332:345 – Linear Systems & Signals
Block Diagram Realizations – Fall 2009 – S. J. Orfanidis

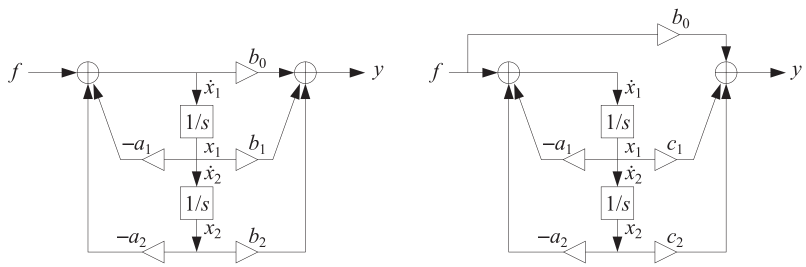
Consider the second-order transfer function example:

$$H(s)=\frac{b_0 s^2+b_1 s+b_2}{s^2+a_1 s+a_2}=b_0+\frac{c_1 s+c_2}{s^2+a_1 s+a_2}=b_0+\frac{r_1}{s-p_1}+\frac{r_2}{s-p_2}$$

where p_1, p_2 are the system poles, assumed to be disctinct, $p_1 \neq p_2$. Note that

$$c_1 = b_1 - b_0 a_1$$
$$c_2 = b_2 - b_0 a_2$$

Controller Canonical Form



$$\dot{x}_1 = -a_1 x_1 - a_2 x_2 + f$$
$$\dot{x}_2 = x_1$$
$$\begin{aligned} y &= b_0\dot{x}_1 + b_1 x_1 + b_2 x_2 \\ &= b_0(-a_1 x_1 - a_2 x_2 + f) + b_1 x_1 + b_2 x_2 \\ &= c_1 x_1 + c_2 x_2 + b_0 f \end{aligned}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-a_1 & -a_2\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}f$$

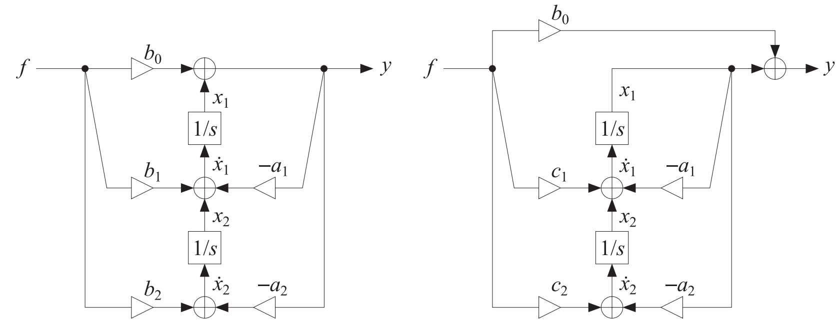
$$y=[c_1,\, c_2]\begin{bmatrix}x_1\\ x_2\end{bmatrix}+b_0 f$$

Also known as the "phase-variable canonical" form. Often written in a reversed way by renumbering the states in a reversed fashion, $x_1 \to x_2$ and $x_2 \to x_1$ and writing the above equations in reverse order (as is done in the textbook):

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -a_2 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}f$$

$$y=[c_2,\, c_1]\begin{bmatrix}x_1\\ x_2\end{bmatrix}+b_0 f$$

Observer Canonical Form



$$y = b_0 f + x_1$$
$$\dot{x}_1 = -a_1 y + x_2 + b_1 f = -a_1(b_0 f + x_1) + x_2 + b_1 f = -a_1 x_1 + x_2 + c_1 f$$
$$\dot{x}_2 = -a_2 y + b_2 f = -a_2(b_0 f + x_1) + b_2 f = -a_2 x_1 + c_2 f$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-a_1 & 1\\ -a_2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}c_1\\ c_2\end{bmatrix}f$$

$$y=[1,\, 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix}+b_0 f$$

Also know as the "dual" of the phase-variable canconical form. It is the transposed of the controller form obtained by applying the four transposition rules:

- replace adders by nodes
- replace nodes by adders
- reverse all flows
- exchange input with output

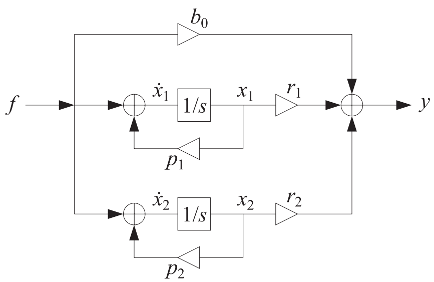
The $[A, B, C, D]$ state-space parameters are obtained by the replacements:

$$A \to A^T, \quad B \to C^T, \quad C \to B^T, \quad D \to D$$

The above (scalar) transfer function remains invariant because of the identity:

$$H(s) = C(sI - A)^{-1}B + D = B^T(sI - A^T)^{-1}C^T + D$$

Parallel Pole Form



$$\begin{aligned}
\dot{x}_1 &= p_1 x_1 + f \\
\dot{x}_2 &= p_2 x_2 + f \\
y &= r_1 x_1 + r_2 x_2 + b_0 f
\end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} f$$

$$y = [r_1, \, r_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 f$$

Valid in this form only when poles are distinct. If poles are complex-valued, then the states are too. Note that the state matrix A is diagonal. The parameters can be obtained by performing partial fraction expansion on $H(s)$.

In MATLAB, the call,

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[A,B,C,D] = tf2ss([b0,b1,b2], [1,a1,a2]);
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produces the controller canonical state-space form, whereas the call,

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[r,p,k] = residue([b0,b1,b2], [1,a1,a2]);
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produces the vectors of residues and poles for the paralle/pole form:

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \quad b_0 = k$$