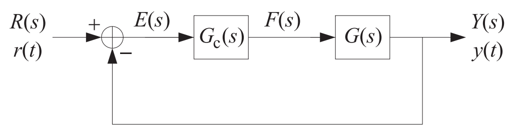
332:345 – Linear Systems & Signals – Fall 2009
Sample Exam-2 Questions

1. For the PID controller example shown below, choose the following system and PID transfer functions:

$$G(s) = \frac{1}{s+a}$$

$$G_c(s) = k_p + \frac{k_i}{s} + k_d s$$



   (a) Derive expressions for the following closed-loop transfer functions in terms of the parameters a, k_p, k_i, k_d:

$$H(s) = \frac{Y(s)}{R(s)}, \quad H_{\text{err}}(s) = \frac{E(s)}{R(s)}$$

   (b) Set $a = 2$ and $k_d = 1$. What should be the numerical values of k_p, k_i in order for the poles of the closed-loop system $H(s)$ to be positioned at the s-plane locations $s = -4$ and $s = -5$? [*Ans.* $k_p = 16$, $k_i = 40$.] What are the transfer functions $H(s)$, $H_{\text{err}}(s)$ in this case?

   (c) For arbitrary values of a and k_d, and arbitrary settings for the closed-loop pole locations $s = -a_1$ and $s = -a_2$, where a_1, a_2 are both positive numbers, how would one calculate the desired settings k_p, k_i of the controller? Repeat if the desired closed-loop poles are required to be at the conjugate locations $s = -a \pm jb$, with a, b positive.

   (d) For the numerical values derived in part (b), use partial fraction expansions to determine the unit-step output, that is, $y(t)$ when $r(t) = u(t)$. [*Ans.* $y(t) = u(t) + e^{-4t}u(t) - 1.5e^{-5t}u(t)$.]

2. Determine the causal impulse response $h(t)$ of the following system (where a, b are positive and $a \neq b$):

$$H(s) = \frac{1}{(s+a)(s+b)}$$

   Determine $h(t)$ also in the case when $a = b$.

3. Show that the zero-order-hold discretized version $H(z)$ of the transfer function $H(s)$ of the previous problem (with $a \neq b$) has the following form:

$$H(z) = \frac{A(1-z^{-1})}{1-e^{-aT}z^{-1}} + \frac{B(1-z^{-1})}{1-e^{-bT}z^{-1}} + C$$

   where T is the sampling time interval. Determine the coefficients A, B, C in terms of a, b.

4. The signal $f(t) = e^{j\omega_0 t}u(t)$ is sent to the input of the system in problem 2 (with $a \neq b$). Show that the output will have the following form for all t,

$$y(t) = H_0 e^{j\omega_0 t}u(t) + Ae^{-at}u(t) + Be^{-bt}u(t)$$

   Determine the coefficients H_0, A, B in terms of ω_0, a, b. Identify the steady-state and transient terms in this output.

   If $a = 2$ and $b = 3$, determine the 40-dB effective time constant t_{eff} of this system. [*Ans.* $t_{\text{eff}} = 2.3$.]

5. For any stable and causal system (i.e., with poles strictly in the left-hand s-plane), show that its 60-dB and 40-dB time constants are related by

$$t_{60} = 1.5\, t_{40}$$

6. Determine the inverse z-transform of the following:

$$F(z) = \frac{19 - 9z^{-1} - 9z^{-2} + 4z^{-3}}{(1-0.5z^{-1})(1-0.8z^{-1})} = \frac{19 - 9z^{-1} - 9z^{-2} + 4z^{-3}}{1 - 1.3z^{-1} + 0.4z^{-2}}$$

   You must use long division to reduce the order of the numerator (from order 3 to order 1) and then apply partial fraction expansion.

   [*Ans.* $f(n) = 10\delta(n) + 10\delta(n-1) + 5(0.5)^n u(n) + 4(0.8)^n u(n)$.]

7. Consider the periodic function $f(t)$ defined over one period T by:

$$f(t) = 1 - \frac{2|t|}{T}, \quad -\frac{T}{2} \le t \le \frac{T}{2}$$

   Sketch $f(t)$. Then, determine the coefficients c_m of its Fourier series expansion:

$$f(t) = \sum_{m=-\infty}^{\infty} c_m e^{2\pi jmt/T}$$

8. Repeat the previous problem if $f(t)$ is defined by:

$$f(t) = \begin{cases} 1, & |t| < \frac{T}{4} \\ 0, & \frac{T}{4} < |t| < \frac{T}{2} \end{cases}$$

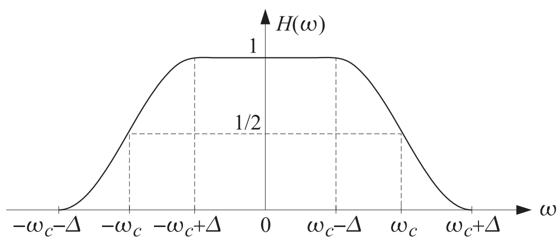
9. Using the results of the previous two problems and the Parseval identity for periodic signals, prove the following two infinite series results:

$$\sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{8}, \qquad \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^4} = \frac{\pi^4}{96}$$

10. The raised-cosine filter is used very widely in digital data transmission systems. Its frequency response is bandlimited and is defined by,

$$H(\omega)=\begin{cases}1\,, & |\omega| \le \omega_c-\Delta \\ \frac{1}{2}\left[1+\cos\left(\frac{\pi}{2\Delta}\left(|\omega|-\omega_c+\Delta\right)\right)\right], & \omega_c-\Delta \le |\omega| \le \omega_c+\Delta \\ 0\,, & |\omega| > \omega_c+\Delta\end{cases}$$

where $\Delta < \omega_c$. It has a flat response over the interval $|\omega| \le \omega_c - \Delta$, and beyond that, it tapers to zero following a cosine curve. It is depicted below.



By direct calculation of the inverse Fourier transform, show that the impulse response of this filter is given by,

$$h(t)=\frac{\sin(\omega_c t)}{\pi t}\cdot\frac{\cos(t\Delta)}{1-4t^2\Delta^2/\pi^2}$$

What is its value at $t=0$ and at $t=\pm\pi/2\Delta$? To do the Fourier integral, first note that because $H(\omega)$ is even in ω, the integral simplifies into,

$$h(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}H(\omega)e^{j\omega t}d\omega=\frac{1}{\pi}\int_0^{\infty}H(\omega)\cos(\omega t)\,d\omega$$

Then, split the integral as follows,

$$h(t)=\frac{1}{\pi}\int_0^{\omega_c-\Delta}H(\omega)\cos(\omega t)\,d\omega+\frac{1}{\pi}\int_{\omega_c-\Delta}^{\omega_c+\Delta}H(\omega)\cos(\omega t)\,d\omega$$

Show that the first term is:

$$\frac{\sin\big((\omega_c-\Delta)t\big)}{\pi t}$$

and the second,

$$\frac{4t^2\Delta^2\sin\big((\omega_c-\Delta)t\big)+\pi^2\cos(\omega_c t)\sin(t\Delta)}{\pi t(\pi^2-4t^2\Delta^2)}$$

Hint: You may use the indefinite integral:

$$\int\cos(ax)\cos(bx)\,dx=\frac{\sin\big((a+b)x\big)}{2(a+b)}+\frac{\sin\big((a-b)x\big)}{2(a-b)}$$

11. Using only the transform pairs and properties listed on the table of Fourier transforms given in class, work out the Fourier or inverse Fourier transforms of the following cases:

(a) $f(t)=te^{-at}u_h(t) \quad\Rightarrow\quad F(\omega)=?$

(b) $f(t)=e^{-at}e^{j\omega_0 t}u_h(t) \quad\Rightarrow\quad F(\omega)=?$

(c) $f(t)=e^{-at}\cos(\omega_0 t)\,u_h(t) \quad\Rightarrow\quad F(\omega)=?$

(d) $f(t)=e^{-at}\sin(\omega_0 t)\,u_h(t) \quad\Rightarrow\quad F(\omega)=?$

(e) $F(\omega)=\dfrac{e^{-j(\omega-\omega_0)t_0}}{a+j(\omega-\omega_0)} \quad\Rightarrow\quad f(t)=?$

(f) $F(\omega)=\dfrac{j\omega+b}{j\omega+a} \quad\Rightarrow\quad f(t)=?$

(g) $F(\omega)=\dfrac{j(\omega-\omega_0)+b}{j(\omega-\omega_0)+a} \quad\Rightarrow\quad f(t)=?$

(h) $F(\omega)=\dfrac{j(\omega-\omega_0)+b}{j(\omega-\omega_0)+a}\,e^{-j(\omega-\omega_0)t_0} \quad\Rightarrow\quad f(t)=?$

(i) $F(\omega)=e^{-a|\omega-\omega_0|} \quad\Rightarrow\quad f(t)=?$

(j) $f(t)=e^{-a|t-t_0|} \quad\Rightarrow\quad F(\omega)=?$

(k) $f(t)=\cos(\omega_0 t)\dfrac{\sin(\omega_c t)}{\pi t},\quad \omega_0>\omega_c \quad\Rightarrow\quad F(\omega)=?$

(l) $f(t)=\dfrac{t_0^2}{t_0^2+t^2} \quad\Rightarrow\quad F(\omega)=?$

(m) $f(t)=te^{-t^2/2\sigma^2} \quad\Rightarrow\quad F(\omega)=?$

(n) $F(\omega)=\cos\left(\dfrac{\omega^2}{2\beta}\right),\quad \beta=\text{real} \quad\Rightarrow\quad f(t)=?$

(o) $F(\omega)=\sin\left(\dfrac{\omega^2}{2\beta}\right),\quad \beta=\text{real} \quad\Rightarrow\quad f(t)=?$

(p) $f(t)=\cos\left(\dfrac{\beta t^2}{2}-\dfrac{\pi}{4}\right),\quad \beta=\text{real}, \quad\Rightarrow\quad F(\omega)=?$

(q) $F(\omega)=\dfrac{4}{(3+j\omega)^2+16} \quad\Rightarrow\quad f(t)=?$

(r) $F(\omega)=\dfrac{3+j\omega}{(3+j\omega)^2+16} \quad\Rightarrow\quad f(t)=?$

(s) $f(t)=\cos(\omega_0 t+\phi),\quad \phi$ is a constant $\quad\Rightarrow\quad F(\omega)=?$

12. Prove the following properties of Fourier transforms listed on the table posted on the course web page: (a) reflection, (b) duality, (c) conjugation, (d) hermitian, (e) delay, (f) modulation, (g) time differentiation, (h) convolution, and finally the following property:

$$f(t) \longleftrightarrow F(\omega) \quad \Rightarrow \quad e^{j\omega_0 t} f(t-t_0) \longleftrightarrow e^{-j(\omega-\omega_0)t_0} F(\omega-\omega_0)$$

13. Using the properties listed on your table of Fourier transforms, show the following property: If $f(t)$ is real-valued and even in t, i.e., $f(t) = f(-t)$, then its Fourier transform is also real-valued and even, i.e., $F(\omega) (= F(-\omega)$. Moreover, show that $f(t)$ can be recovered from $F(\omega)$ by,

$$f(t) = \frac{1}{\pi}\int_0^\infty F(\omega)\cos(\omega t)\,d\omega$$

14. Consider the two gaussian pulses of widths τ_1 and τ_2:

$$f_1(t) = \frac{1}{\sqrt{2\pi\tau_1^2}}\exp\left(-\frac{t^2}{2\tau_1^2}\right), \quad f_2(t) = \frac{1}{\sqrt{2\pi\tau_2^2}}\exp\left(-\frac{t^2}{2\tau_2^2}\right)$$

Working Fourier transforms, show that the convolution of $f_1(t)$ and $f_2(t)$ is also a gaussian pulse of the form:

$$f_1(t) * f_2(t) = \frac{1}{\sqrt{2\pi\tau^2}}\exp\left(-\frac{t^2}{2\tau^2}\right)$$

and determine its width τ in terms of τ_1, τ_2.

15. Consider the sinc-pulse of time-width τ:

$$f(t) = \frac{\sin(\pi t/\tau)}{\pi t}$$

This pulse is sent through an ideal bandlimited channel of bandwidth ω_B rads/sec with frequency response:

$$H(\omega) = \mathrm{rect}_{2\omega_B}(\omega) = \begin{cases} 1, & |\omega| \le \omega_B \\ 0, & |\omega| > \omega_B \end{cases}$$

(a) Using Fourier transforms, show that if $\omega_B < \pi/\tau$, the output signal will be given by:

$$y(t) = \frac{\sin(\omega_B t)}{\pi t}$$

How does the effective duration of $y(t)$ compare to that of $f(t)$?

(b) Show that if $\omega_B \ge \pi/\tau$, then $y(t) = f(t)$.

16. Using Fourier transforms or their inverses, determine the values of the the following integrals without actually performing the indicated integrations:

$$\int_{-\infty}^\infty e^{-t^2/2\tau^2}dt, \qquad \int_{-\infty}^\infty \frac{\sin(\omega_c t)}{\pi t}dt, \qquad \int_{-\infty}^\infty \frac{1}{\pi}\frac{a}{a^2+\omega^2}d\omega$$

17. Consider the linear-FM-modulated pulse

$$E(t) = f(t)\,e^{j\omega_0 t + j\dot\omega t^2/2}$$

where $f(t)$ is a pulse of effective duration of T seconds. Define the effective bandwidth due to FM modulation by $\omega_B = \dot\omega_0 T$, or in units of Hz, $B = \omega_B/2\pi$. A pulse compression filter that compresses the pulse $E(t)$ into a new pulse of compressed effective duration of $T_{\text{compr}} = 1/B$ is defined by its frequency and impulse responses:

$$H_{\text{compr}}(\omega) = e^{j(\omega-\omega_0)^2/2\dot\omega_0} \quad \Leftrightarrow \quad h_{\text{compr}}(t) = \sqrt{\frac{j\dot\omega_0}{2\pi}}\,e^{j\omega_0 t - j\dot\omega_0 t^2/2}$$

The output of this filter is the compressed signal obtained by convolving the input $E(t)$ with the impulse response:

$$E_{\text{compr}}(t) = \int_{-\infty}^\infty h_{\text{compr}}(t-t')E(t')\,dt'$$

Let $F(\omega)$ be the Fourier transform of the pulse envelope $f(t)$. By explicit manipulation of the convolution integral, show that the output signal is expressible in terms of $F(\omega)$ by,

$$E_{\text{compr}}(t) = \sqrt{\frac{j\dot\omega_0}{2\pi}}\,e^{j\omega_0 t - j\dot\omega_0 t^2/2}F(-\dot\omega_0 t)$$

where the last factor is $F(\omega)$ with ω substituted by $-\dot\omega_0 t$.

18. Apply the above result to the following three cases for the envelope $f(t)$, for which you have explicit expressions for $F(\omega)$:

$$f(t) = e^{-t^2/2T^2}, \quad f(t) = e^{-|t|/T}, \quad f(t) = \frac{T^2}{T^2+t^2}$$

where in all cases $f(t)$ was normalized to unity at $t = 0$. (The first case was worked out in class.)

In each case, derive the corresponding output $E_{\text{compr}}(t)$ of the pulse-compression filter and show that, indeed, it has an effective duration of the order of $T_{\text{compr}} = 1/\omega_B$. Note also that in all cases the effective height of the compressed pulse is increased by a factor of $\sqrt{\omega_B T}$.

In class, we discussed the reasons why one wants to have both a large T and a large ω_B resulting in a large time-bandwidth product $\omega_B T$.