332:345 – Linear Systems & Signals – Fall 2009
Sample Final Exam Questions

1. Please review all sample questions for exams 1 & 2.

2. Consider a second-order analog audio parametric equalizer filter,

$$H(s) = \frac{G\alpha s + G_0(s^2 + \omega_0^2)}{s^2 + \alpha s + \omega_0^2}, \quad \alpha = \sqrt{\frac{G_B^2 - G_0^2}{G^2 - G_B^2}}\,\Delta\omega \tag{1}$$

where G, G_0, G_B are the peak, reference, and bandwidth gains, and $\Delta\omega$ is the bandwidth measured at level G_B.

(a) Often the bandwidth gain is defined to be the geometric mean of the peak and reference gains, that is, $G_B = (GG_0)^{1/2}$. Show that this choice corresponds to the arithmetic mean of the gains in dB units. (The dB gains are defined by $G_{\text{dB}} = 20\log_{10}(G)$, etc.)

(b) Set $G_0 = 1$. For the above geometric-mean choice for G_B, show that an equalizer of boost gain G_{dB} at center frequency ω_0 and width $\Delta\omega$, is the exact inverse of an equalizer with a cut gain by an equal an opposite amount $-G_{\text{dB}}$ at the same center frequency and width, i.e., show that the corresponding boost and cut transfer functions will be related by,

$$H_{\text{boost}}(s)H_{\text{cut}}(s) = 1$$

(c) Set $G_0 = 1$. For the above geometric-mean choice for G_B, show that the equalizer corresponding to peak gain G and center frequency ω_0 and width $\Delta\omega$ is given by:

$$H(s) = \frac{s^2 + \Delta\omega\, G^{1/2} s + \omega_0^2}{s^2 + \Delta\omega\, G^{-1/2} s + \omega_0^2}$$

You may use this result to prove part(b).

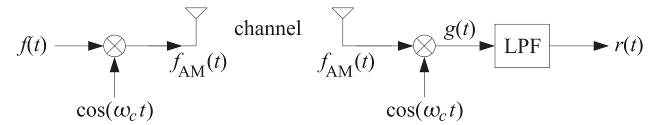
(d) Consider again the more general version of the equalizer given by Eq. (1). In audio work the bandwidth is usually expressed in octaves, that is, a bandwidth of say B octaves means that the ratio of the upper and lower bandedge frequencies is given by

$$\frac{\omega_2}{\omega_1} = 2^B \quad \Rightarrow \quad B = \log_2\left(\frac{\omega_2}{\omega_1}\right) = \text{octave bandwidth}$$

Using the fact that $\omega_2\omega_1 = \omega_0^2$, show that the actual bandwidth $\Delta\omega$ in units of rads/sec is related to the octave bandwidth B by

$$\Delta\omega = 2\omega_0 \sinh\left(\frac{1}{2}B\ln(2)\right)$$

3. The transmitter and receiver of an AM communication system are depicted below. The baseband signal to be transmitted is $f(t)$ and has Fourier transform $F(\omega)$ assumed to be bandlimited in the interval $|\omega| \le \omega_B$.



The modulated transmitted signal and its demodulated version at the receiver will be:

$$f_{\text{AM}}(t) = \cos(\omega_c t) f(t)$$
$$g(t) = \cos(\omega_c t) f_{\text{AM}}(t)$$

(a) Express the spectra of $f_{\text{AM}}(t)$ and $g(t)$ in terms of the original spectrum $F(\omega)$ and make a rough sketch of them versus ω, assuming that $\omega_c \gg \omega_B$.

(b) Discuss the need for the indicated lowpass filter (LPF) and explain how to pick its specifications.

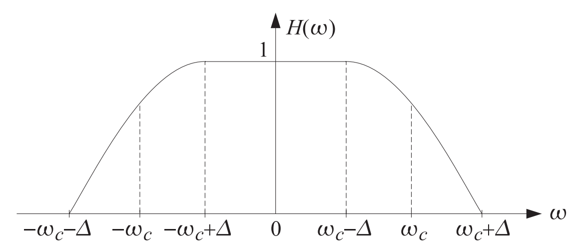
(c) Explain why the above system would fail if the receive-carrier $\cos(\omega_c t)$ were to be replaced by $\sin(\omega_c t)$, i.e., $g(t) = \sin(\omega_c t) f_{\text{AM}}(t)$, having a 90° phase difference compared to the transmit-carrier. (In practice, any possible phase offset in the carrier signals is compensated using a phase-locked loop.)

[*Hint:* trig identity $2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta)$.]

4. The square–root raised-cosine filter is probably the most widely used pulse shaping filter in digital communication systems. Its frequency response is bandlimited and is defined by,

$$P(\omega) = \begin{cases} 1, & |\omega| \le \omega_c - \Delta \\ \cos\left(\dfrac{\pi}{4\Delta}(|\omega| - \omega_c + \Delta)\right), & \omega_c - \Delta \le |\omega| \le \omega_c + \Delta \\ 0, & |\omega| > \omega_c + \Delta \end{cases}$$

where $\Delta < \omega_c$. It has a flat response over the interval $|\omega| \le \omega_c - \Delta$, and beyond that, it tapers to zero following a cosine curve. It is depicted below.



(a) Verify that $P(\omega)$ is the square root of the raised-cosine filter defined in the sample problems for exam-2.

(b) By direct calculation of the inverse Fourier transform, show that the impulse response of this filter is given by,

$$p(t)=\frac{\sin\big((\omega_c-\Delta)t\big)+\dfrac{4t\Delta}{\pi}\cos\big((\omega_c+\Delta)t\big)}{\pi t\cdot\left[1-\left(\dfrac{4t\Delta}{\pi}\right)^2\right]} \tag{2}$$

(c) Show that its value at $t=0$ is,

$$p(0)=\frac{\pi\omega_c+(4-\pi)\Delta}{\pi^2}$$

and that its value at $t=\pm\pi/(4\Delta)$ is,

$$\frac{\sqrt{2}\Delta}{2\pi^2}\left[(\pi-2)\cos\left(\frac{\pi\omega_c}{4\Delta}\right)+(\pi+2)\sin\left(\frac{\pi\omega_c}{4\Delta}\right)\right]$$

(d) To do the required Fourier integral for obtaining Eq. (2), first note that because $P(\omega)$ is even in ω, the integral simplifies into,

$$p(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}P(\omega)e^{j\omega t}d\omega=\frac{1}{\pi}\int_0^{\infty}P(\omega)\cos(\omega t)\,d\omega$$

Then, split the integral as follows,

$$p(t)=\frac{1}{\pi}\int_0^{\omega_c-\Delta}P(\omega)\cos(\omega t)\,d\omega+\frac{1}{\pi}\int_{\omega_c-\Delta}^{\omega_c+\Delta}P(\omega)\cos(\omega t)\,d\omega$$

Show that the first term is:

$$\frac{\sin\big((\omega_c-\Delta)t\big)}{\pi t}$$

and the second,

$$\frac{4\Delta}{\pi^2}\cdot\frac{\cos\big((\omega_c+\Delta)t\big)+\dfrac{4t\Delta}{\pi}\sin\big((\omega_c-\Delta)t\big)}{1-\left(\dfrac{4t\Delta}{\pi}\right)^2}$$

Then, obtain Eq. (2) by adding up these two terms.

Hint: You may use the indefinite integral:

$$\int\cos(ax)\cos(bx)\,dx=\frac{\sin\big((a+b)x\big)}{2(a+b)}+\frac{\sin\big((a-b)x\big)}{2(a-b)}$$

5. Show that the inverse of an invertible 2×2 matrix is given explicitly as follows:

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\quad\Rightarrow\quad A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

Then, show that its characteristic polynomial is given by

$$p(s)=\det(sI-A)=s^2-\mathrm{tr}(A)s+\det(A)$$

where I is the unit 2×2 matrix, and $\mathrm{tr}(A),\det(A)$ denote the trace and determinant of the matrix A. This is a general result for any 2×2 matrix.

Finally, verify the Cayley-Hamilton theorem by explicit matrix multiplication, that is, show that

$$p(A)=A^2-\mathrm{tr}(A)A+\det(A)I=0$$

6. Consider the second-order linear system:

$$\ddot{y}(t)+a_1\dot{y}(t)+a_2y(t)=b_0\ddot{f}(t)+b_1\dot{f}(t)+b_2f(t)$$

where the input $f(t)$ is assumed to be a causal function that does not have any impulsive terms (e.g., $\delta(t),\dot{\delta}(t)$, etc.).

(a) Show the following relationships between the output values at times $t=0-$ and $t=0+$:

$$\begin{aligned}y(0+)-y(0-)&=b_0f(0+)\\ \dot{y}(0+)-\dot{y}(0-)&=(b_1-b_0a_1)f(0+)+b_0\dot{f}(0+)\end{aligned} \tag{3}$$

where because of the causality of $f(t)$, we used the result that $f(0-)=\dot{f}(0-)=0$. Eqs. (3) allow one to map the initial conditions at $0-$ to those at $0+$. *Hint:* Integrate the differential equation once and then twice over the interval $[0-,t]$ and then take the limit $t\to0+$:

$$\int_{0-}^{t}\big[\ddot{y}(\tau)+a_1\dot{y}(\tau)+a_2y(\tau)\big]d\tau=\int_{0-}^{t}\big[b_0\ddot{f}(\tau)+b_1\dot{f}(\tau)+b_2f(\tau)\big]d\tau$$

$$\int_{0-}^{t}dt'\int_{0-}^{t'}\big[\ddot{y}(\tau)+a_1\dot{y}(\tau)+a_2y(\tau)\big]d\tau=\int_{0-}^{t}dt'\int_{0-}^{t'}\big[b_0\ddot{f}(\tau)+b_1\dot{f}(\tau)+b_2f(\tau)\big]d\tau$$

(b) Consider the controller canonical state-space realization (actually, any other realization) of the above second-order system, and differentiate the output equation once to get the system:

$$\begin{aligned}\dot{\mathbf{x}}&=A\mathbf{x}+Bf\\ y&=C\mathbf{x}+Df\\ \dot{y}&=C\dot{\mathbf{x}}+D\dot{f}=CA\mathbf{x}+CBf+D\dot{f}\end{aligned}$$

From the last two equations, argue that:

$$y(0+)-y(0-)=Df(0+)$$

$$\dot{y}(0+)-\dot{y}(0-)=CBf(0+)+D\dot{f}(0+)$$

and show that these are equivalent to Eqs. (3).

7. Consider an $[A,B,C,D]$ state-space representation of a single-input single-output (SISO) order-p system:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bf$$

$$y = C\mathbf{x} + Df$$

where the state vector $\mathbf{x}$ is a p-dimensional column vector. Given an arbitrary $p \times p$ invertible matrix V, define the transformed state vector $\mathbf{x}' = V\mathbf{x}$, and the transformed matrices:

$$A' = VAV^{-1}, \quad B' = VB, \quad C' = CV^{-1}, \quad D' = D$$

Show that these define a new state-space representation that satisfies:

$$\dot{\mathbf{x}}' = A'\mathbf{x}' + B'f$$

$$y = C'\mathbf{x}' + D'f$$

Moreover, show that $C'B' = CB$. How is this result relevant to the previous problem?

8. Consider the linear system described by the following second-order differential equation:

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\ddot{f}(t)+f(t) \qquad (4)$$

(a) Find its transfer function $H(s)$ and draw the controller canonical realization form. *Answer:* $H(s)=\dfrac{s^2+1}{s^2+3s+2}$.

(b) Apply long-division to put $H(s)$ in the form,

$$H(s)=\frac{s^2+1}{s^2+3s+2}=b_0+\frac{c_1 s+c_2}{s^2+3s+2}$$

and determine the numerical values of the coefficients b_0, c_1, c_2. Then draw the corresponding controller canonical realization form.

(c) Applying the four transposition rules, draw the transposed realization of that of part (a), that is, the observer canonical form.

(d) For the controller realization of part (a), define appropriate internal states and derive the corresponding $[A,B,C,D]$ state-space realization.

Answers: $A=\begin{bmatrix}-3 & -2\\ 1 & 0\end{bmatrix}$, $B=\begin{bmatrix}1\\0\end{bmatrix}$ $C=[-3,-1]$, $D=1$.

(e) Calculate the Laplace transform of the state transition matrix, i.e.,

$$\Phi(s)=(sI-A)^{-1}$$

where I is the 2×2 unit matrix. *Answer:* $\dfrac{1}{s^2+3s+2}\begin{bmatrix}s & -2\\ 1 & s+3\end{bmatrix}$.

(f) By explicit matrix multiplication, calculate the system transfer function and show that it is correctly given by parts (a,b), that is, show that

$$H(s)=C(sI-A)^{-1}B+D$$

(g) Perform a partial-fraction expansion on each entry of the matrix $\Phi(s)$ and then perform an inverse Laplace transform, thereby, obtaining the state transition matrix:

$$\phi(t)=e^{At}, \quad t\ge 0$$

Answer: $\phi(t)=\begin{bmatrix}2e^{-2t}-e^{-t} & 2e^{-2t}-2e^{-t}\\ e^{-t}-e^{-2t} & 2e^{-t}-e^{-2t}\end{bmatrix}$.

(h) By explicit matrix multiplication, determine the impulse response of this system, that is,

$$h(t)=Ce^{At}B+D\delta(t)=C\phi(t)B+D\delta(t)$$

Answer: $h(t)=(2e^{-t}-5e^{-2t})u(t)+\delta(t)$.

(i) Alternatively, obtain $h(t)$ by performing the inverse Laplace transform of $H(s)$ using partial-fraction expansions of part (b).

(j) Next, assume that the input to Eq. (4) is $f(t)=e^{-3t}u(t)$. Using standard Laplace transform methods and partial-fraction expansions, solve Eq. (4) subject to arbitrary initial conditions, that is, arbitrary values for $y(0-)$ and $\dot{y}(0-)$. You should find the answer to be:

$$y(t)=(2y_0+\dot{y}_0+1)e^{-t}-(y_0+\dot{y}_0+5)e^{-2t}+5e^{-3t} \qquad (5)$$

for $t\ge 0$, where we used the shorthand notation $y_0=y(0-)$ and $\dot{y}_0=\dot{y}(0-)$.

(k) Read sections 7.1, 7.4, 7.5, and 7.11 of your text on the issues that arise when the differential equation of the system contains time-derivatives of the input (which cause delta-function impulses). Using the techniques illustrated by Examples 7.5 and 7.21, solve part (j) by writing $y(t)$ as the sum of a homogeneous solution and a particular solution of the form:

$$y(t) = y_h(t)+y_p(t)$$

$$y_h(t) = Ae^{-t}+Be^{-2t}$$

$$y_p(t) = \int_{0-}^{t} h(t-\tau)f(\tau)\,d\tau$$

Fix A, B by imposing the initial conditions $y_0, \dot{y}_0$. Verify that the resulting $y(t)$ is the same as that of Eq. (5). *Answers:*

$$y_h(t) = (2y_0 + \dot{y}_0)e^{-t} - (y_0 + \dot{y}_0)e^{-2t}$$
$$y_p(t) = e^{-t} - 5e^{-2t} + 5e^{-3t}$$

(l) Consider the conventional approach of solving a differential equation as the sum of a homogeneous solution and a particular solution. For an input of the form $f(t) = e^{s_1 t}u(t)$, (e.g., $s_1 = -3$ in this example,) the particular solution is found to be $H(s_1)e^{s_1 t}$, where $H(s_1)$ is the value of $H(s)$ at $s = s_1$. Applying this to this example, we write the solution on the form:

$$y(t) = Ae^{-t} + Be^{-2t} + H(-3)e^{-3t}, \quad t \geq 0$$

By setting $t = 0+$, we obtain a relation between A, B. Another relation is obtained by differentiating with respect to t and then setting $t = 0+$. By solving these two equations for A, B show that $y(t)$ has the following form for $t \geq 0$:

$$y(t) = \big(2y(0+) + \dot{y}(0+) + 5\big)e^{-t} - \big(y(0+) + \dot{y}(0+) + 10\big)e^{-2t} + 5e^{-3t}$$

By applying the results of Eqs. (3), show that this solution is equivalent to that of Eq. (5).

(m) Determine the observability matrix (and its inverse) for the second-order system of Eq. (4), that is,

$$F = \begin{bmatrix} C \\ CA \end{bmatrix}$$

Answer: $F = \begin{bmatrix} -3 & -1 \\ 8 & 6 \end{bmatrix}, \quad F^{-1} = \frac{1}{10}\begin{bmatrix} -6 & -1 \\ 8 & 3 \end{bmatrix}.$

(n) For the state-space form of part (d), determine the initial state-vector $\mathbf{x}_0 = \mathbf{x}(0-) = \mathbf{x}(0+) = \mathbf{x}(0)$ by mapping the given initial conditions with the help of the observability matrix F, as discussed in class:

$$\begin{bmatrix} y_0 \\ \dot{y}_0 \end{bmatrix} = F\mathbf{x}_0 \quad \Rightarrow \quad \mathbf{x}_0 = F^{-1}\begin{bmatrix} y_0 \\ \dot{y}_0 \end{bmatrix}$$

Answer: $\mathbf{x}_0 = \frac{1}{10}\begin{bmatrix} -6y_0 - \dot{y}_0 \\ 8y_0 + 3\dot{y}_0 \end{bmatrix}.$

(o) Using the state-transition matrix of part (g), calculate the homogeneous and forced parts of the state vector, that is,

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_f(t) = e^{At}\mathbf{x}_0 + \int_0^t e^{A(t-\tau)}Bf(\tau)\,d\tau$$

Answers:

$$\mathbf{x}_h(t) = e^{At}\mathbf{x}_0 = \frac{1}{10}\begin{bmatrix} 4(y_0 + \dot{y}_0)e^{-t} - 5(2y_0 + \dot{y}_0)e^{-2t} \\ 5(2y_0 + \dot{y}_0)e^{-t} - 2(y_0 + \dot{y}_0)e^{-2t} \end{bmatrix}$$
$$\mathbf{x}_f(t) = \int_0^t e^{A(t-\tau)}Bf(\tau)\,d\tau = \frac{1}{2}\begin{bmatrix} 4e^{-2t} - e^{-t} - 3e^{-3t} \\ e^{-t} - 2e^{-2t} + e^{-3t} \end{bmatrix}$$

(p) Calculate $y(t)$ using the state output equation:

$$y(t) = C\mathbf{x}(t) + Df(t)$$

and verify that it is again given by Eq. (5).

(q) For the observer canonical form obtained in part (c), derive the corresponding $[A, B, C, D]$ state-space realization and write down the corresponding differential equations. Indicate on the block diagram exactly what signals are chosen to be the components of the state vector.

(r) Consider the following state-space realization obtained by applying long-division to put $H(s)$ in the form,

$$H(s) = \frac{s^2 + 1}{s^2 + 3s + 2} = b_0 + \frac{A_1}{s+1} + \frac{A_2}{s+2} \tag{6}$$

and determine the values of the coefficients b_0, A_1, A_2. By choosing appropriate internal states, derive the corresponding state-space parameters $[A, B, C, D]$ that represent Eq. (6) and write down the corresponding state equations. The state matrix A and transition matrix e^{At} are diagonal here. *Answers:*

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = [2, -5], \quad D = 1$$

(s) Carry out questions (m–o) for the diagonal realization of part (q). Verify that the resulting output is correctly given by Eq. (5). *Answers:*

$$F = \begin{bmatrix} 2 & -5 \\ -2 & 10 \end{bmatrix}, \quad F^{-1} = \begin{bmatrix} 1 & 0.5 \\ 0.2 & 0.2 \end{bmatrix}, \quad \mathbf{x}_0 = \frac{1}{10}\begin{bmatrix} 5(2y_0 + \dot{y}_0) \\ 2(y_0 + \dot{y}_0) \end{bmatrix}$$
$$\mathbf{x}_f(t) = \frac{1}{2}\begin{bmatrix} e^{-t} - e^{-3t} \\ 2e^{-2t} - 2e^{-3t} \end{bmatrix}, \quad \mathbf{x}_h(t) = \frac{1}{10}\begin{bmatrix} 5(2y_0 + \dot{y}_0)e^{-t} \\ 2(y_0 + \dot{y}_0)e^{-2t} \end{bmatrix}$$

(t) Write the decomposition of $H(s)$ given in part (b) as follows:

$$H(s) = b_0 + \frac{c_1 s + c_2}{s^2 + 3s + 2} = b_0 + \left(\frac{1}{s+1}\right)\cdot\left(\frac{c_1 s + c_2}{s+2}\right)$$

Draw block diagram realizations for the two factors,

$$\left(\frac{1}{s+1}\right), \quad \left(\frac{c_1 s + c_2}{s+2}\right)$$

and connect them in cascade (i.e., in series) and in parallel with the term b_0 to obtain a new block diagram realization of $H(s)$. Then, introduce appropriate internal states and derive the corresponding $[A, B, C, D]$ state-space representation. *Answers:* if the two factors are realized in their controller canonical forms,

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [-3, 5], \quad D = 1$$

(u) Determine the matrices $(sI - A)^{-1}$ and e^{At} of the previous part. *Answers:*

$$\frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & 0 \\ 1 & s+1 \end{bmatrix}, \qquad \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-2t} & e^{-2t} \end{bmatrix}$$

9. Repeat all questions (a–u) of the previous problem for the following system described by the transfer function:

$$H(s) = \frac{2s+1}{s^2 + 3s + 2}$$

You will need to determine first the difference equation between input $f(t)$ and output $y(t)$. One simplification here is that $D = 0$ and the impulse response has no impulsive terms.