Homework Problems – Week 4

440:127 – Spring 2015 – S. J. Orfanidis

1. End-of-Chapter Problem 5.12.
2. End-of-Chapter Problem 5.17. Plot parts (a,b,c,d) in separate figure windows, not as subplots.
3. End-of-Chapter Problem 5.20.
4. End-of-Chapter Problem 5.31.
5. This is a continuation of Problem 5.17. When you get a loan or mortgage payable in a fixed number of years, the payment amount (per payment period) can be calculated from the formula:

$$x = \frac{r y_0 (1+r)^N}{(1+r)^N - 1} \tag{1}$$

where y_0 is the loan amount and, assuming monthly payments, $r = R/12$, where R is the annual interest rate (e.g., for a 6% annual rate, $R = 0.06$ and $r = R/12 = 0.005$), and N is the total number of payment periods (e.g., $N = 12 \times 5 = 60$ months for a 5-year loan). Let y_n denote the loan balance at the end of the n-th payment period. At the next period, $n+1$, your payment of x dollars is subtracted from the present balance of y_n, but before this is done, the bank charges you interest $i_n = r y_n$, therefore, only the amount $b_n = x - r y_n$ is used to reduce your balance, so that the next balance is:

$$y_{n+1} = y_n - b_n = y_n - (x - r y_n) = (1+r) y_n - x$$

This recursion can be solved analytically, and after using Eq. (1), we obtain the solution:

$$y_n = \frac{(1+r)^N - (1+r)^n}{(1+r)^N - 1}\, y_0\,, \quad n = 0, 1, 2, \ldots, N \tag{2}$$

The solution correctly gives $y_n = y_0$ at $n = 0$, and $y_n = 0$ at $n = N$. Consider the numerical values $y_0 = 10000$, $R = 0.06$, $N = 60$.

a. Compute the payment amount x and the values of the balance y_n for $n = 0, 1, \ldots, N$, as well as the interest i_n paid at the n-th period, and the amount b_n used to reduce the balance, and using **fprintf** make a table exactly like the one shown below (you must print all 61 entries).

```
  n      yn        bn        in
---------------------------------
  0   10000.00   143.33    50.00
  1    9856.67   144.04    49.28
  2    9712.63   144.76    48.56
  3    9567.86   145.49    47.84
  4    9422.37   146.22    47.11

 ------------ etc. -------------

 56     763.74   189.51     3.82
 57     574.23   190.46     2.87
 58     383.78   191.41     1.92
 59     192.37   192.37     0.96
 60       0.00     -         -
```

Note that initially the interest payment is high but it gets smaller with time, while the amount that goes to repay the loan increases. Banks usually give you a payment schedule just like this table.

b. Using the function **sum** compute the total amount that went to re-pay the loan (i.e., the sum of the b_ns), and the total amount of interest paid.

   Make a plot of y_n versus n and observe how it is driven to zero at $n = N$.

c. The payment amount x given in Eq. (1) can be thought of as function of two independent variables N, R. Using **meshgrid** and **surf**, make a surface plot like the one shown below by assuming that N and R range over $10 \le N \le 60$ months, and $2 \le R \le 8$ annual percentage rates.

6. A pitcher throws a ball to a batter located at a distance d, with the ball being launched from height h_1 with velocity v_0 and angle θ_0, and arriving t seconds later at the batter's position at height h_2. The ball's $x-y$ trajectory (neglecting air resistance and curved balls) is described by,

$$x = v_0\, t \cos\theta_0$$
$$y = h_1 + v_0\, t \sin\theta_0 - \frac{1}{2} g t^2 \tag{3}$$

where g is the acceleration of gravity. In this problem, we wish to plot the ball's trajectory, and moreover, calculate and plot the travel time and pitch angle as functions of the pitcher's velocity v_0.

Substituting $t = x/(v_0 \cos\theta_0)$ and using the trigonometric identity, $1 + \tan^2\theta_0 = 1/\cos^2\theta_0$, we may re-write the y-equation in the following form, which shows that the trajectory is parabolic:

$$y = h_1 + x \tan\theta_0 - \frac{g\, x^2}{2 v_0^2}\left(1 + \tan^2\theta_0\right) \tag{4}$$

Given the pitcher's velocity v_0 and setting $x = d$ and $y = h_2$, we obtain two conditions from which one can calculate the required launch angle θ_0 and travel time t in terms of v_0:

$$h_2 = h_1 + d \tan\theta_0 - \frac{g\, d^2}{2 v_0^2}\left(1 + \tan^2\theta_0\right), \quad t = \frac{d}{v_0 \cos\theta_0} \tag{5}$$

The first equation is quadratic in the variable $\tan\theta_0$ and can be solved using the quadratic formula. It has two solutions which are both valid, but the one that has the smaller θ_0 leads to a faster travel time. In order for the solutions to exist as real numbers (i.e. the ball can reach the batter), it is necessary that v_0 be larger than a certain minimum value, which can be determined by setting the discriminant of the quadratic equation (5) to zero. This gives the minimum velocity:

$$v_{0,\min} = \sqrt{-gh + g\sqrt{h^2 + d^2}}, \quad h = h_1 - h_2 \tag{6}$$

a. Let $h_1 = 5$, $h_2 = 2$, $d = 60$ feet and $g = 32$ ft/sec^2. First, calculate the minimum velocity (6) in miles per hour†. Then, for the pitcher velocity of $v_0 = 50$ miles/hr, calculate the two possible launch angles θ_0 in degrees and the corresponding travel times t in seconds from Eqs. (5).

   For the two found angles θ_0 make a plot of the trajectories in Eq. (4) versus x in the range $0 \le x \le d$, placing both trajectories on the same graph.

b. Next choose a vector of equally-spaced velocity values in the interval, $40 \le v_0 \le 100$ mph, and calculate the corresponding vectors of angles θ_0 in degrees (using the fast solution), and travel times t in seconds.

   Plot θ_0 versus v_0, then, plot t versus v_0. In both cases, use units of mph for v_0 in the plots.

   Determine the speed v_z in mph at which the pitch angle goes from positive to negative, and place that point on the angle graph. Calculate also the corresponding travel time for this speed and place it on the time graph.

c. For a fast pitcher (e.g., $v_0 > 95$ mph), because the distance $d = 60$ feet is fairly short, the effect of gravity will be minimal. For the same velocity vector v_0 of part (b), calculate and plot the travel times in the case when gravity is completely ignored, but on the same graph place the travel times of part (b) for comparison.

† 1 mile = 5280 feet

Hints: $\theta_0 = \arctan\left(\dfrac{v_0^2 \pm \sqrt{v_0^4 + 2v_0^2 gh - g^2 d^2}}{gd}\right), \quad v_z = \sqrt{\dfrac{gd^2}{2h}}$

7. This is a continuation of the hoops movie example and explains the math and physics contained in the file `hoops.m`. Anyone who has shot a basketball knows that the shooting speed and shooting angle must be interelated in order to reach the hoop. Moreover, most people try to shoot at the lowest possible speed, because it makes it easier to aim.

   At time t during the flight, the x, y coordinates of the moving basketball are given by the typical projectile equations, shown below in Eq. (7), where v, θ are the initial lauch speed and angle and g is the acceleration of gravity, $g = 9.81$ m/s^2.

$$x = v\cos\theta\, t$$
$$y = x\tan\theta - \frac{1}{2}\frac{g\,x^2}{v^2\cos^2\theta} \tag{7}$$

   The condition to reach the hoop is that $y = h$ when $x = d$, which gives the speed-angle relationship:

$$d = h\tan\theta - \frac{1}{2}\frac{gd^2}{v^2\cos^2\theta} \quad\Rightarrow\quad v = \sqrt{\frac{gd/2}{\cos^2\theta\,(\tan\theta - h/d)}} \tag{8}$$

   The corresponding time of flight to reach the hoop is given by,

$$t = \frac{d}{v\cos\theta} = \sqrt{\frac{2d}{g}(\tan\theta - h/d)} \tag{9}$$

   It is clear from Eq. (8) that for v to be real-valued, the angle θ must satisfy the restriction $\tan\theta \geq h/d$, or, $\theta \geq \theta_{\min} = \arctan(h/d)$. The geometrical meaning of $\theta_{\min}$ is shown in the above figure. Clearly, if the baskeball is shot below the line connecting the launch point and the hoop, the ball will fly below the hoop, no matter the initial speed. Note also that $v = \infty$ at $\theta = \theta_{\min}$ and at $\theta = 90^{\circ}$. Therefore, there is

a minimum value of v somewhere in the range, $\theta_{\text{min}} < \theta < 90^{\text{o}}$. From calculus, the minimum of v can be determined by differentiating Eq. (8) with respect to θ and setting the derivative to zero. This leads to the optimum values for θ and v,

$$\theta_{\text{opt}} = \arctan\left[\frac{h}{d} + \sqrt{1 + \frac{h^2}{d^2}}\right], \quad v_{\text{opt}} = \sqrt{g\, d \tan\theta_{\text{opt}}} \tag{10}$$

a. Set $h = 1$ and $d = 10$ meters. Define anonymous functions $v(\theta)$ and $t(\theta)$ for Eq. (8) and (9). Plot $v(\theta)$ over the range $\theta_{\text{min}} < \theta < 90^{\text{o}}$. To avoid infinities, do the plot over the more restricted range $1.01\,\theta_{\text{min}} \le \theta \le 89.9^{\text{o}}$. [You need to exercise some care in converting the angles between radians and degrees]. Place on the graph the optimum point $\theta_{\text{opt}}, v_{\text{opt}}$. Note how broad the minimum is, which explains the fact that there is typically quite a leeway in choosing a low shooting speed and corresponding angle.

b. Plot $t(\theta)$ versus θ over the same range as in part (a), and add the point that corresponds to θ_{opt}.

c. Determine the optimum point $\theta_{\text{opt}}, v_{\text{opt}}$ in two other ways, (i) using the **min** function, and (ii) using the **fminbnd** function, and compare them with the exact values given in Eq. (10).

travel time vs. angle
t(θ)
θopt = 47.86°
t (sec)
θ (degrees)
0
2
4
6
8
10
0
10
20
30
40
50
60
70
80
90