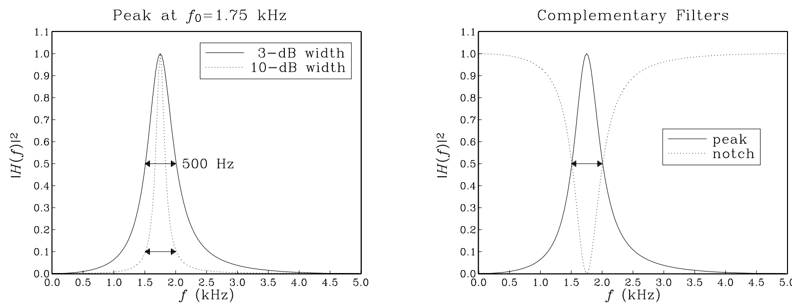

Fig. 11.3.3 Peaking and complementary notch filters.

The zeros of the denominator, $1 - 0.7838z^{-1} + 0.7265z^{-2} = 0$, determine the poles of the transfer function. They are:

$$p, p^* = 0.3919 \pm j0.7569 = 0.8524e^{\pm j0.3479\pi}$$

The poles are not exactly at the desired frequency $\omega_0 = 0.35\pi$. Naive pole placement would have placed them there. For example, choosing the same radius $R = 0.8524$, we would have in that case:

$$p, p^* = 0.8524e^{\pm j0.35\pi} = 0.3870 \pm j0.7597$$

corresponding to the denominator polynomial $1 - 0.7739z^{-1} + 0.7265z^{-2}$. The bilinear transformation method places the poles at appropriate locations to achieve the desired peak and width. □

11.4 Parametric Equalizer Filters

Frequency equalization (EQ) is a common requirement in audio systems—analog, digital, home, car, public, or studio recording/mixing systems [255].

Graphic equalizers are the more common type, in which the audio band is divided into a *fixed* number of frequency bands, and the amount of equalization in each band is controlled by a bandpass filter whose gain can be varied up and down. The center frequencies of the bands and the filter 3-dB widths are fixed, and the user can vary only the overall gain in each band. Usually, second-order bandpass filters are adequate for audio applications.

A more flexible equalizer type is the *parametric equalizer*, in which all three filter parameters—gain, center frequency, and bandwidth—can be varied. Cascading four or five such filters together can achieve almost any desired equalization effect.

Figure 11.4.1 shows the frequency response of a typical second-order parametric equalizer. The specification parameters are: a *reference* gain G_0 (typically taken to be unity for cascadable filters), the center frequency ω_0 of the boost or cut, the filter gain G at ω_0, and a desired width $\Delta\omega$ at an appropriate bandwidth level G_B that lies

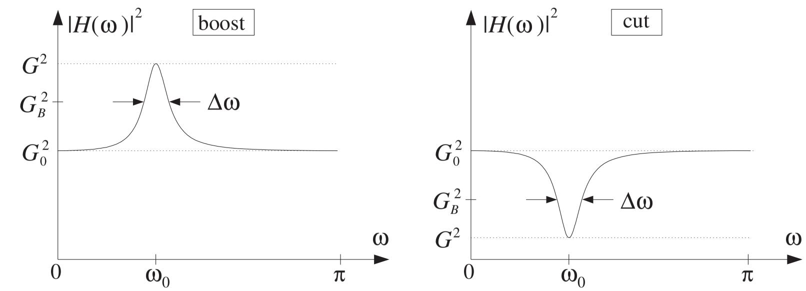

Fig. 11.4.1 Parametric EQ filter with boost or cut.

between G_0 and G. As shown in Fig. 11.4.1, the relative gains must be chosen as follows, depending on whether we have a boost or a cut:

$$\boxed{\begin{array}{ll} G_0^2 < G_B^2 < G^2 & \text{(boost)} \\ G^2 < G_B^2 < G_0^2 & \text{(cut)} \end{array}} \tag{11.4.1}$$

The notch and peak filters of the previous section can be thought of as special cases of such a filter. The peaking filter corresponds to $G_0 = 0$, $G = 1$ and the notching filter to $G_0 = 1$, $G = 0$.

The definition of $\Delta\omega$ is arbitrary, and not without ambiguity. For example, we can define it to be the 3-dB width. But, what exactly do we mean by "3 dB"?

For the boosting case, we can take it to mean 3 dB *below the peak*, that is, choose $G_B^2 = G^2/2$; alternatively, we can take it to mean 3 dB *above the reference*, that is, $G_B^2 = 2G_0^2$. Moreover, because G_B^2 must lie between G_0^2 and G^2, the first alternative implies that $G_0^2 < G_B^2 = G^2/2$, or $2G_0^2 < G^2$, and the second $2G_0^2 = G_B^2 < G^2$. Thus, either alternative requires that $G^2 > 2G_0^2$, that is, the boost gain must be at least 3 dB higher than the reference. So, what do we do when $G_0^2 < G^2 < 2G_0^2$? In that case, any G_B^2 that lies in $G_0^2 < G_B^2 < G^2$ will do. A particularly interesting choice is to take it to be the arithmetic mean of the end values:

$$G_B^2 = \frac{G_0^2 + G^2}{2} \tag{11.4.2}$$

Another good choice is the geometric mean, $G_B^2 = GG_0$, corresponding to the arithmetic mean of the dB values of the endpoints [265,268] (see Problem 11.4.)

Similar ambiguities arise in the cutting case: we can take 3 dB to mean 3 dB *above the dip*, that is, $G_B^2 = 2G^2$, or, alternatively, we can take it to mean 3 dB *below the reference*, $G_B^2 = G_0^2/2$. Either alternative requires that $G^2 < G_0^2/2$, that is, the cut gain must be at least 3 dB below the reference. If $G_0^2/2 < G^2 < G_0^2$, we may again use the average (11.4.2). To summarize, some possible (but not necessary) choices for G_B^2 are as follows:

$$G_B^2 = \begin{cases} G^2/2, & \text{if } G^2 > 2G_0^2 \quad \text{(boost, alternative 1)} \\ 2G_0^2, & \text{if } G^2 > 2G_0^2 \quad \text{(boost, alternative 2)} \\ (G_0^2 + G^2)/2, & \text{if } G_0^2 < G^2 < 2G_0^2 \quad \text{(boost)} \\ \\ 2G^2, & \text{if } G^2 < G_0^2/2 \quad \text{(cut, alternative 1)} \\ G_0^2/2, & \text{if } G^2 < G_0^2/2 \quad \text{(cut, alternative 2)} \\ (G_0^2 + G^2)/2, & \text{if } G_0^2/2 < G^2 < G_0^2 \quad \text{(cut)} \end{cases} \tag{11.4.3}$$

The filter design problem is to determine the filter's transfer function in terms of the specification parameters: $\{G_0, G, G_B, \omega_0, \Delta\omega\}$. In this section, we present a simple design method based on the bilinear transformation, which is a variation of the methods in [260–268].

We define the parametric equalizer filter as the following *linear combination* of the notching and peaking filters of the previous section:

$$\boxed{H(z) = G_0 H_{\text{notch}}(z) + G H_{\text{peak}}(z)} \tag{11.4.4}$$

At ω_0 the gain is G, because the notch filter vanishes and the peak filter has unity gain. Similarly, at DC and the Nyquist frequency, the gain is equal to the reference G_0, because the notch is unity and the peak vanishes. From the complementarity property (11.3.23) it follows that when $G = G_0$ we have $H(z) = G_0$, that is, no equalization. Inserting the expressions (11.3.5) and (11.3.18) into Eq. (11.4.4), we obtain:

$$\boxed{H(z) = \frac{\left(\dfrac{G_0 + G\beta}{1+\beta}\right) - 2\left(\dfrac{G_0\cos\omega_0}{1+\beta}\right)z^{-1} + \left(\dfrac{G_0 - G\beta}{1+\beta}\right)z^{-2}}{1 - 2\left(\dfrac{\cos\omega_0}{1+\beta}\right)z^{-1} + \left(\dfrac{1-\beta}{1+\beta}\right)z^{-2}}} \tag{11.4.5}$$

The parameter β is a generalization of Eqs. (11.3.4) and (11.3.19) and is given by:

$$\boxed{\beta = \sqrt{\frac{G_B^2 - G_0^2}{G^2 - G_B^2}}\,\tan\left(\frac{\Delta\omega}{2}\right)} \tag{11.4.6}$$

Note that because of the assumed inequalities (11.4.1), the quantity under the square root is always positive. Also, for the special choice of G_B^2 of Eq. (11.4.2), the square root factor is *unity*. This choice (and those of Problem 11.4) allows a smooth transition to the no-equalization limit $G \to G_0$. Indeed, because β does not depend on the G's, setting $G = G_0$ in the numerator of Eq. (11.4.5) gives $H(z) = G_0$.

The design equations (11.4.5) and (11.4.6) can be justified as follows. Starting with the same linear combination of the analog versions of the notching and peaking filters given by Eqs. (11.3.2) and (11.3.16), we obtain the analog version of $H(z)$:

$$\boxed{H_a(s) = G_0 H_{\text{notch}}(s) + G H_{\text{peak}}(s) = \frac{G_0(s^2 + \Omega_0^2) + G\alpha s}{s^2 + \alpha s + \Omega_0^2}} \tag{11.4.7}$$

Then, the bandwidth condition $|H_a(\Omega)|^2 = G_B^2$ can be stated as:

$$|H_a(\Omega)|^2 = \frac{G_0^2(\Omega^2 - \Omega_0^2)^2 + G^2\alpha^2\Omega^2}{(\Omega^2 - \Omega_0^2)^2 + \alpha^2\Omega^2} = G_B^2 \tag{11.4.8}$$

It can be cast as the quartic equation:

$$\Omega^4 - \left(2\Omega_0^2 + \frac{G^2 - G_B^2}{G_B^2 - G_0^2}\alpha^2\right)\Omega^2 + \Omega_0^4 = 0$$

Proceeding as in the previous section and using the geometric-mean property $\Omega_1\Omega_2 = \Omega_0^2$ and Eq. (11.3.15), we find the relationship between the parameter α and the analog bandwidth $\Delta\Omega = \Omega_2 - \Omega_1$:

$$\alpha = \sqrt{\frac{G_B^2 - G_0^2}{G^2 - G_B^2}}\,\Delta\Omega = \sqrt{\frac{G_B^2 - G_0^2}{G^2 - G_B^2}}\,(1 + \Omega_0^2)\tan\left(\frac{\Delta\omega}{2}\right) \equiv (1 + \Omega_0^2)\beta$$

This defines β. Then, the bilinear transformation of Eq. (11.4.7) leads to Eq. (11.4.5).

Example 11.4.1: Design the following six parametric EQ filters operating at 10 kHz rate that satisfy the specifications: $G_0 = 1$ and

(a) center frequency of 1.75 kHz, 9-dB boost gain, and 3-dB width of 500 Hz defined to be 3 dB below the peak (alternative 1).

(b) same as (a), except the width is 3 dB above the reference (alternative 2).

(c) center frequency of 3 kHz, 9-dB cut gain, and 3-dB width of 1 kHz defined to be 3 dB above the dip (alternative 1).

(d) same as (c), except the width is 3 dB below the reference (alternative 2).

(e) center frequency of 1.75 kHz, 2-dB boost, and 500 Hz width defined by Eq. (11.4.2).

(f) center frequency of 3 kHz, 2-dB cut, and 1 kHz width defined by Eq. (11.4.2).

Solution: The boost examples (a), (b), and (e) have digital frequency and width:

$$\omega_0 = \frac{2\pi\cdot 1.75}{10} = 0.35\pi, \quad \Delta\omega = \frac{2\pi\cdot 0.5}{10} = 0.1\pi$$

and the cut examples (c), (d), and (f) have:

$$\omega_0 = \frac{2\pi\cdot 3}{10} = 0.6\pi, \quad \Delta\omega = \frac{2\pi\cdot 1}{10} = 0.2\pi$$

Normally, a "3-dB" change means a change by a factor of 2 in the magnitude square. Here, for plotting purposes, we take "3 dB" to mean literally 3 dB, which corresponds to changes by $10^{3/10} = 1.9953 \simeq 2$. Therefore, in case (a), a boost gain of 9 dB above the reference G_0 corresponds to the value:

$$G = 10^{9/20}G_0 = 2.8184, \qquad G^2 = 7.9433 \quad \text{(instead of 8)}$$

The bandwidth level is defined to be 3 dB below the peak, that is, $A_B = 9 - 3 = 6$ dB, and therefore:

$$G_B = 10^{-3/20}G = 10^{6/20}G_0 = 1.9953, \quad G_B^2 = 3.9811$$

With these values of $\{G_0, G, G_B, \omega_0, \Delta\omega\}$, we calculate the value of β from Eq. (11.4.6):

$$\beta = \sqrt{\frac{G_B^2 - G_0^2}{G^2 - G_B^2}}\,\tan\left(\frac{\Delta\omega}{2}\right) = \sqrt{\frac{3.9811 - 1}{7.9433 - 3.9811}}\,\tan\left(\frac{0.1\pi}{2}\right) = 0.1374$$

We calculate also $\cos\omega_0 = \cos(0.35\pi) = 0.4540$. The transfer function of filter (a), obtained from Eq. (11.4.5), is then:

$$H_a(z) = \frac{1.2196 - 0.7983z^{-1} + 0.5388z^{-2}}{1 - 0.7983z^{-1} + 0.7584z^{-2}}$$

For filter (b), the width is defined to be 3 dB above the reference, that is, $A_B = 3$ dB:

$$G_B = 10^{3/20}G_0 = 10^{3/20} = 1.4125, \quad G_B^2 = 1.9953$$

From Eq. (11.4.6), we calculate $\beta = 0.0648$, and from Eq. (11.4.5) the filter:

$$H_b(z) = \frac{1.1106 - 0.8527z^{-1} + 0.7677z^{-2}}{1 - 0.8527z^{-1} + 0.8783z^{-2}}$$

For filter (c), we have a 9-dB cut gain, that is, 9 dB below the reference:

$$G = 10^{-9/20}G_0 = 0.3548, \quad G^2 = 0.1259$$

and the bandwidth level is 3 dB above this dip, that is, $A_B = -9 + 3 = -6$ dB:

$$G_B = 10^{3/20}G = 10^{-6/20}G_0 = 0.5012, \quad G_B^2 = 0.2512$$

Then, we calculate $\cos\omega_0 = \cos(0.6\pi) = -0.3090$ and $\beta = 0.7943$, and the transfer function:

$$H_c(z) = \frac{0.7144 + 0.3444z^{-1} + 0.4002z^{-2}}{1 + 0.3444z^{-1} + 0.1146z^{-2}}$$

For filter (d), the width is 3 dB below the reference, that is, $A_B = 0 - 3 = -3$ dB:

$$G_B = 10^{-3/20}G_0 = 0.7079, \quad G_B^2 = 0.5012$$

We calculate $\beta = 0.3746$ and the transfer function:

$$H_d(z) = \frac{0.8242 + 0.4496z^{-1} + 0.6308z^{-2}}{1 + 0.4496z^{-1} + 0.4550z^{-2}}$$

The four filters (a)–(d) are shown in the left graph of Fig. 11.4.2. The magnitude responses are plotted in dB, that is, $20\log_{10}|H(\omega)|$. The reference level $G_0 = 1$ corresponds to 0

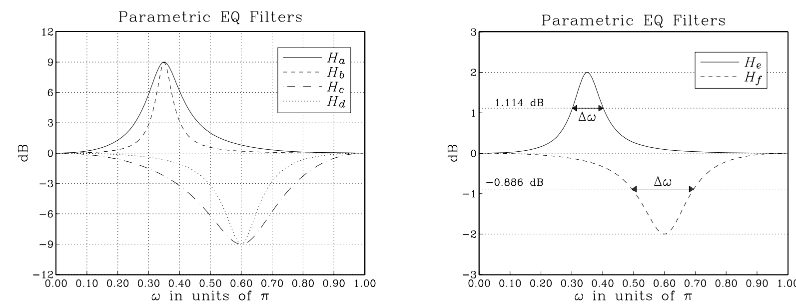

Fig. 11.4.2 Parametric EQ filters of Example 11.4.1.

dB. Notice the horizontal grid lines at 6 dB, 3 dB, −3 dB, and −6 dB, whose intersections with the magnitude responses define the corresponding bandwidths $\Delta\omega$.

For filter (e), the boost gain is 2 dB and therefore, the bandwidth level cannot be chosen to be 3 dB below the peak or 3 dB above the reference. We must use an intermediate level between 0 and 2 dB. In particular, we may use Eq. (11.4.2). Thus, we calculate the parameters:

$$G = 10^{2/20}G_0 = 1.2589, \quad G_B^2 = \frac{G_0^2 + G^2}{2} = 1.2924$$

corresponding to $A_B = 10\log_{10}(G_B^2) = 1.114$ dB. The square root factor in the definition of β is unity, therefore, we calculate:

$$\beta = \tan\left(\frac{\Delta\omega}{2}\right) = \tan\left(\frac{0.1\pi}{2}\right) = 0.1584$$

and the transfer function:

$$H_e(z) = \frac{1.0354 - 0.7838z^{-1} + 0.6911z^{-2}}{1 - 0.7838z^{-1} + 0.7265z^{-2}}$$

Finally, in case (f), we have a 2-dB cut, giving the values:

$$G = 10^{-2/20}G_0 = 0.7943, \quad G_B^2 = \frac{G_0^2 + G^2}{2} = 0.8155$$

corresponding to $A_B = 10\log_{10}(G_B^2) = -0.886$ dB. The parameter β is now $\beta = \tan(\Delta\omega/2) = \tan(0.2\pi/2) = 0.3249$, resulting in the transfer function:

$$H_f(z) = \frac{0.9496 + 0.4665z^{-1} + 0.5600z^{-2}}{1 + 0.4665z^{-1} + 0.5095z^{-2}}$$

Filters (e) and (f) are shown in the right graph of Fig. 11.4.2. The vertical scales are expanded compared to those of the left graph. The horizontal lines defining the bandwidth levels $A_B = 1.114$ dB and $A_B = -0.886$ dB are also shown.

In practice, parametric EQ filters for audio have cut and boost gains that vary typically from −18 dB to 18 dB with respect to the reference gain. □

Example 11.4.2: Instead of specifying the parameters $\{\omega_0, \Delta\omega\}$, it is often convenient to specify either one or both of the corner frequencies $\{\omega_1, \omega_2\}$ that define the width $\Delta\omega = \omega_2 - \omega_1$.

Design four parametric EQ filters that have a 2.5-dB cut and bandwidth defined at 1 dB below the reference, and have center or corner frequencies as follows:

(a) Center frequency $\omega_0 = 0.6\pi$ and right corner $\omega_2 = 0.7\pi$. Determine also the left corner ω_1 and the bandwidth $\Delta\omega$.

(b) Center frequency $\omega_0 = 0.6\pi$ and left corner $\omega_1 = 0.5\pi$. Determine also the right corner ω_2 and the bandwidth $\Delta\omega$.

(c) Left and right corner frequencies $\omega_1 = 0.5\pi$ and $\omega_2 = 0.7\pi$. Determine also the center frequency ω_0.

(d) Compare the above to the standard design that has $\omega_0 = 0.6\pi$, and $\Delta\omega = 0.2\pi$. Determine the values of ω_1, ω_2.

Solution: Assuming $G_0 = 1$, the cut and bandwidth gains are:

$$G = 10^{-2.5/20} = 0.7499, \quad G_B = 10^{-1/20} = 0.8913$$

Note that G_B was chosen arbitrarily in this example and not according to Eq. (11.4.2). For case (a), we are given ω_0 and ω_2. Under the bilinear transformation they map to the values:

$$\Omega_0 = \tan(\omega_0/2) = 1.3764, \quad \Omega_2 = \tan(\omega_2/2) = 1.9626$$

Using the geometric-mean property (11.3.11), we may solve for ω_1:

$$\tan\left(\frac{\omega_1}{2}\right) = \Omega_1 = \frac{\Omega_0^2}{\Omega_2} = 0.9653 \quad \Rightarrow \quad \omega_1 = 0.4887\pi$$

Thus, the bandwidth is $\Delta\omega = \omega_2 - \omega_1 = 0.2113\pi$. The design equations (11.4.5) and (11.4.6) give then $\beta = 0.3244$ and the transfer function:

$$H_a(z) = \frac{0.9387 + 0.4666z^{-1} + 0.5713z^{-2}}{1 + 0.4666z^{-1} + 0.5101z^{-2}}$$

For case (b), we are given ω_0 and ω_1 and calculate ω_2:

$$\Omega_1 = \tan\left(\frac{\omega_1}{2}\right) = 1, \quad \tan\left(\frac{\omega_2}{2}\right) = \Omega_2 = \frac{\Omega_0^2}{\Omega_1} = 1.8944 \;\Rightarrow\; \omega_2 = 0.6908\pi$$

where Ω_0 was as in case (a). The width is $\Delta\omega = \omega_2 - \omega_1 = 0.1908\pi$. Then, we find $\beta = 0.2910$ and the transfer function:

$$H_b(z) = \frac{0.9436 + 0.4787z^{-1} + 0.6056z^{-2}}{1 + 0.4787z^{-1} + 0.5492z^{-2}}$$

The magnitude responses (in dB) of cases (a) and (b) are shown in the left graph of Fig. 11.4.3. The bandwidths are defined by the intersection of the horizontal grid line at −1 dB and the curves.

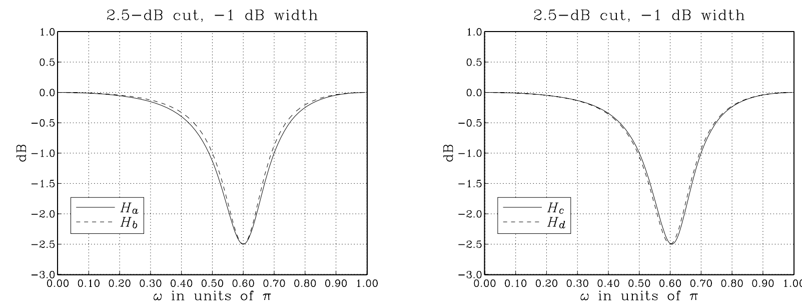

Fig. 11.4.3 Parametric EQ filters.

For case (c), we are given ω_1 and ω_2. Their bilinear transformations are:

$$\Omega_1 = \tan(\omega_1/2) = 1, \quad \Omega_2 = \tan(\omega_2/2) = 1.9626$$

The center frequency is computed from:

$$\tan(\omega_0/2) = \Omega_0 = \sqrt{\Omega_1\Omega_2} = 1.4009 \quad \Rightarrow \quad \omega_0 = 0.6053\pi$$

Using the calculated ω_0 and the width $\Delta\omega = \omega_2 - \omega_1 = 0.2\pi$, we find $\cos\omega_0 = -0.3249$, $\beta = 0.3059$, and the transfer function:

$$H_c(z) = \frac{0.9414 + 0.4976z^{-1} + 0.5901z^{-2}}{1 + 0.4976z^{-1} + 0.5315z^{-2}}$$

Finally, in the standard case (d), we start with ω_0 and $\Delta\omega$. We find $\cos\omega_0 = -0.3090$, $\beta = 0.3059$, and the transfer function:

$$H_d(z) = \frac{0.9414 + 0.4732z^{-1} + 0.5901z^{-2}}{1 + 0.4732z^{-1} + 0.5315z^{-2}}$$

With $\Omega_0 = \tan(\omega_0/2) = 1.3764$, the exact values of ω_1 and ω_2 are obtained by solving the system of equations:

$$\Omega_1\Omega_2 = \Omega_0^2 = 1.8944, \quad \Omega_2 - \Omega_1 = \Delta\Omega = (1 + \Omega_0^2)\tan(\Delta\omega/2) = 0.9404$$

which have positive solutions $\Omega_1 = 0.9843$, $\Omega_2 = 1.9247$. It follows that

$$\omega_1 = 2\arctan(\Omega_1) = 0.494951\pi, \quad \omega_2 = 2\arctan(\Omega_2) = 0.694951\pi$$

where as expected $\Delta\omega = \omega_2 - \omega_1 = 0.2\pi$. The magnitude responses are shown in the right graph of Fig. 11.4.3. Note that cases (c) and (d) have the same β because their widths $\Delta\omega$ are the same. But, the values of $\cos\omega_0$ are different, resulting in different values for the coefficients of z^{-1}; the other coefficients are the same. □

In addition to parametric equalizers with variable center frequencies ω_0, in audio applications we also need lowpass and highpass filters, referred to as "shelving" filters, with adjustable gains and cutoff frequencies. Such filters can be obtained from Eq. (11.4.5) by replacing $\omega_0 = 0$ for the lowpass case and $\omega_0 = \pi$ for the highpass one.

In the lowpass limit, $\omega_0 = 0$, we have $\cos\omega_0 = 1$ and the numerator and denominator of Eq. (11.4.5) develop a common factor $(1 - z^{-1})$. Canceling this factor, we obtain the *lowpass shelving filter*:

$$\boxed{H_{\text{LP}}(z) = \frac{\left(\dfrac{G_0 + G\beta}{1+\beta}\right) - \left(\dfrac{G_0 - G\beta}{1+\beta}\right)z^{-1}}{1 - \left(\dfrac{1-\beta}{1+\beta}\right)z^{-1}}} \tag{11.4.9}$$

where β is still given by Eq. (11.4.6), but with $\Delta\omega$ replaced by the filter's cutoff frequency ω_c and with G_B replaced by the defining level G_c of the cutoff frequency:

$$\boxed{\beta = \sqrt{\frac{G_c^2 - G_0^2}{G^2 - G_c^2}}\,\tan\left(\frac{\omega_c}{2}\right)} \tag{11.4.10}$$

In the highpass limit, $\omega_0 = \pi$, we have $\cos\omega_0 = -1$ and the numerator and denominator of Eq. (11.4.5) have a common factor $(1 + z^{-1})$. Canceling it, we obtain the *highpass shelving filter*:

$$\boxed{H_{\text{HP}}(z) = \frac{\left(\dfrac{G_0 + G\beta}{1+\beta}\right) + \left(\dfrac{G_0 - G\beta}{1+\beta}\right)z^{-1}}{1 + \left(\dfrac{1-\beta}{1+\beta}\right)z^{-1}}} \tag{11.4.11}$$

It can also be obtained from Eq. (11.4.9) by the replacement $z \to -z$. The parameter β is obtained from Eq. (11.4.6) by the replacements $G_B \to G_c$ and $\Delta\omega \to \pi - \omega_c$. The latter is necessary because $\Delta\omega$ is measured from the center frequency $\omega_0 = \pi$, whereas ω_c is measured from the origin $\omega = 0$. Noting that $\tan\big((\pi - \omega_c)/2\big) = \cot(\omega_c/2)$, we have:

$$\boxed{\beta = \sqrt{\frac{G_c^2 - G_0^2}{G^2 - G_c^2}}\,\cot\left(\frac{\omega_c}{2}\right)} \tag{11.4.12}$$

For both the lowpass and highpass cases, the filter specifications are the parameters $\{G_0, G, G_c, \omega_c\}$. They must satisfy Eq. (11.4.1) for boosting or cutting. Figure 11.4.4 depicts these specifications. Some possible choices for G_c^2 are still given by Eq. (11.4.3).

The limiting forms of the corresponding analog filter (11.4.7) can be obtained by taking the appropriate limits in the variable $\Omega_0 = \tan(\omega_0/2)$. For the lowpass case, we have the limit $\Omega_0 \to 0$ and for the highpass case, the limit $\Omega_0 \to \infty$. We must also replace $\alpha = (1 + \Omega_0^2)\beta$ before taking these limits.

Taking the limits, we obtain the analog filters whose bilinear transformations are the shelving filters Eq. (11.4.9) and (11.4.11):

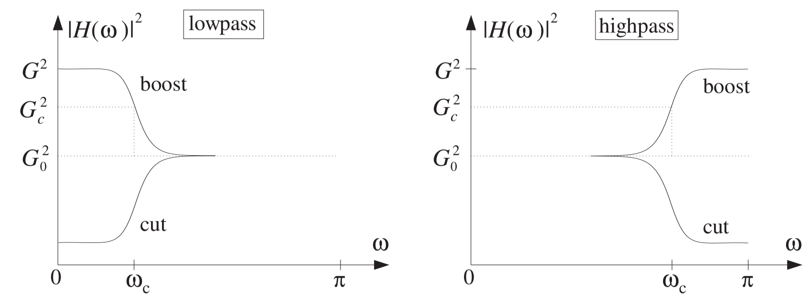

Fig. 11.4.4 Lowpass and highpass shelving filters with boost or cut.

$$H_{\text{LP}}(s) = \frac{G_0 s + G\beta}{s + \beta}\,, \quad H_{\text{HP}}(s) = \frac{G_0 + G\beta s}{1 + \beta s} \tag{11.4.13}$$

The special case $G_0 = 0$, $G = 1$ was considered in Section 11.2. In that section, the lowpass case corresponded to $\alpha = \beta$ and the highpass to $\alpha = 1/\beta$.

Setting $\Omega = \Omega_c = \tan(\omega_c/2)$, the following bandwidth conditions may be solved for β, resulting into the design equations (11.4.10) and (11.4.12):

$$|H_{\text{LP}}(\Omega)|^2 = \frac{G_0^2\Omega^2 + G^2\beta^2}{\Omega^2 + \beta^2} = G_c^2\,, \quad |H_{\text{HP}}(\Omega)|^2 = \frac{G_0^2 + G^2\beta^2\Omega^2}{1 + \beta^2\Omega^2} = G_c^2$$

11.5 Comb Filters

The lowpass and highpass shelving filters of the previous section can be turned into *periodic comb or notch filters* with adjustable gains and peak widths. This can be accomplished by the replicating transformation of Eq. (8.3.30), that is, $z \to z^D$, which shrinks the frequency response by a factor of D and replicates it D times such that D copies of it fit into the Nyquist interval. Under this transformation, the lowpass filter $H_{\text{LP}}(z)$ of Eq. (11.4.9) becomes the comb filter:

$$\boxed{H(z) = \frac{b - cz^{-D}}{1 - az^{-D}}}\,, \quad a = \frac{1-\beta}{1+\beta}\,,\; b = \frac{G_0 + G\beta}{1+\beta}\,,\; c = \frac{G_0 - G\beta}{1+\beta} \tag{11.5.1}$$

This transfer function can also be obtained from the analog lowpass shelving filter $H_{\text{LP}}(s)$ of Eq. (11.4.13) by the generalized bilinear transformation:

$$s = \frac{1 - z^{-D}}{1 + z^{-D}}\,, \quad \Omega = \tan\left(\frac{\omega D}{2}\right) = \tan\left(\frac{\pi f D}{f_s}\right) \tag{11.5.2}$$

The DC peak of the lowpass filter $H_{\text{LP}}(z)$ has full width $2\omega_c$, counting also its symmetric negative-frequency side. Under D-fold replication, the symmetric DC peak will be shrunk in width by a factor of D and replicated D times, with replicas centered at the Dth root-of-unity frequencies $\omega_k = 2\pi k/D$, $k = 0, 1, \dots, D-1$. Thus, the full width