332:521 – Digital Signals and Filters
Computer Experiment 3 – Due October 21, 2010

In this experiment, you will study: (a) some examples of signal enhancement and noise reduction filters, (b) the interplay between steady-state and transient response and the trade-off between time constant and sharpness of filter specifications, and (c) the effect of coefficient quantization on the performance and stability of filters.

Problem 3.1: Coefficient Quantization Effects

Please do Problems 7.20 and 7.21 of the text.

Problem 3.2: FIR Bandpass Filter

A signal $x(n)$ is the sum of a desired signal $s(n)$ and interference $v(n)$:

$$x(n) = s(n) + v(n)$$

where

$$s(n) = \sin(\omega_2 n)\,, \quad v(n) = \sin(\omega_1 n) + \sin(\omega_3 n)$$

with

$$\omega_1 = 0.05\pi\,, \quad \omega_2 = 0.2\pi\,, \quad \omega_3 = 0.35\pi$$

In order to remove $v(n)$, the signal $x(n)$ is filtered through a bandpass FIR filter that is designed to pass the frequency ω_2 and reject the interfering frequencies ω_1, ω_3. An example of such a filter of order $M = 100$ was designed with the methods of Chapter 11 (using a Hamming-window design) and has impulse response:

$$h(n) = w(n)\left[\frac{\sin\big(\omega_b(n - M/2)\big) - \sin\big(\omega_a(n - M/2)\big)}{\pi(n - M/2)}\right], \quad 0 \le n \le M$$

where $\omega_a = 0.15\pi$, $\omega_b = 0.25\pi$, and $w(n)$ is the Hamming window:

$$w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{M}\right), \quad 0 \le n \le M$$

To avoid a computational issue at $n = M/2$, you may use MATLAB's built-in function `sinc`, which is defined as follows:

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

a. On the same graph plot $x(n)$ and $s(n)$ versus n

b. Filter $x(n)$ through the filter $h(n)$ using a circular-buffer implementation of the filter, and plot the filtered output $y(n)$, together with $s(n)$.

   Apart from an overall delay introduced by the filter, $y(n)$ should resemble $s(n)$ after the M initial transients

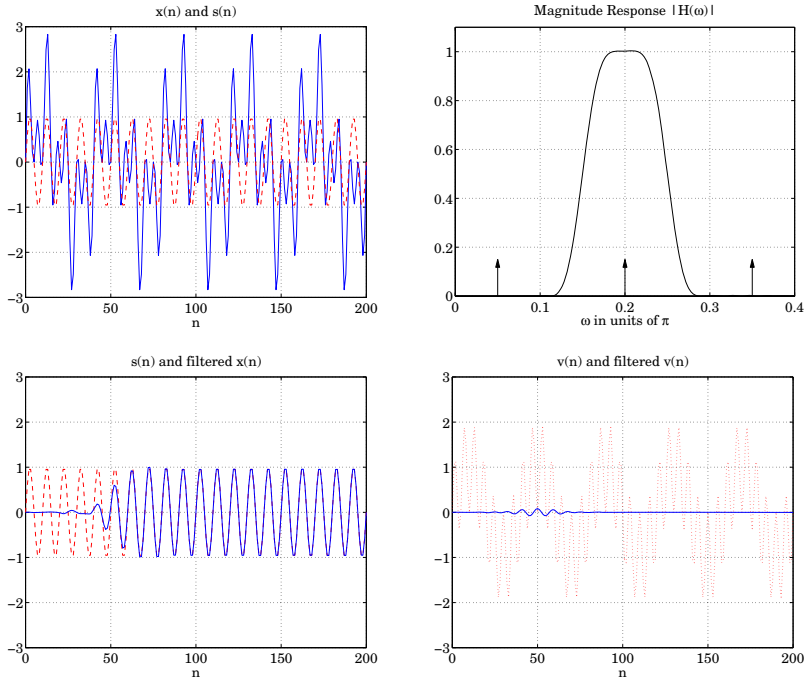
c. To see what happened to the interference, filter the signal $v(n)$ separately through the filter and plot the output, on the same graph with $v(n)$ itself.

d. Using the built-in MATLAB function `freqz`, or the textbook function `dtft`, calculate and plot the magnitude response of the filter over the frequency interval $0 \le \omega \le 0.4\pi$:

$$|H(\omega)| = \left|\sum_{n=0}^{M} h(n)e^{-j\omega n}\right|$$

   Indicate on that graph the frequencies $\omega_1, \omega_2, \omega_3$.

e. Redesign the filter with $M = 200$ and repeat parts [a-d]. Discuss the effect of choosing a longer filter length.



Problem 3.3: Resonator Bandpass Filter

Consider $N = 300$ samples of a noisy sinusoidal signal of frequency $f_0 = 500$ Hz sampled at a rate of $f_s = 10$ kHz:

$$x(n) = \cos(\omega_0 n) + v(n), \quad 0 \le n \le N-1$$

where $\omega_0 = 2\pi f_0 / f_s$ is the digital frequency and $v(n)$ is white noise with zero mean and unit variance.

The following 2-pole resonator filter, tuned to the frequency ω_0, will extract the sinusoid (except for a time delay), while at the same time reducing the noise:

$$H(z) = \frac{G}{(1 - Re^{j\omega_0} z^{-1})(1 - Re^{-j\omega_0} z^{-1})} = \frac{G}{1 - 2R\cos\omega_0 z^{-1} + R^2 z^{-2}}$$

where $0 < R < 1$. The gain G is fixed so as to ensure $|H(\omega_0)| = 1$. This gives the following expression for G:

$$G = (1-R)(1 - 2R\cos(2\omega_0) + R^2)^{1/2}$$

The white noise component $v(n)$ has power that is spread equally over the entire Nyquist interval. After filtering, the overall noise power will be reduced because only the amount of power that resides within the resonator's peak will survive the filtering process. If we denote by $y_v(n)$ the filtered version of $v(n)$, then the amount of noise reduction is quantified by the noise-reduction-ratio, defined to be the ratio of the output to the input noise variances:

$$\mathcal{R} = \frac{\sigma_{y_v}^2}{\sigma_v^2} = \sum_n h^2(n) = \oint_{u.c.} H(z) H(z^{-1}) \frac{dz}{2\pi j z}$$

a. Show that the causal/stable impulse response of this filter is given by:

$$h(n) = \frac{G}{\sin(\omega_0)} R^n \sin(\omega_0 n + \omega_0), \qquad n = 0, 1, 2, \ldots$$

b. Show that the noise-reduction-ratio is given by:

$$\mathcal{R} = \frac{G^2(1 + R^2)}{(1 - R^2)(1 - 2R\cos\omega_0 + R^2)(1 + 2R\cos\omega_0 + R^2)}$$

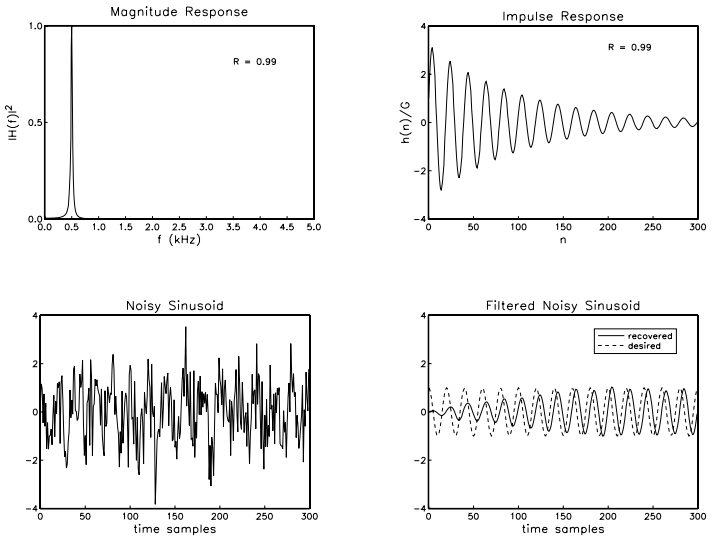
c. For each of the three values $R = 0.95$, $R = 0.97$, $R = 0.99$, plot $h(n)/G$ versus n. Also, plot the magnitude response squared $|H(f)|^2$ over the frequency range $0 \le f \le 5$ kHz. (Recall that the peak width is controlled by R through $\Delta\omega - 2(1-R)$.)

d. Generate $N = 300$ zero-mean, unit-variance, gaussian, white-noise samples $v(n)$, then, compute the input signal samples $x(n) = \cos(\omega_0 n) + v(n)$, and plot them versus n using vertical scales $[-4, 4]$. (Make sure you initialize your random-number generator with a seed in order to get reproducible results.)

Next, filter $x(n)$ through the above filter for each value of R. Plot the resulting output signal $y(n)$ on the same graph as the noise-free desired signal $s(n) = \cos(\omega_0 n)$, again using vertical scales $[-4, 4]$. Comment on the duration of transients versus the narrowness of the resonator peak versus the amount of noise reduction.

e. Filter the noise signal $v(n)$ separately through the filter (for each R), and compute the corresponding filtered output noise $y_v(n)$. On two separate graphs, plot $v(n)$ and $y_v(n)$ versus n. Explain why the filtered noise looks more like a sinusoid than noise.

Using the actual samples $v(n), y_v(n)$, for $n = 0, 1, \ldots, N-1$, calculate their sample variances $\hat{\sigma}_v^2, \hat{\sigma}_{y_v}^2$, and the corresponding estimated noise-reduction ratio $\hat{\mathcal{R}} = \hat{\sigma}_{y_v}^2 / \hat{\sigma}_v^2$, and compare it with the theoretical value $\mathcal{R}$ computed from part (b).



Problem 3.4: Notch and Peak Filters

We saw in Chap. 6 that the sinusoidal response of a second-order filter with poles at p_1, p_2 has the following exact form, for $n \geq 0$:

$$x(n) = \cos(\omega_0 n) \quad \Rightarrow \quad y(n) = |H(\omega_0)| \cos(\omega_0 n + \theta_0) + B_1 p_1^n + B_2 p_2^n$$

where the phase shift θ_0 is value of the phase response of the filter at ω_0, and B_1, B_2 depend on the particulars of the transfer function. In this part, you will study how well the steady-state term represents the output of a short-duration signal and the effect of the transient terms on the time constant of the filter.

Consider the following signal of duration of 12 seconds defined as three concatenated four-second unity-amplitude sinusoidal signals of frequencies $f_1 = 2$, $f_2 = 4$, and $f_2 = 6$ Hz:

$$x(t) = \begin{cases} \cos(2\pi f_1 t), & 0 \leq t < 4 \text{ sec} \\ \cos(2\pi f_2 t), & 4 \leq t < 8 \text{ sec} \\ \cos(2\pi f_3 t), & 8 \leq t < 12 \text{ sec} \end{cases}$$

This signal is sampled at a rate of $f_s = 200$ samples/sec. The following two filters, operating at the rate f_s, are notch filters that have been designed to have a notch at $f_2 = 6$ Hz, therefore, they will knock out the middle portion of $x(t)$:

$$H_1(z) = \frac{0.969531 - 1.923772\, z^{-1} + 0.969531\, z^{-2}}{1 - 1.923772\, z^{-1} + 0.939063\, z^{-2}}$$

$$H_2(z) = \frac{0.996088 - 1.976468\, z^{-1} + 0.996088\, z^{-2}}{1 - 1.936468\, z^{-1} + 0.992177\, z^{-2}}$$

The first filter has a 3-dB width of $\Delta f = 2$ Hz, and the second, a width of $\Delta f = 0.25$ Hz. The magnitude responses $|H(f)|$ are shown below. We will learn in Chap. 11 how to design such filters. They have been designed by the MATLAB function `parmeq` invoked with the parameters:

```
[a,b] = parmeq(1, 0, 1/sqrt(2), 2*pi*f2/fs, 2*pi*Df/fs);
```

where **a**, **b** are the denominator and numerator filter coefficient row vectors. In this experiment, we study the interplay between notch width and transient time constant. The first filter has a wide width and a short time constant, whereas the second filter has a narrow width and a long time constant.

a. Calculate the 40-dB time constants of both of these filters in seconds.

b. Let $x(t_n)$ denote the sampled input $x(t)$. Plot $x(t_n)$ versus t_n over the period of 6 seconds. Filter this input through $H_1(z)$ and plot the output $y(t_n)$ versus t_n.

   Notice how quickly the middle portion of $x(t_n)$ is notched out. Notice also that the f_1 and f_3 portions no longer have unity-amplitudes. Verify that the (steady-state) amplitudes are given respectively by the magnitude response numbers $|H_1(f_1)|$ and $|H_1(f_3)|$.

   Do you observe a phase shift? Is the observed transient response consistent with the calculation of the time constant of part (a)?

c. Repeat part (b) using the second filter $H_2(z)$.

d. On two separate graphs, plot the magnitude responses $|H_1(f)|$ and $|H_2(f)|$ versus f in the range $0 \leq f \leq 10$ Hz. The expected graphs are shown below. The values at f_1 and f_3 have been indicated on these graphs.

e. For each filter, calculate the corresponding left and right 3-dB frequencies, say, f_L and f_R, and indicate them on your graphs of part (f) by connecting them with a horizontal segment at the 3-dB level. Verify that the difference $f_R - f_L$ is equal to the given 3-dB widths of 2 Hz and 0.25 Hz, respectively.

   (This is a hard question. You can determine these frequencies analytically if you read Sect. 11.3 of the text, otherwise, you can determine them by trial and error.)

f. Next, consider the following two peaking filters, which are complementary to the above notch filters. They have a peak at f_2 and the same 3-dB widths of 2 and 0.25 Hz, respectively:

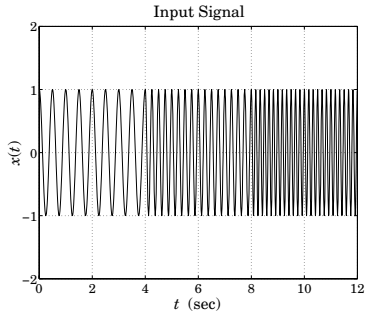
$$H_1(z) = \frac{0.030469(1 - z^{-2})}{1 - 1.923772\, z^{-1} + 0.939063\, z^{-2}}$$

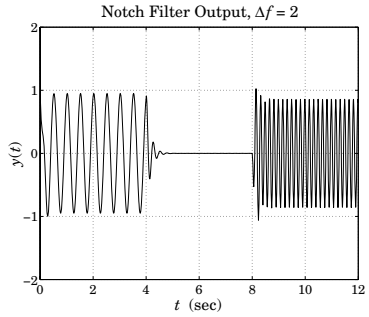
$$H_2(z) = \frac{0.003912(1 - z^{-2})}{1 - 1.976468\, z^{-1} + 0.992177\, z^{-2}}$$

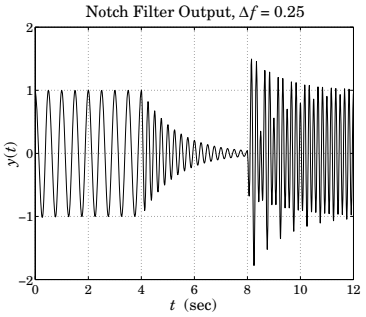
They can also be designed with the `parmeq` function:

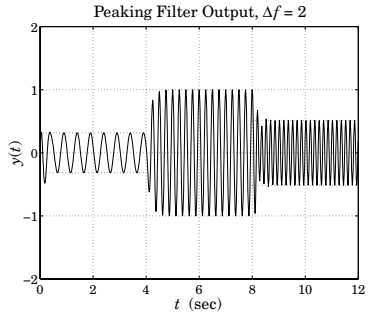
```
[a,b] = parmeq(0, 1, 1/sqrt(2), 2*pi*f2/fs, 2*pi*Df/fs);
```

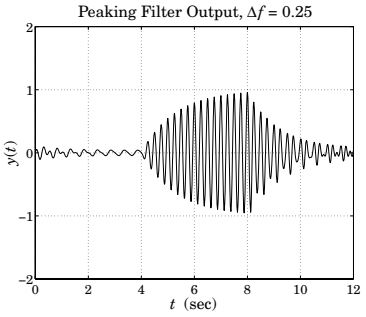
Repeat questions (a-e) for these filters. The peaking filter is supposed to extract the middle portion of the input and remove the f_1 and f_3 portions. Discuss how well each filter accomplishes this goal and correlate what you see in the time-dependence of the output signals with the magnitude responses of the peaking filters.

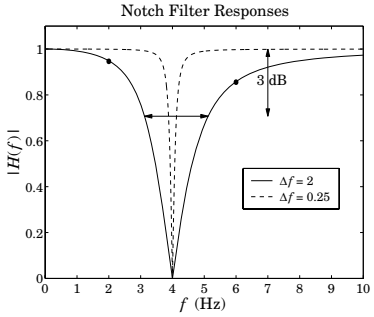


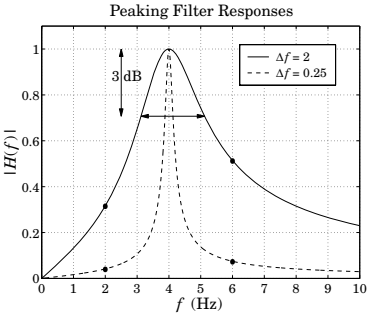

Notch Filter Output, $\Delta f = 2$
$y(t)$
t (sec)


Notch Filter Output, $\Delta f = 0.25$
$y(t)$
t (sec)


Peaking Filter Output, $\Delta f = 2$
$y(t)$
t (sec)


Peaking Filter Output, $\Delta f = 0.25$
$y(t)$
t (sec)


Notch Filter Responses
$|H(f)|$
3 dB
$\Delta f = 2$
$\Delta f = 0.25$
f (Hz)


Peaking Filter Responses
$|H(f)|$
3 dB
$\Delta f = 2$
$\Delta f = 0.25$
f (Hz)