EEG Sparse Source Localization
via Range Space Rotation

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Abstract—The problem of sparse Electroencephalography (EEG) source localization can be formulated as a sparse signal recovery problem. However, the dictionary matrix (Lead Field) of a realistic head model has high coherence, indicating that the sparse signal, corresponding to brain activations might not be recoverable via \(\ell_1\)-norm minimization techniques. In spite of the high coherence in the EEG dictionary matrix, we can still estimate the support of the underlying source signal as long as the problem satisfies the Range Space Property (RSP). In this paper, we show that one can use an initial estimate of the sparse solution to rotate the range of the sensing matrix transpose and obtain high quality source localization. We derive the conditions which the rotation matrix should meet in order to make the unique least \(\ell_1\)-norm solution support match the actual source support. We validate the proposed approach using simulations and a real EEG experiment, and compare the results with those obtained by other methods that have been previously proposed for EEG source localization.

I. INTRODUCTION

EEG based source localization is important for clinical purposes. EEG is less expensive as compared to other functional brain imaging techniques, such as Functional Magnetic Resonance Imaging (fMRI) [1], and even provides better temporal resolution as compared to fMRI. The main drawback of EEG as compared to fMRI is its low spatial resolution. The process of source localization suffers from many problems including, low Signal to Noise Ratio (SNR), and noise that is spatially correlated and temporarily non-stationary. Existing EEG based source localization techniques include the MUltiple Signal Classification (MUSIC) [2], and the Linearly Constrained Minimum Variance (LCMV) methods [3], which rely on second-order statistics of the data. However, it is hard to obtain good estimates of those statistics due to the non-stationarity of the data [4]. Another class of source localization methods is the class that fits the observations to a linear system model, i.e., \(y = Ax + \text{noise}\), where \(A\) is the lead field matrix (see description in Section III), and \(x\) is the source vector. The small number of obtained recordings at a given time, as compared to the internal mesh size of the brain [5] makes the source estimation problem under-determined with infinitely many solutions [6]. However, at a given time a small number of sources inside the brain are active above a certain threshold, thus making \(x\) a sparse vector. EEG source localization methods that exploit the sparsity of \(x\) include \(\ell_1\)-norm minimization [7]–[10], FOCUSS [11], Matching Pursuit (MP) [12], Order Recursive Matching Pursuit (ORMP) [12], and Source Deflated Matching Pursuit (SDMP) [13].

When viewing the source estimation as a sparse recovery problem, one typically employs \(\ell_1\)-norm minimization, hoping that there is a unique least \(\ell_1\)-norm solution, which is also the unique least \(\ell_0\)-norm solution (the unique sparsest solution); in other words, hoping that the \(\ell_1\)-norm minimization problem and the \(\ell_0\)-norm minimization problem are strictly equivalent. Conditions for the latter to hold include the mutual coherence [14], the RIP [15], and the null space property [16]. However, in many cases, those conditions do not hold. For example, the coherence histogram of a real head lead field matrix obtained from BrainStorm [17] is shown in Fig. 1 (the diagonal elements are excluded), where one can see that approximately 15% of columns exhibit correlation factor bigger than 0.7. The recently introduced Range-Space Property (RSP) and full rank property [18] conditions refer to cases in which the least \(\ell_0\)-norm solution is not unique, and provide the conditions for one of the sparsest solutions to be the unique least \(\ell_1\)-norm solution (see Theorem 1 of this paper).

In this paper, we address the problem of recovering the source vector from EEG recordings via \(\ell_1\)-norm minimization, where, due to the high lead filed matrix coherence, there is no guarantee that this minimization will yield the desired source vector. In particular, we show that by weighting the \(A\) matrix with an appropriately designed diagonal matrix \(W\), we can transform the original problem into a problem that satisfies Theorem 1 on a vector that has the same support as the source vector. Thus, the least \(\ell_1\)-norm solution of the transformed problem will indicate the locations of the activated sources.

II. BACKGROUND THEORY

The sparsest solution of the underdetermined problem \(y = Ax\) can be obtained by solving the \(\ell_0\)-norm minimization problem, i.e.,

\[
\min \{ \|x\|_0 \} \quad \text{s.t.} \quad y = Ax \tag{1}
\]
However, since this is an NP-hard problem, its $\ell_1$ approximation is used [14], [15], [19], i.e.,

$$\min \{ \|x\|_1 \} \quad \text{s.t.} \quad y = Ax \quad (2)$$

Under certain conditions, the $\ell_1$ and $\ell_0$ problems are strongly equivalent, i.e., there is a unique solution to the problem of (1), which coincides with the unique solution of the problem of (2) [11], [14]–[16]. Sometimes, (1) has many sparsest solutions. When one of those sparsest solution coincides with the unique solution of the problem of (2), then, the problems of (2) and (1) are referred to as equivalent [18].

Recent work [18] focusing on the case in which a unique solution to the $\ell_0$-norm minimization problem does not exist, provides a set of conditions so that a sparsest solution is the unique solution of the $\ell_1$-norm minimization problem. These conditions are given in the following theorem.

**Theorem 1** Let $x \in \mathbb{R}^n$ be a sparsest solution to the system $Ax = y$. Then $x$ is the unique least $\ell_1$-norm solution to this system if and only if $x$ satisfies the following: [18]

(i): There exists a vector $u$ such that:

$$\begin{cases} 
  u \in \mathbb{R}(A^T) \\
  u_i = 1 \quad x_i > 0 \\
  u_i = -1 \quad x_i < 0 \\
  |u_i| < 1 \quad x_i = 0
\end{cases}$$

(ii): The matrix $[A_+ \ A_-]$ has full column rank where $A_+$ are the columns associated with positive $x$, $A_-$ are the columns in $A$ that associated with negative $x$.

The first condition is called the Range Space Property (RSP), and the second condition is called the full rank property.

### III. THE PROPOSED APPROACH

In this paper, the current-based dipole model was adopted, in which the activated sources are modeled as dipoles [20]. By segmenting the cortex into $m$ nodes, a dipole vector, called the lead field vector (LFV), is assigned to each node. With $n$ number of electrodes on the scalp, at given time instant $t$, the EEG model can be described as

$$y(t) = Ax(t) + n_o(t) \quad (3)$$

where $y(t) \in \mathbb{R}^n$ represents the electrode readings at time instant $t$, $x(t) \in \mathbb{R}^{3m}$ denotes the dipole source vector at time instant $t$, $A \in \mathbb{R}^{n \times 3m}$ is the lead field matrix and $n_o(t) \in \mathbb{R}^n$ denotes the noise vector. It is clear that the system described by (3) is under-determined, i.e., for the same electrode readings $y(t)$, infinite solutions for $x(t)$ can be obtained.

Here we only consider activations in the cortical regions. In response to simple tasks, a small number of cortical nodes can be assumed to be activated, thus $x(t)$ has a sparse structure, and therefore, sparse signal recovery theory can be used.

As it was already noted, the lead field matrix has high coherence. Thus, there is no guarantee that applying $\ell_1$-norm solution will solve for the underlying sparse vector. In the following, we show that by weighting matrix $A$ with a diagonal matrix $W$, formed based on an approximation of the actual solution, we transform the original problem into a problem that satisfies Theorem 1 on a vector that has the same support as the true solution. Thus, the least $\ell_1$-norm solution of the transformed problem will indicate the locations of the activated sources.

We now provide a sufficient condition for Theorem 1 to hold. In the following, $\prec$ represents element-wise less than, $\succ$ represents element-wise greater than, and $(\cdot)^+$ represents the Moore-Penrose pseudoinverse.

**Theorem 2** If for the sparsest solution $x$ it holds that $|A_{j0}^T(A_{j0}^T)^+ u_k| = 1$, then $x$ is the unique least $\ell_1$-norm solution to the problem $y = Ax$, where $1$ is a vector of ones of appropriate size, $A_{j0} = [A_+ \ A_-]$, $A_{j0}$ are the columns of $A$ associated with zero values in $x$, and $u_k^T = [u_p, u_n]^T$ where $u_p$ and $u_n$ are row vectors corresponding to positive and negative $x_i$'s respectively.

**Proof:** See Appendix.

Let $W$ be a diagonal matrix, which is nonzero over the support of $x$, such that $x = Wq$. Eq. (1) can be written as

$$\min \{ \|Wq\|_0 \} \quad \text{s.t.} \quad y = AWq \quad (4)$$

Since $\|Wq\|_0 = \|q\|_0$, we can write (4) as:

$$\min \{ \|q\|_0 \} \quad \text{s.t.} \quad y = AWq \quad (5)$$

The corresponding $\ell_1$-norm minimization problem becomes:

$$\min \{ \|q\|_1 \} \quad \text{s.t.} \quad y = AWq \quad (6)$$

If we obtain the solution of the problem of 6, i.e., $q$, we will be able to determine the support of $x$, as $x$ and $q$ have the same support. Next, we show how to select $W$, so that problem (6) satisfies Theorem 2 and thus a sparsest solution of (5) is the unique solution to problem (6).

For the discussion below, we will assume that the observation is a linear combination of independent columns of $A$.

Based on Theorem 2, we need to show that we can find a $W$ such that

$$|W_{j0}A_{j0}^T(A_{j0}^T)^+ W_{j0}^{-1} u_k| = 1, \quad (7)$$

where $W_{j0}$ and $A_{j0}$ present the corresponding elements of $W$ that are associated with zero and non-zero elements of $x$, respectively. The above equation can be rewritten as:

$$|w_{j0}A_{j0}^T(A_{j0}^T)^+ W_{j0}^{-1} u_k| = 1, \quad (8)$$

where $w_{j0}$ represents the $i^{th}$ diagonal element of $W_{j0}$, $A_{j0}$ is the $i^{th}$ column of $A_{j0}$, $i = 1, 2, ..., r$, and $r$ is the number of zero elements in $x$.

It can be easily seen that if one selects the $W$ matrix such that its $w_{j0}$ elements are close to zero, and its $W_{j0}$ elements have large values, (8) will hold, and thus Theorem 2 will hold. An example of a good $W$ would be $W = \text{diag}(|q|)$, where $q_0 = \{ 1 \ x_0 \neq 0 \}$, or $W = [\text{diag}(|x_0|)]$, where $x_0$ is the solution of the original problem. In Theorem 2 and (7), it is obvious that there are infinitely many weighting matrices that can solve for the true solution. All we need is an estimate of the true solution.

Based on the above discussion, one can find the support of the source vector by selecting $W$ based on an estimate of the source vector, and then solve the problem of (6). A coarse estimate of the desirable solution can be obtained via
one of the existing methods, such as MUSIC [2], LCMV [3], LORETTA [21], etc. To account for noise in the observations, we can set the problem to minimize the tradeoff between the objective function \( \| q \|_1 \) and the fitting error \( \| y - AWq \|_2 \), i.e., \( \min \{ \| q \|_1 + \lambda \| y - AWq \|_2 \} \), where \( \lambda = \frac{\alpha}{\sigma^2} \), \( \sigma^2 \) is the noise variance, and \( \alpha \) is a positive constant.

In [22], a recursive re-weighted \( \ell_1 \)-norm algorithm was proposed to enhance the sparsity of the solution of the least \( \ell_1 \)-norm problem. The algorithm takes a non-convex approach; there is no guarantee for equivalence between \( \ell_0 \)- and \( \ell_1 \)-norm problems, and no guarantee of convergence. Our proposed algorithm addresses these issues, by minimizing a convex function with convergence guarantee, and by providing equivalence guarantee between \( \ell_0 \)- and \( \ell_1 \)-norm solutions by choosing a weighting matrix \( W \) that satisfies Theorem 2.

IV. SIMULATION RESULTS

In this section, we conduct simulations based on the EEG sensing matrix obtained from the BrainStom software [17]. The “success rate” is used as performance metric. For \( k \) sources, we declare an estimation to be successful if the distance between the locations of the \( k \) largest estimated sources and the locations of the actual sources is less than a predefined threshold value \( d \). MUSIC estimation is used to construct the initial weighting matrix for FOCCUS and our proposed algorithm. The number of snapshots that is used in MUSIC estimation is 50. We perform 100 Monte Carlo runs for each case. In each iteration, \( k \) sources are placed at random locations and white noise is added to the observations.

The threshold for distance \( d \) is set to 1 cm. Fig. 2 shows the performance of our proposed algorithm, FOCCUS, ORMP and MUSIC versus SNR for the case of two sources. One can see that our algorithm is the most robust as compared to the other methods. Fig. 3 compares the performance of all methods for different number of sources, where the proposed algorithm shows the best success rate compared to FOCCUS, MUSIC, and ORMP as the number of sources increases. We also tested the proposed algorithm on real EEG data. EEG experiments for eliciting auditory evoked potentials (AEPs) [13] were conducted with one volunteer who had provided his written informed consent. The stimulus was a pure tone of 1000 Hz that was presented to the left ear of the participant for the duration of 100 ms. The inter-stimulus interval (ISI) was set to be 700 ms. The experiment lasted for 15 minutes (476 trials with stimuli, and 540 control trials with no 1 KHz tone). Brain potentials were recorded using a 64-channel EEG system (Brain Products, Germany). The collected recordings were preprocessed using Matlab Toolbox EEGLAB [23] and then averaged across trials. The resulting AEP waveforms are shown in Fig. 4. For this experiment, activations in both left and right primary auditory cortices are expected to occur [13], [24]. To estimate the location of activities, data corresponding to the peak range of AEP, over the duration of [126 – 158] ms, was used in the MUSIC algorithm which was used as the weighting matrix for our proposed algorithm and FOCCUS. Modified ORMP is used in the experimental estimation [13]; the time samples used in ORMP are between 148 ms to 152 ms. Before applying the algorithms, the AEP was pre-whitening using control trials. Fig. 5 shows the estimated source locations using the proposed algorithm (red circles), FOCCUS (blue squares) and average estimated sources from 148 ms to 152 ms for ORMP (green diamonds). For ORMP, we restricted the number of sources (sparsity level) to 4. As shown in Fig. 5, our proposed algorithm shows activations in temporal and parietal regions, ORMP has captured activities in temporal, parietal and occipital lobes, while FOCCUS shows strong activity in many regions. As shown in Fig. 5 our algorithm recovers a sparser signal compared to FOCCUS and ORMP.
FOCUSS (blue squares), and ORMP (green diamonds), where the symbol size is proportional to its magnitude.

Fig. 5. The Estimated Sources Using the Proposed Algorithm (red circles), FOCUSS (blue squares), and ORMP (green diamonds), where the symbol size is proportional to its magnitude.

V. CONCLUSIONS

An EEG source localization algorithm has been proposed in this paper. We have shown that the weighting matrix has the effect of rotating the range space of the transposed sensing matrix to make the \( \ell_1 \) unique minimization solution match the desirable \( \ell_0 \)-norm solution. The impact of SNR and number of sources on the performance of the proposed algorithm were explored through Monte Carlo simulations, and results were compared with those of FOCUSS and ORMP. The proposed approach appears to be more robust in terms of SNR as compared to FOCUSS, and in terms of ORMP. The proposed approach appears to be more robust in terms of SNR as compared to FOCUSS, and in terms of ORMP. The proposed approach appears to be more robust in terms of SNR as compared to FOCUSS, and in terms of ORMP. We have validated the performance of the algorithm using real EEG data involving auditory stimuli.

APPENDIX

To prove Theorem 2 we will prove its contrapositive, i.e., we will show that if a solution does not satisfy Theorem 1, then \( |A_{J0}^T A_{J0}^{-1} u_s| \geq 1 \).

Let \( x \) satisfy all RSP conditions except the fourth of Theorem 1(i). Define \( v \) such that \( A^T v = u \). The above conditions can be rewritten as

\[
A^T_{J_0} v = 1, A^T_{J} v = -1, |A^T_{J0} v| > 1
\]  

(9)

The first and the second conditions in (9) can be grouped to get an under-determined system, which can be written as:

\[
A^T_{J} v = u \text{ where: } A_s = [ A_{J+}, A_{J-} ] , u_s = [1, -1]^T
\]

The above system is an under-determined system, and the least square solution \( v_{LS} = (A^T_{J})^T u_s \) is a solution to the above system.

Substituting \( v_{LS} \) in the third condition of (9), we get:

\[
|A^T_{J0} (A^T_{J})^T u_s| > 1.
\]

This completes the proof of Theorem 2.

REFERENCES


