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DIGITAL LOGIC DESIGN

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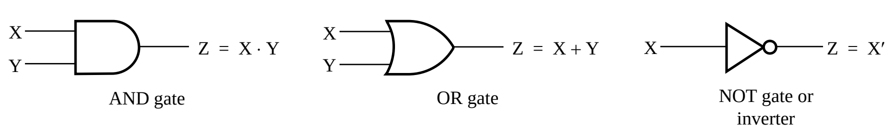
Lecture #2: Binary Number System
Complement Number Representation

Why Binary Number System?

- Because computers work with binary values
  0 & 1, LOW & HIGH, TRUE & FALSE
- Recall the basic building blocks
  -- AND, OR, NOT logic gates



- ⇒ We need to learn to work with the Binary Number System

Number Systems

A positional number system has a
radix (or base of the number) any integer $r \geq 2$

$$d_{p-1} \ldots d_k \ldots d_1 d_0 \,.\, d_{-1} \ldots d_{-j} \ldots d_{-n}$$

Most Significant Digit

r^k

radix point

r^{-j}

Least Significant Digit

$$D = \sum_{i=-n}^{p-1} d_i \cdot r^i \qquad 0 \leq d_i \leq (r-1)$$

Example: 25.375 radix 10

$$\mathbf{2} \cdot 10^1 + \mathbf{5} \cdot 10^0 + \mathbf{3} \cdot 10^{-1} + \mathbf{7} \cdot 10^{-2} + \mathbf{5} \cdot 10^{-3}$$

The integer and the fractional part are processed separately.

Powers of 2: 2^n

It will be convenient to remember these powers:

n	2^n
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024

n	2^n
–1	0.5
–2	0.25
–3	0.125

Integer Part

$$\sum_{i=0}^{p-1} d_i \cdot r^i = ((\cdots(d_{p-1} \cdot \mathbf{r} + d_{p-2})\cdot \mathbf{r} + \cdots + d_2)\cdot \mathbf{r} + d_1)\cdot \mathbf{r} + d_0$$

Divide by **r**, the remainder is $d_0, d_1, d_2, \ldots$
from the least significant to the most significant digit.

Will use r = 2, binary conversion: $d_i = \{0, 1\}$

Example: $25_{10} = ?_2$

25:2 =	12R1	$d_0 = 1$
12:2 =	6R0	$d_1 = 0$
6:2 =	3R0	$d_2 = 0$
3:2 =	1R1	$d_3 = 1$
1:2 =	0R1	$d_4 = 1$

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Integer Part

$$\sum_{i=0}^{p-1} d_i \cdot r^i = ((\cdots(d_{p-1} \cdot \mathbf{r} + d_{p-2})\cdot \mathbf{r} + \cdots + d_2)\cdot \mathbf{r} + d_1)\cdot \mathbf{r} + d_0$$

Divide by **r**, the remainder is $d_0, d_1, d_2, \ldots$
from the least significant to the most significant digit.

Will use r = 2, binary conversion: $d_i = \{0, 1\}$

Example: $25_{10} = ?_2$

25:2 =	12R1	$d_0 = 1$	← LSB (Least Significant Bit)
12:2 =	6R0	$d_1 = 0$	
6:2 =	3R0	$d_2 = 0$	
3:2 =	1R1	$d_3 = 1$	
1:2 =	0R1	$d_4 = 1$	← MSB (Most Significant Bit)

$\Rightarrow \boxed{25_{10} = \mathbf{11001}_2}$

Verify the result: $\mathbf{1}\cdot 2^4 + \mathbf{1}\cdot 2^3 + \mathbf{1}\cdot 2^0 = 16 + 8 + 1 = 25$

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Fractional Part

$$\sum_{i=-n}^{-1} d_i \cdot r^i = r^{-1} \cdot (d_{-1} + r^{-1} \cdot (d_{-2} + \cdots)))$$

Same like before, but now we *multiply* with the radix.

Example: $0.375_{10} = ?_2$

$$\begin{array}{lccl} 0.375 \times 2 = 0.750 & < 1 & \Rightarrow & d_{-1} = 0 \\ 0.750 \times 2 = 1.500 & > 1 & \Rightarrow & d_{-2} = 1 \\ 0.500 \times 2 = 1.000 & & \Rightarrow & d_{-3} = 1 \end{array}$$

Fractional Part

$$\sum_{i=-n}^{-1} d_i \cdot r^i = r^{-1} \cdot (d_{-1} + r^{-1} \cdot (d_{-2} + \cdots)))$$

Same like before, but now we *multiply* with the radix.

Example: $0.375_{10} = ?_2$

$$\begin{array}{lcclr} 0.375 \times 2 = 0.750 & < 1 & \Rightarrow & d_{-1} = 0 & \leftarrow \text{MSB} \\ 0.750 \times 2 = 1.500 & > 1 & \Rightarrow & d_{-2} = 1 & \\ 0.500 \times 2 = 1.000 & & \Rightarrow & d_{-3} = 1 & \leftarrow \text{LSB} \end{array}$$

$\Rightarrow \boxed{0.375_{10} = \mathbf{0.011}_2}$

Verify the result: $1 \cdot 2^{-2} + 1 \cdot 2^{-3} = 0.25 + 0.125 = 0.375$

Important Radices

Radices important to computer engineers are: r = **2**, **8**, **16**

Binary	Decimal	Octal	3-Bit String	Hexadecimal	4-Bit String
0	0	0	000	0	0000
1	1	1	001	1	0001
10	2	2	010	2	0010
11	3	3	011	3	0011
100	4	4	100	4	0100
101	5	5	101	5	0101
110	6	6	110	6	0110
111	7	7	111	7	0111
1000	8	10	—	8	1000
1001	9	11	—	9	1001
1010	10	12	—	A	1010
1011	11	13	—	B	1011
1100	12	14	—	C	1100
1101	13	15	—	D	1101
1110	14	16	—	E	1110
1111	15	17	—	F	1111

Example: $11100001.011_2 = \underbrace{011}\,\underbrace{100}\,\underbrace{001}\,.\,\underbrace{011}_2 = 341.3_8$

$341.3_8 = 3 \cdot 8^2 + 4 \cdot 8^1 + 1 \cdot 8^0 + 3 \cdot 8^{-1} = 225.375_{10}$

Fourth digit was added to the fractional part

$11100001.011_2 = 1110\ 0001\ .\ 0110_2 = E1.6_{16}$

$E1.6_{16} = 14 \cdot 16^1 + 1 \cdot 16^0 + 6 \cdot 16^{-1} = 225.375_{10}$

Binary Addition

c_{in}	x	y	c_{out}	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

s = sum
c_{in} = carry in
c_{out} = carry out

X, Y, $C_{in} \rightarrow$ s, C_{out}

Decimal:

$1_{10} + 1_{10} = 2_{10}$

Binary:

$1_2 + 1_2 = 10_2$

$1_2 + 1_2 = 10_2$

sum

carry out

$X + Y + C_{in} = s$	C_{out}
0 + 1 + 0 = 1	0
1 + 0 + 1 = 0	1
1 + 1 + 1 = 1	1

Binary Addition

c_{in}	x	y	c_{out}	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

c_{in} = carry in
c_{out} = carry out

X, Y, C_{in} → s, C_{out}

augend
\+ addend
= sum

Example addition:

X 1 1 0 1 1 0
\+ Y 1 0 1 1 1
\+ carries
sum =

X 1 1 0 1 1 0
\+ Y 1 0 1 1 1
\+ carries 1 1
sum = 1 1 0 1

X 1 1 0 1 1 0
\+ Y 1 0 1 1 1
\+ carries 1 1 1 1
sum = 1 0 0 1 1 0 1

Subtraction

Decimal:

25_{10}
$- 7_{10}$

borrow 10_{10} from the next leftward digit

15
25
– 7
= 8

borrow

25
– 17
= 18

minuend
subtrahend
difference

Binary:

100_2
$- 1_2$

borrow 10_2 $(=2_{10})$ from the next leftward digit

10
100
– 1
= 1

because $10_2 - 1_2 = 1_2$

borrow

10
100
– 11
= 11

borrow

100
– 1 1
= 011

Binary Subtraction

b_{in}	x	y	b_{out}	d
0	0	0	0	0
0	0	1	1	1
0	1	0	0	1
0	1	1	0	0
1	0	0	1	1
1	0	1	1	0
1	1	0	0	0
1	1	1	1	1

b_{in} = borrow in
b_{out} = borrow out

$X, Y, B_{in} \rightarrow s, B_{out}$

$X - Y - B_{in} = d$	B_{out}
$0 - 1 - 0 = 1$	1
$1 - 0 - 1 = 0$	1
$1 - 1 - 1 = 1$	0

Example Subtraction (1)

Must borrow 1, yielding the new subtraction $10_2 - 1_2 = 1_2$

After the first borrow, the new subtraction for the column is (1 – **1**) – 1, so we must borrow again.

The borrow ripples through leftwards until there is a non-zero digit from which to borrow.

minuend	X	229	1 1 1 0 0 1 0 1
subtrahend	Y	– 46	– 0 0 1 0 1 1 1 0
borrows			1 1 1 1 1
difference	X – Y	183	1 0 1 1 0 1 1 1

10_2 10_2 10_2 10_2 10_2

Example Subtraction (2)

Verify the result:

$$
\begin{array}{r}
110110 \\
+\ \ 10111 \\
\hline
1001101
\end{array}
$$

minuend	X	77
subtrahend	Y	− 23
difference	X – Y	54

$$
\begin{array}{cccccccc}
 & & 10_2 & 10_2 & & 10_2 & 10_2 & \\
 & 1 & \cancel{0} & \cancel{0} & 1 & \cancel{1} & \cancel{0} & 1 \\
- & & & 1 & 0 & 1 & 1 & 1 \\
\text{borrows} & \boxed{1} & \boxed{1} & & \boxed{1} & \boxed{1} & & \\
\hline
 & & 1 & 1 & 0 & 1 & 1 & 0
\end{array}
$$

Signed-Magnitude Representation

Use the MSB for the sign:

n bits:

$\underbrace{d_{n-1}}_{\text{sign}}\ \underbrace{d_{n-2} \cdots d_0}_{\text{magnitude}}$

$d_{n-1} = 0 \quad \leftarrow \quad$ positive number

$d_{n-1} = 1 \quad \leftarrow \quad$ negative number

n bits represent 2^n numbers

largest positive number: $0\overbrace{11\cdots1}^{n-1 \text{ bits}}$ $\quad \sum_{i=0}^{n-2} 1 \cdot 2^i = 2^{n-1} - 1$

smallest negative number: $111\cdots1$ $\quad -(2^{n-1} - 1)$

two representations for zero: $000\cdots0$ and $100\cdots0$

Signed-Magnitude Arithmetic

Arithmetic operations must process the sign separately.
For example, subtraction: $A - B$

1. Compare the magnitudes $A \overset{?}{\lesseqgtr} B$
2. Subtract smaller magnitude from larger magnitude
3. If $B > A$, then change the sign of the result

… too complicated … will NOT use it for computations.

Instead, we use two's complement representation …

Radix-Complement Representation

Assumptions:

- fixed number of digits, n
- $D = d_{n-1} \dots d_k \dots d_1 d_0$, radix r

radix-complement representation of D:

$$[D]_r = r^n - D$$

The involution property: $[[D]_r]_r = r^n - (r^n - D) = D$

How to compute it?
(would like to avoid subtraction)

Radix-Complement Computation

$$[D]_r = r^n - D$$

- Rewrite $r^n = (r^n - 1) + 1$

  Then, $[D]_r = r^n - D = ((r^n - 1) - D) + 1$

- Observe that $(r^n - 1)$ has the form $\underbrace{mm \ldots mm}_{n}$ where $m = r - 1$

  For example, for $r = 10$ and $n = 4$, $(r^n - 1) = 9999$
  for $r = 2$ and $n = 5$, $(r^n - 1) = 11111$

- Define the $\boxed{\textit{complement}}$ of a digit d to be $d'_r = r - 1 - d$

  For example, for $r = 10$, the complements of 3, 5, and 8 are
  $3'_{10} = 10 - 1 - 3 = 6$ $\quad 5'_{10} = 10 - 1 - 5 = 4$ $\quad 8'_{10} = 10 - 1 - 8 = 1$

- Then, the complement of D is obtained by complementing individual digits of D and adding 1

2's-Complement Representation

n-bit 2's-complement representation of D:

$$[D]_2 = 2^n - D_2$$

Compute two's complement as:

$$[D]_2 = (2^n - 1 - D_2) + 1$$

$$\begin{array}{rl} 2^n - 1: & 1 \quad 1 \quad \cdots \; 1 \quad \leftarrow \text{ n bits} \\ -\,D: & -\,d_{n-1}d_{n-2} \cdots d_0 \\ \hline & d'_{n-1}d'_{n-2} \cdots d'_0 \\ & + \qquad\qquad 1 \\ \hline & [D]_2 \end{array}$$

$$1 - d_i = d'_i \quad \begin{array}{l} \nearrow 1 - 0 = 1 \\ \searrow 1 - 1 = 0 \end{array}$$

2's-Complement Computation

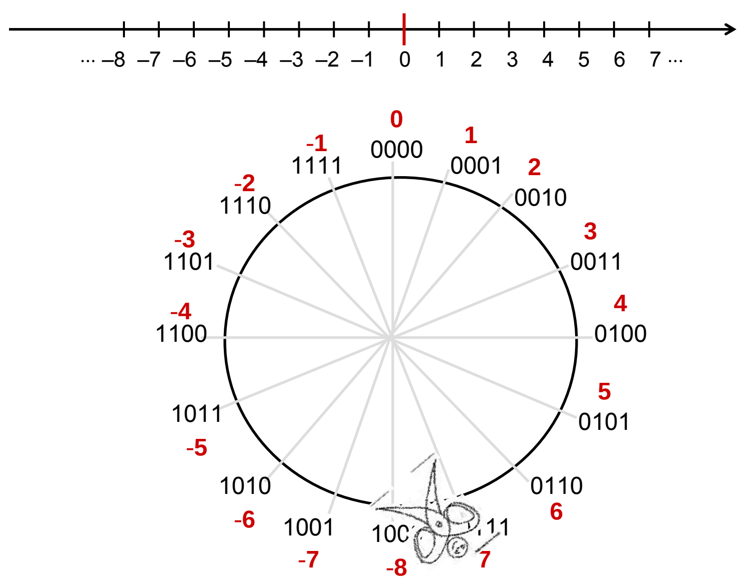
1. Complement the digits
2. Add 1 to the Least Significant Bit
3. Discard carry out from Most Significant Bit

$$[D]_2 = (2^n - 1 - D_2) + 1$$

$$\begin{array}{rr}
2^n - 1: & 1 \quad 1 \quad \cdots \; 1 \quad \leftarrow \text{n bits} \\
- D: & - d_{n-1} d_{n-2} \cdots d_0 \\
\hline
 & d'_{n-1} d'_{n-2} \cdots d'_0 \\
 & + \qquad\qquad 1 \\
\hline
 & [D]_2
\end{array}$$

$1 - d_i = d'_i$: $1 - 0 = 1$, $1 - 1 = 0$

Two's Complement Number System



2's-Complement Representation (2)

Range of n-bit 2's complement: $-2^{n-1} \le A \le 2^{n-1} - 1$

Example: n = 5
represent -13_{10} in 2's complement:

$$\begin{array}{rr} 13_{10} = 01101_2 \rightarrow & 10010 \\ & 1 \\ \hline & 10011 \end{array} = -13_{10}$$

What decimal number is represented in 5-bit 2's complement:
11010

?

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2's-Complement Representation (2)

Range of n-bit 2's complement: $-2^{n-1} \le A \le 2^{n-1} - 1$

Example: n = 5
represent -13_{10} in 2's complement:

$$\begin{array}{rr} 13_{10} = 01101_2 \rightarrow & 10010 \\ & 1 \\ \hline & 10011 \end{array} = -13_{10}$$

What decimal number is represented in 5-bit 2's complement:

11010

negative number — for magnitude: 00101

complement the digits and add 1

$$\begin{array}{r} 00101 \\ +\quad 1 \\ \hline 00110 \end{array} \leftarrow \text{ that is } 6_{10}$$

so the number is: $\mathbf{-6_{10}}$

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So, what is this number?

$1011001_2 = ?_{10}$

Answer: depends on the representation!

Unsigned: $1011001_2 = 89_{10}$

Signed-magnitude: $1011001_2 = -25_{10}$

Two's complement: $1011001_2 = -39_{10}$