

332:347 – Linear Systems Lab – Lab 1

1. This lab illustrates the definition of the Dirac delta function,  $\delta(t)$ , as a limit of ordinary functions. Consider the following four limiting forms:

$$\delta(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} P_{\epsilon}(t), \quad \delta(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \Delta_{2\epsilon}(t)$$

$$\delta(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-t^2/2\epsilon}$$

$$\delta(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + t^2}$$

$$\delta(t) = \lim_{\epsilon \rightarrow 0} \frac{\sin(t/\epsilon)}{\pi t}$$

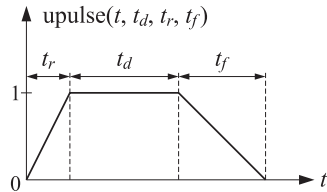
where  $P_{\epsilon}(t)$  and  $\Delta_{2\epsilon}(t)$  are the unit rectangular and unit triangular pulses as defined in the text (Eqs. (2.9) and (2.13)).

- (a) For each form, select two successively smaller values of  $\epsilon$  and evaluate the above functions over 1000 equally-spaced time instants in the range  $-1 \leq t \leq 1$ , for example,

$$t = \text{linspace}(-1, 1, 1001);$$

Plot the corresponding functions versus time and note how they resemble the ideal  $\delta(t)$  as  $\epsilon$  gets smaller.

*Notes:* Vectorize your calculations. Use MATLAB's built-in function `sinc` to evaluate the fourth case (it handles the case  $t = 0$ ). Also, you may use the supplied function `upulse`, whose input parameters are shown below, to generate the rectangular and triangular pulses.



One way to verify the correctness of the above limiting forms is to compute the Fourier transforms of the ordinary functions on the right-hand sides and show that each tends to unity as a function of frequency  $\omega$  in the limit  $\epsilon \rightarrow 0$ . This is so because the Fourier transform of  $\delta(t)$  is  $\Delta(\omega) = 1$ . We will discuss this approach later, but for completeness, we give the corresponding Fourier transforms below:

$$\frac{1}{\epsilon} P_{\epsilon}(t) \xrightarrow{FT} \frac{\sin(\omega\epsilon/2)}{\omega\epsilon/2}, \quad \frac{1}{\sqrt{2\pi\epsilon}} e^{-t^2/2\epsilon} \xrightarrow{FT} e^{-\epsilon\omega^2/2}$$

$$\frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + t^2} \xrightarrow{FT} e^{-\epsilon|\omega|}, \quad \frac{\sin(t/\epsilon)}{\pi t} \xrightarrow{FT} P_{2/\epsilon}^h(\omega) = \begin{cases} 1, & |\omega| < 1/\epsilon \\ 0.5, & |\omega| = 1/\epsilon \\ 0, & |\omega| > 1/\epsilon \end{cases}$$

2. Consider the following differential equation describing a first-order system, such as an RC circuit:

$$\dot{y}(t) + ay(t) = f(t) \quad (1)$$

We saw in class that its solution subject to zero initial conditions,  $y(0) = 0$ , is given in terms of the input  $f(t)$  by,

$$y(t) = \int_0^t e^{-a(t-\tau)} f(\tau) d\tau, \quad t \geq 0 \quad (2)$$

We also saw that Eq. (2) can be integrated numerically by the following difference equation,

$$y_n = a_1 y_{n-1} + b_0 f_n + b_1 f_{n-1} \quad (3)$$

where  $f_n = f(nT)$  and  $y_n$  is the numerical approximation to  $y(nT)$ , with coefficients given by,

$$a_1 = \frac{1 - paT}{1 + qaT}, \quad b_0 = \frac{qT}{1 + qaT}, \quad b_1 = \frac{pT}{1 + qaT} \quad (4)$$

where  $p + q = 1$  corresponding to the cases (i)  $p = 1, q = 0$ , forward Euler integration rule, (ii)  $p = 0, q = 1$ , backward Euler integration rule, and (iii)  $p = q = 1/2$ , trapezoidal integration rule. A simple way to implement the iteration (3) is by the loop:

initialize at  $w = v = 0$ , then,  
for each  $n = 0, 1, 2, \dots$ , do:  
 $y_n = a_1 w + b_0 f_n + b_1 v$   
 $w = y_n$   
 $v = f_n$

$$(5)$$

Another integration rule that is widely used in linear systems is: (iv) the zero-order hold, which for Eq. (1) leads to the difference equation:

$$y_n = a_1 y_{n-1} + b_1 f_{n-1}, \quad a_1 = e^{-aT}, \quad b_1 = \frac{1 - e^{-aT}}{a} \quad (6)$$

It can be iterated as a special case of Eq. (5) with  $b_0 = 0$ .

- (a) Consider the special case  $a = 2$  and  $f(t) = e^{-t}u(t)$ . By performing the integral in Eq. (2) show that the exact solution is:

$$y(t) = e^{-t} - e^{-2t}, \quad t \geq 0 \quad (7)$$

- (b) Write and execute a MATLAB script that iterates the difference equation (3) for  $n = 0, 1, 2, \dots$ , with zero initial conditions, and determines the solution over the interval  $0 \leq t \leq 5$ . Use the value  $T = 0.1$  for the sampling time.

On the same graph, plot the exact  $y(t)$  versus  $t$ , as well as  $y_n$  versus  $t_n = nT$ , for all four integration rules (i-iv) and assess which rule appears to be better.

To unclutter the graphs you might want to plot only two discretization rules plus the exact output per graph.

Note that you may define the sampled  $t$  points over  $[0, 5]$ , and the corresponding input samples  $f(t)$ , by

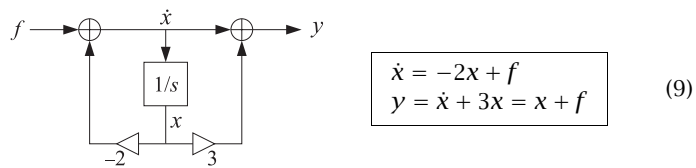
$$t = 0:T:5; \quad f = \exp(-t);$$

- (c) Repeat with  $T = 0.05$  and  $T = 0.01$  and note the improvement of the numerical approximation as  $T$  decreases.

3. Consider a different linear system that now has derivatives of the input signal in its right-hand side:

$$\dot{y} + 2y = \dot{f} + 3f \quad (8)$$

To solve this system, we convert it first into its (so-called controllable) state-space form, described by the following system of equations, and shown in block-diagram form below:



where  $x$  is the state and the block  $1/s$  represents an integrator. This could be obtained with the help of the MATLAB function `tf2ss` (transfer function to state-space), which is not really needed here:

```
b = [1,3]; a = [1,2]; % transfer function, H(s) = (s+3)/(s+2)
[A,B,C,D] = tf2ss(b,a); % x-dot = Ax + Bf, y = Cx + Df
```

- (a) Verify that the system of equations (9) implies Eq. (8).  
 (b) Consider the same input as in the previous problem,  $f(t) = e^{-t}u(t)$ . By applying Eq. (2) to the signal  $x(t)$ , show that the exact solution for the state and for  $y(t)$  are:

$$\begin{aligned} x(t) &= e^{-t} - e^{-2t}, \quad t \geq 0 \\ y(t) &= 2e^{-t} - e^{-2t}, \quad t \geq 0 \end{aligned} \quad (10)$$

- (c) Using similar discretization schemes as in the previous problem, integrate the state equation  $\dot{x} = -2x + f$  numerically over the time interval  $t \in [0, 5]$ . Use the values  $T = 0.1, 0.05, 0.01$ . *Hint:* This part is already done.

On the same graph, for each value of  $T$ , plot the exact  $y(t)$  together with the results of the four integration rules (i-iv). Make such graphs for each of the three values of  $T$ . Comment on the numerical integration accuracy.

## Typical Outputs

