Fall 2009 – 332:347 – Linear Systems Lab – Lab 3

1. Consider the following linear system:

$$\ddot{y}(t)+4\dot{y}(t)+3y(t)=\dot{f}(t)+2f(t)$$

Assuming zero-initial conditions, show that the following three input signals produce the indicated outputs expressed in the s-domain by:

$$f(t)=\delta(t) \quad \Rightarrow \quad Y(s)=\frac{s+2}{(s+1)(s+3)}=\frac{1/2}{s+1}+\frac{1/2}{s+3}$$

$$f(t)=u(t) \quad \Rightarrow \quad Y(s)=\frac{s+2}{s(s+1)(s+3)}=\frac{2/3}{s}-\frac{1/2}{s+1}-\frac{1/6}{s+3}$$

$$f(t)=e^{-4t}u(t) \quad \Rightarrow \quad Y(s)=\frac{s+2}{(s+4)(s+1)(s+3)}=-\frac{2/3}{s+4}+\frac{1/6}{s+1}+\frac{1/2}{s+3}$$

For each case, verify the indicated partial fraction expansion coefficients using the function:

```
[r,p,k] = residue(num,den);
```

Then, determine the analytical expressions for the time-domain output signals $y(t)$ and plot them versus t in the range $0 \le t \le 10$.

2. A signal $f(t)$ consists of a sinusoid plus random noise:

$$f(t)=\sin(\omega_0 t)+v(t) \tag{1}$$

It is desired to process $f(t)$ through a bandpass filter $H(s)$ that lets the sinusoid pass through unchanged, while it substantially attenuates the noise component, so that the output signal would have the form:

$$y(t)=\sin(\omega_0 t)+y_v(t) \tag{2}$$

where $y_v(t)$ denotes the filtered noise, which must be much weaker than the input noise, i.e., the RMS value of $y_v(t)$ must be much less than the RMS value of $v(t)$, or in terms of their variances, $\sigma_{y_v}^2 \ll \sigma_v^2$. Such a bandpass filter can be designed to have transfer function and I/O differential equation:

$$H(s)=\frac{\alpha s}{s^2+\alpha s+\omega_0^2} \quad \Leftrightarrow \quad \ddot{y}(t)+\alpha\dot{y}(t)+\omega_0^2 y(t)=\alpha f(t) \tag{3}$$

This is complementary to the notch filter discussed in lab-2. Its magnitude frequency response, obtained by setting $s=j\omega$, is given by:

$$|H(\omega)|^2=\frac{\alpha^2\omega^2}{\left(\omega^2-\omega_0^2\right)^2+\alpha^2\omega^2} \tag{4}$$

It has a narrow peak centered at ω_0 and unity gain there, i.e., $H(\omega_0)=1$. Its 3-dB width is $\Delta\omega=\alpha$ (see graphs at end). Its impulse response is given by:

$$h(t)=\alpha e^{-\alpha t/2}\left[\cos(\omega_r t)-\frac{\alpha}{2\omega_r}\sin(\omega_r t)\right]u(t)\,, \quad \omega_r=\sqrt{\omega_0^2-\frac{\alpha^2}{4}} \tag{5}$$

Assuming the noise component to be white noise with broadband flat spectrum, the narrow peak of the filter will only let through a small part of the noise (whatever lies within the effective width of the peak), so that the output noise power will be proportional to the bandwidth parameter α — it can be shown that $\sigma_{y_v}^2/\sigma_v^2 \approx T\alpha/2$. Thus, the smaller the α, the more noise reduction. On the other hand, as can be seen from Eq. (5), the smaller the α, the longer the effective time constant $t_{\text{eff}}=2/\alpha$ of the filter, resulting in longer transients. In this lab, you will study this tradeoff between noise reduction and speed of response.

(a) Start with the values $\omega_0=5$ and $\alpha=1$. Plot the magnitude response squared $|H(\omega)|^2$ versus ω in the interval $0 \le \omega \le 10$. Then, plot the phase response $\text{Arg}\big[H(\omega)\big]$ versus the same values of ω.

(b) Generate 2001 equally-spaced noisy sinusoidal samples of $f(t)$ in the interval $0 \le t \le 40$, e.g., using the code:

```
w0 = 5; a = 1;
t = linspace(0,40,2001);
seed = 10000; randn('state',seed);   % initialize random number generator
v = randn(size(t));                  % generate white gaussian noise samples
f = sin(w0*t) + v;                   % noisy sinusoid
```

Compute the filter output samples $y(t)$ using the function `lsim`,

```
y = lsim(tf(num,den),f,t, [0;0], 'zoh');
```

where this syntax, as opposed to `y = lsim(num,den,f,t)`, forces the use of the zero-order-hold method of integration.

On two separate graphs, plot $f(t)$ and $y(t)$ versus t. Observe the initial transients and the steady-state output (it's not quite equal to the sinusoid because a small amount of noise survives the filtering process.)

(c) To observe what happens to the noise itself, filter the noise signal $v(t)$ through this filter to obtain the filtered noise $y_v(t)$. On two separate graphs, but using the same vertical scales, plot the signals $v(t)$ and $y_v(t)$ versus t.

(d) Repeat parts (a–c) for the values $\alpha=1/2$ and $\alpha=1/10$, discussing the tradeoffs between noise reduction, speed of response, and quality of resulting desired signal.

(e) The zero-order-hold method implemented by the function `lsim` is equivalent to replacing the continuous-time transfer function $H(s)$

of Eq. (3) by the following discrete-time transfer function and corresponding input/output equation:

$$
\begin{aligned}
H(z) &= \frac{Gz^{-1}(1-z^{-1})}{1+a_1z^{-1}+a_2z^{-2}} \\
y_n + a_1y_{n-1} + a_2y_{n-2} &= G(f_{n-1} - f_{n-2})
\end{aligned}
\tag{6}
$$

with coefficients:

$$
\begin{aligned}
G &= \frac{\alpha}{\omega_r}\, e^{-\alpha T/2}\sin(\omega_r T) \\
a_1 &= -2e^{-\alpha T/2}\cos(\omega_r T) \\
a_2 &= e^{-\alpha T}
\end{aligned}
\tag{7}
$$

We will derive this result in class later. For the values $\omega_0 = 5$, $\alpha = 1/2$, $T = t(2)-t(1)$, compute the output samples $y_n = y(t_n)$ for the input samples $f_n = f(t_n)$ by writing a repetitive loop that solves the difference equation (6) (as was done in Lab-2), for example:

initialize $w_1 = w_2 = 0$, then,
for each $n = 0, 1, 2, \dots$ do:
  $w_0 = f_n - a_1w_1 - a_2w_2$
  $y_n = G(w_1 - w_2)$
  $w_2 = w_1$
  $w_1 = w_0$

Plot $y(t_n)$ and compare it with that obtained using `lsim` in part (b). You may check the output of your loop by comparing it with the output of the function `filter`.

Typical Outputs