4. The signal \( f(t) = e^{j\omega_0 t}u(t) \) is sent to the input of the system in problem 2 (with \( a \neq b \)). Show that the output will have the following form for all \( t \),
\[
y(t) = H_0 e^{j\omega_0 t}u(t) + Ae^{-at}u(t) + Be^{-bt}u(t)
\]
Determine the coefficients \( H_0, A, B \) in terms of \( \omega_0, a, b \). Identify the steady-state and transient terms in this output.
If \( a = 2 \) and \( b = 3 \), determine the 40-dB effective time constant \( t_{\text{eff}} \) of this system. \([\text{Ans. } t_{\text{eff}} = 2.3]\)

5. For any stable and causal system (i.e., with poles strictly in the left-hand \( s \)-plane), show that its 60-dB and 40-dB time constants are related by
\[
t_{60} = 1.5 t_{40}
\]

6. Determine the inverse \( z \)-transform of the following:
\[
F(z) = \frac{19 - 9z^{-1} - 9z^{-2} + 2z^{-3}}{(1 - 0.5z^{-1})(1 - 0.8z^{-1})} = \frac{19 - 9z^{-1} - 9z^{-2} + 4z^{-3}}{1 - 1.3z^{-1} + 0.4z^{-2}}
\]
You must use long division to reduce the order of the numerator (from order 3 to order 1) and then apply partial fraction expansion.
\([\text{Ans. } f(n) = 10\delta(n) + 10\delta(n - 1) + 5(0.5)^n u(n) + 4(0.8)^n u(n)\])

7. Consider the periodic function \( f(t) \) defined over one period \( T \) by:
\[
f(t) = 1 - \frac{2|t|}{T}, \quad -\frac{T}{2} \leq t \leq \frac{T}{2}
\]
Sketch \( f(t) \). Then, determine the coefficients \( c_m \) of its Fourier series expansion:
\[
f(t) = \sum_{m=-\infty}^{\infty} c_m e^{2\pi jmt/T}
\]

8. Repeat the previous problem if \( f(t) \) is defined by:
\[
f(t) = \begin{cases} 
1, & |t| < \frac{T}{4} \\
0, & \frac{T}{4} < |t| < \frac{T}{2}
\end{cases}
\]

9. Using the results of the previous two problems and the Parseval identity for periodic signals, prove the following two infinite series results:
\[
\sum_{m=1,3,5,...}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{8}, \quad \sum_{m=1,3,5,...}^{\infty} \frac{1}{m^4} = \frac{\pi^4}{96}
\]
10. The raised-cosine filter is used very widely in digital data transmission systems. Its frequency response is bandlimited and is defined by,

\[
H(\omega) = \begin{cases} 
1, & |\omega| \leq \omega_c - \Delta \\
\frac{1}{2} \left[ 1 + \cos \left( \frac{\pi}{2\Delta} (|\omega| - \omega_c + \Delta) \right) \right], & \omega_c - \Delta \leq |\omega| \leq \omega_c + \Delta \\
0, & |\omega| > \omega_c + \Delta
\end{cases}
\]

where \( \Delta < \omega_c \). It has a flat response over the interval \( |\omega| \leq \omega_c - \Delta \), and beyond that, it tapers to zero following a cosine curve. It is depicted below.

By direct calculation of the inverse Fourier transform, show that the impulse response of this filter is given by,

\[
h(t) = \frac{\sin(\omega_c t)}{\pi t} \cdot \cos(t\Delta) \left( 1 - \frac{t^2}{\Delta^2} \right)
\]

What is its value at \( t = 0 \) and at \( t = \pm \pi/2\Delta \)?

Then, split the integral as follows,

\[
h(t) = \frac{1}{\pi t} \int_{-\infty}^{\omega_c - \Delta} H(\omega) \cos(\omega t) d\omega + \frac{1}{\pi t} \int_{\omega_c - \Delta}^{\omega_c + \Delta} H(\omega) \cos(\omega t) d\omega
\]

Show that the first term is:

\[
\frac{\sin((\omega_c - \Delta) t)}{\pi t}
\]

and the second,

\[
\frac{4t^2\Delta^2 (\omega_c - \Delta) t + \pi^2 \cos(\omega_c t) \sin(t\Delta)}{\pi t (\pi^2 - 4t^2\Delta^2)}
\]

Hint: You may use the indefinite integral:

\[
\int \cos(ax) \cos(bx) dx = \frac{\sin((a + b)x)}{2(a + b)} + \frac{\sin((a - b)x)}{2(a - b)}
\]

11. Using only the transform pairs and properties listed on the table of Fourier transforms given in class, work out the Fourier or inverse Fourier transforms of the following cases:

- (a) \( f(t) = te^{-at}u_h(t) \) \( \Rightarrow F(\omega) = \?)
- (b) \( f(t) = e^{-at}e^{j\omega_0 t}u_h(t) \) \( \Rightarrow F(\omega) = \?)
- (c) \( f(t) = e^{-at}\cos(\omega_0 t)u_h(t) \) \( \Rightarrow F(\omega) = \?)
- (d) \( f(t) = e^{-at}\sin(\omega_0 t)u_h(t) \) \( \Rightarrow F(\omega) = \?)
- (e) \( F(\omega) = e^{-j(\omega_0 - \omega_0)t_0} \) \( \Rightarrow f(t) = \?)
- (f) \( F(\omega) = \frac{j\omega + b}{j\omega + a} \) \( \Rightarrow f(t) = \?)
- (g) \( F(\omega) = \frac{j(\omega - \omega_0) + b}{j(\omega - \omega_0) + a} \) \( \Rightarrow f(t) = \?)
- (h) \( F(\omega) = e^{-j(\omega - \omega_0)t} \) \( \Rightarrow f(t) = \?)
- (i) \( F(\omega) = e^{-a(\omega - \omega_0)} \) \( \Rightarrow f(t) = \?)
- (j) \( f(t) = e^{-a|r-t_0|} \) \( \Rightarrow F(\omega) = \?)
- (k) \( f(t) = \cos(\omega_0 t) \) \( \Rightarrow F(\omega) = \?)
- (l) \( f(t) = \frac{t_0^2}{t_0^2 + t^2} \) \( \Rightarrow F(\omega) = \?)
- (m) \( f(t) = te^{-t^2/2}\beta^2 \) \( \Rightarrow F(\omega) = \?)
- (n) \( F(\omega) = \cos \left( \frac{\omega^2}{2\beta} \right), \ \beta = \text{real} \Rightarrow f(t) = \?)
- (o) \( F(\omega) = \sin \left( \frac{\omega^2}{2\beta} \right), \ \beta = \text{real} \Rightarrow f(t) = \?)
- (p) \( f(t) = \cos \left( \frac{\beta t^2}{2} - \frac{\pi}{4} \right), \ \beta = \text{real} \Rightarrow F(\omega) = \?)
- (q) \( F(\omega) = \frac{4}{(3 + j\omega)^2 + 16} \) \( \Rightarrow f(t) = \?)
- (r) \( F(\omega) = \frac{3 + j\omega}{(3 + j\omega)^2 + 16} \) \( \Rightarrow f(t) = \?)
- (s) \( f(t) = \cos(\omega_0 t + \phi), \ \phi \text{ is a constant} \Rightarrow F(\omega) = \?)
12. Prove the following properties of Fourier transforms listed on the table posted on the course web page: (a) reflection, (b) duality, (c) conjugation, (d) hermitian, (e) delay, (f) modulation, (g) time differentiation, (h) convolution, and finally the following property:

\[ f(t) \leftrightarrow F(\omega) \quad \Rightarrow \quad e^{j\omega_0 t} f(t - t_0) \rightarrow e^{-j(\omega - \omega_0)t_0} F(\omega - \omega_0) \]

13. Using the properties listed on your table of Fourier transforms, show the following property: If \( f(t) \) is real-valued and even in \( t \), i.e., \( f(t) = f(-t) \), then its Fourier transform is also real-valued and even, i.e., \( F(\omega) = F(-\omega) \). Moreover, show that \( f(t) \) can be recovered from \( F(\omega) \) by,

\[ f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(\omega) \cos(\omega t) d\omega \]

14. Consider the two gaussian pulses of widths \( \tau_1 \) and \( \tau_2 \):

\[ f_1(t) = \frac{1}{\sqrt{2\pi \tau_1^2}} \exp\left( -\frac{t^2}{2\tau_1^2} \right), \quad f_2(t) = \frac{1}{\sqrt{2\pi \tau_2^2}} \exp\left( -\frac{t^2}{2\tau_2^2} \right) \]

Working Fourier transforms, show that the convolution of \( f_1(t) \) and \( f_2(t) \) is also a gaussian pulse of the form:

\[ f_1(t) * f_2(t) = \frac{1}{\sqrt{2\pi \tau^2}} \exp\left( -\frac{t^2}{2\tau^2} \right) \]

determine its width \( \tau \) in terms of \( \tau_1, \tau_2 \).

15. Consider the sinc-pulse of time-width \( \tau \):

\[ f(t) = \frac{\sin(\pi t/\tau)}{\pi t} \]

This pulse is sent through an ideal bandlimited channel of bandwidth \( \omega_B \) rads/sec with frequency response:

\[ H(\omega) = \text{rect}_{2\omega_B}(\omega) = \begin{cases} 1, & |\omega| \leq \omega_B \\ 0, & |\omega| > \omega_B \end{cases} \]

(a) Using Fourier transforms, show that if \( \omega_B < \pi/\tau \), the output signal will be given by:

\[ y(t) = \frac{\sin(\omega_B t)}{\pi t} \]

How does the effective duration of \( y(t) \) compare to that of \( f(t) \)?

(b) Show that if \( \omega_B \geq \pi/\tau \), then \( y(t) = f(t) \).

16. Using Fourier transforms or their inverses, determine the values of the following integrals without actually performing the indicated integrations:

\[ \int_{-\infty}^{\infty} e^{-t^2/2\tau^2} dt, \quad \int_{-\infty}^{\infty} \frac{\sin(\omega t)}{\pi t} dt, \quad \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{a}{a^2 + \omega^2} d\omega \]

17. Consider the linear-FM-modulated pulse

\[ E(t) = f(t) e^{j\omega_0 t + j\omega^2 t^2/2} \]

where \( f(t) \) is a pulse of effective duration of \( T \) seconds. Define the effective bandwidth due to FM modulation by \( \omega_F = \omega_0 T \), or in units of Hz, \( B = \omega_F/2\pi \). A pulse compression filter that compresses the pulse \( E(t) \) into a new pulse of compressed effective duration of \( T_{\text{compr}} = 1/B \) is defined by its frequency and impulse responses:

\[ H_{\text{compr}}(\omega) = e^{j(\omega-\omega_0)^2/2\omega_0} \Leftrightarrow h_{\text{compr}}(t) = \sqrt{\frac{j\omega_0}{2\pi}} e^{j\omega_0 t - j\omega_0^2 t^2 / 2} \]

The output of this filter is the compressed signal obtained by convolving the input \( E(t) \) with the impulse response:

\[ E_{\text{compr}}(t) = \int_{-\infty}^{\infty} h_{\text{compr}}(t - t') E(t') dt' \]

Let \( F(\omega) \) be the Fourier transform of the pulse envelope \( f(t) \). By explicit manipulation of the convolution integral, show that the output signal is expressible in terms of \( F(\omega) \) by,

\[ E_{\text{compr}}(t) = \int_{-\infty}^{\infty} f(t) e^{j\omega_0 t - j\omega_0^2 t^2 / 2} F(-\omega_0 t) \]

where the last factor is \( F(\omega) \) with \( \omega \) substituted by \(-\omega_0 \).

18. Apply the above result to the following three cases for the envelope \( f(t) \), for which you have explicit expressions for \( F(\omega) \):

\[ f(t) = e^{-t^2/2T^2}, \quad f(t) = e^{-|t|/T}, \quad f(t) = \frac{T^2}{T^2 + t^2} \]

where in all cases \( f(t) \) was normalized to unity at \( t = 0 \). (The first case was worked out in class.)

In each case, derive the corresponding output \( E_{\text{compr}}(t) \) of the pulse-compression filter and show that, indeed, it has an effective duration of the order of \( T_{\text{compr}} = 1/\omega_B \). Note also that in all cases the effective height of the compressed pulse is increased by a factor of \( \sqrt{\omega_B T} \).

In class, we discussed the reasons why one wants to have both a large \( T \) and a large \( \omega_B \) resulting in a large time-bandwidth product \( \omega_B T \).