrix} 3 \\ 3 \\ 3 \\ -1 \\ 3 \\ 3 \\ 3 \\ -1 \end{bmatrix} \longleftrightarrow \mathbf{X} = \begin{bmatrix} 8+16j \\ 0 \\ 16 \\ 0 \\ 8j \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Without computing any further DFT/IDFTs, extract from $\mathbf{X}$ the 8-point DFTs of the real and imaginary parts of $\mathbf{x}$ and explain the method that you used.

[*Note:* this can be done by using the results of part (b), but other approaches are possible, also the answer is not as simple as taking the real and imaginary parts of $\mathbf{X}$.]

332:521 – Digital Signals and Filters
Final Exam –December 16, 2009

1. (a) Given the 8-point DFT

$$\mathbf{X} = [0,\, 0,\, 4,\, -8j,\, 0,\, 8j,\, 4,\, 0],$$

express the corresponding time signal $x(n)$, $n = 0, 1, \ldots, 7$, as a sum of real-valued sinusoidal and/or cosinusoidal signals.

(b) Let $x(n)$ for $n = 0, 1, \ldots, N-1$ be a length-N time signal and let $X(\omega_k)$ for $k = 0, 1, \ldots, N-1$, be its N-point DFT. Show that the z-transform of $x(n)$ can be written in the form:

$$X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(\omega_k)}{1 - e^{j\omega_k} z^{-1}}$$

(c) Let ω_k, $k = 0, 1, \ldots, N-1$, be the N DFT frequencies, that is, $\omega_k = 2\pi k/N$. Prove the following identities:

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j\omega_k n} = \delta(k) \quad \text{and} \quad \frac{1}{N} \sum_{k=0}^{N-1} e^{j\omega_k n} = \delta(n)$$

Hint: $\displaystyle\sum_{n=0}^{N-1} x^n = \frac{1 - x^N}{1 - x}$.

2. Consider the filter:

$$H(z) = \frac{0.5 + z^{-2}}{1 + 0.5\, z^{-2}}$$

(a) Show that this is an allpass filter. What is its gain?

(b) Draw the transposed realization form and write the corresponding sample processing algorithm.

(c) Suppose that the following period-4 causal periodic signal is sent to the input of this filter, where the dots represent the repetition of the length-4 period $[1, 2, 3, 4]$:

$$\mathbf{x} = [\underbrace{1, 2, 3, 4}_{\text{one period}}, 1, 2, 3, 4, \ldots\,]$$

Explain why after the initial filter transients die out, the filter output is also going to be a period-4 periodic sequence. What is that sequence? Since that output sequence is different from the input one, how can one reconcile this with the fact that the filter is allpass?

3. In designing digital resonator filters with prescribed peak widths, we use the bilinear transformation method and start from an equivalent analog prototype resonator filter of the form:

$$H(s) = \frac{\alpha s}{s^2 + \alpha s + \Omega_0^2}$$

(a) For this analog filter, show that the 3-dB width of the peak at Ω_0 (measured relative to the peak maximum) is given by $\Delta\Omega = \alpha$.

(b) Show that the 10-dB width of the peak is $\Delta\Omega = 3\alpha$. What is the attenuation level in dB that corresponds to a width of $\Delta\Omega = 2\alpha$?

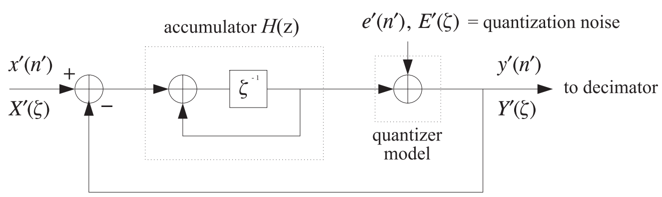
(c) A digital resonator filter $H(z)$ operating at an 8 kHz sampling rate was designed using the above bilinear transformation method. It turned out that the equivalent analog filter was of the form:

$$H(s) = \frac{0.5s}{s^2 + 0.5s + 1}$$

Determine the digital transfer function $H(z)$. Determine the center frequency f_0 and 3-dB peak width Δf of the digital filter in kHz.

4. (a) A model of a first-order noise shaping quantizer is depicted below, where ζ^{-1} is the unit-delay in the fast time scale and n' denotes a fast sampling instant. Show the following input/output equation in the ζ-domain:

$$Y'(\zeta) = \zeta^{-1} X'(\zeta) + (1 - \zeta^{-1}) E'(\zeta)$$



(b) How would you modify this block diagram if you want to implement a second-order noise shaping quantizer, that is, one whose input/output equation has the form:

$$Y'(\zeta) = \zeta^{-1} X'(\zeta) + (1 - \zeta^{-1})^2 E'(\zeta)$$

Verify that your block diagram satisfies this equation.

(c) For a second-order noise shaping quantizer, show from first principles that the savings in bits ΔB is related to the oversampling ratio L by the following approximate equation:

$$2^{2\Delta B} = \frac{5L^5}{\pi^4}$$

Discuss the nature of this approximation.