9.6 Correlation of Discrete-Time Signals

A signal operation similar to signal convolution, but with completely different physical meaning, is signal correlation. The signal correlation operation can be performed either with one signal (autocorrelation) or between two different signals (crosscorrelation). Physically, signal autocorrelation indicates how the signal energy (power) is distributed within the signal, and as such is used to measure the signal power. Typical applications of signal autocorrelation are in radar, sonar, satellite, and wireless communications systems. Devices that measure signal power using signal correlation are known as signal correlators. There are also many applications of signal crosscorrelation in signal processing systems, especially when the signal is corrupted by another undesirable signal (noise) so that the signal estimation (detection) from a noisy signal has to be performed. Signal crosscorrelation can be also considered as a measure of similarity of two signals.

Definition 9.3: Discrete-Time Autocorrelation and Crosscorrelation

Given two discrete-time real signals (sequences) $x[k]$ and $y[k]$. The *autocorrelation and croosscorrelation* functions are respectively defined by

$$R_{xx}[k] = \sum_{m=-\infty}^{\infty} x[m]x[m-k], \qquad R_{yy}[k] = \sum_{m=-\infty}^{\infty} y[m]y[m-k]$$

$$R_{xy}[k] = \sum_{m=-\infty}^{\infty} x[m]y[m-k], \qquad R_{yx}[k] = \sum_{m=-\infty}^{\infty} y[m]x[m-k]$$

where the parameter k is any integer, $-\infty \le k \le \infty$.

Using the definition for the total discrete-time signal energy, we see that for $k = 0$, the autocorrelation function represents the total signal energy, that is

$$R_{xx}[0] = E_{\infty}^{x}, \qquad R_{yy}[0] = E_{\infty}^{y}$$

Naturally, the autocorrelation and crosscorrelation sums are convergent under assumptions that the signals $x[k]$ and $y[k]$ have finite total energy. It can be observed that $R_{xx}[k] \leq R_{xx}[0] = E_{\infty}^{x}$. In addition, it is easy to show that the autocorrelation function is an even function, that is

$$R_{xx}[k] = R_{xx}[-k]$$

Hence, the autocorrelation function is symmetric with respect to the vertical axis. Also, it can shown that

$$R_{xy}[k] = R_{yx}[-k]$$

(see Problem, 9.29).

Problem 9.29

Using the change of variables as $m = n + k$ in the definition formula for the auto-correlation function, we obtain the required result as follows

$$R_{xx}[k] \triangleq \sum_{m=-\infty}^{\infty} x[m]x[m-k] = \sum_{n=-\infty}^{\infty} x[n+k]x[n+k-k]$$

$$= \sum_{n=-\infty}^{\infty} x[n]x[n+k] \triangleq R_{xx}[-k]$$

Using the change of variables as $m = n + k$ in the definition formula for the cross-correlation function, we have

$$R_{xy}[k] \triangleq \sum_{m=-\infty}^{\infty} x[m]y[m-k] = \sum_{n=-\infty}^{\infty} x[n+k]y[n+k-k]$$

$$= \sum_{n=-\infty}^{\infty} y[n]x[n+k] \triangleq R_{yx}[-k]$$

Note that the above defined autocorrelation and crosscorrelation functions are applicable to the energy signals (they have finite total energy). The autocorrelation and crosscorrelation functions can be also defined for power signals (they have infinite energy, but finite power). In such a case, we have to redefine these sums along the definition formula of the discrete-time signal *average* power.

It is interesting to observe that the autocorrelation and cross correlation functions can be evaluated using the discrete-time convolution as follows

$$\boldsymbol{R_{xx}[k] = x[k] * x[-k], \qquad R_{xy}[k] = x[k] * y[-k]}$$

It is left to students as an exercise to establish these results, Problem 9.30.

Problem 9.30

Using the change of variables as $m = k - n$ in the definition formula for the auto-correlation function, we have

$$R_{xx}[k] \triangleq \sum_{m=-\infty}^{\infty} x[m]x[m-k] = \sum_{n=-\infty}^{\infty} x[k-n]x[k-n-k]$$

$$= \sum_{n=-\infty}^{\infty} x[k-n]x[-n] \triangleq x[k] * x[-k]$$

Using the change of variables as $m = n + k$ in the definition formula for the cross-correlation function, we obtain

$$R_{xy}[k] \triangleq \sum_{m=-\infty}^{\infty} x[m]y[m-k] = \sum_{n=-\infty}^{\infty} x[k-n]y[k-n-k]$$

$$= \sum_{n=-\infty}^{\infty} y[-n]x[k-n] \triangleq x[k] * y[-k]$$

As a measure of similarity of two signals, we can use the *correlation coefficient* defined by

$$c_{xy} = \frac{R_{xy}[0]}{\sqrt{R_{xx}[0]R_{yy}[0]}}$$

Note the correlation coefficient satisfies $-1 \leq c_{xy} \leq 1$. This can be established by observing the fact that $R_{xy}[0] = \mathrm{x} \cdot \mathrm{y}$ is the inner product of the vectors that contain respectively the samples of $x[k]$ and $y[k]$. Similarly, the relations $R_{xx}[0] = \mathrm{x} \cdot \mathrm{x} = |\mathrm{x}|^2$ and $R_{yy}[0] = \mathrm{y} \cdot \mathrm{y} = |\mathrm{y}|^2$ represent the square of the corresponding vector Euclidean norm so that the correlation coefficient geometrically represents the angle between the vectors x and y, that is

$$-1 \leq c_{xy} = \frac{R_{xy}[0]}{\sqrt{R_{xx}[0]R_{yy}[0]}} = \frac{\mathrm{x}\cdot\mathrm{y}}{\sqrt{|\mathrm{x}|^2|\mathrm{y}|^2}} = \frac{\mathrm{x}\cdot\mathrm{y}}{|\mathrm{x}||\mathrm{y}|} \triangleq \cos(\mathrm{x},\mathrm{y}) \leq 1$$

When the correlation coefficient is close to one then the signals are similar (they almost overlap) and when the correlation coefficient is close to zero the signals are very different (orthogonal as the matter of fact). When the correlation coefficient is close to minus one the signals are asimilar (opposite direction, but almost the same sample values). Note that the correlation coefficient can be also defined in terms of parameter k, that is

$$-1 \leq c_{xy}[k] = \frac{R_{xy}[k]}{\sqrt{R_{xx}[0]R_{yy}[0]}} \leq 1$$

in which case the same lower and upper bounds hold due to the fact that $|R_{xy}[k]| \leq \sqrt{|R_{xx}[0]||R_{yy}[0]|}$, see Problem 9.31.

Problem 9.31

Using the fact that the sum a nonnegative numbers is nonnegative, that is, staring with the following sum

$$\sum_{m=-\infty}^{\infty} \{x_1[m] - \alpha x_2[m-k]\}^2 \geq 0$$

valid for *any real* α, we have

$$\sum_{m=-\infty}^{\infty} x_1^2[m] + \alpha^2 \sum_{m=-\infty}^{\infty} x_2^2[m-k] - 2\alpha \sum_{m=-\infty}^{\infty} x_1[m]x_2[m-k]$$
$$= R_{x_1}[0] + \alpha^2 R_{x_2}[0] - 2\alpha R_{x_1x_2}[k] \geq 0$$

Taking $\alpha = R_{x_1x_2}[k]/R_{x_2}[0]$, we obtain

$$R_{x_1}[0] + \frac{R^2_{x_1x_2}[k]}{R^2_{x_2}[0]} R_{x_2}[0] - 2\frac{R_{x_1x_2}[k]}{R_{x_2}[0]} R_{x_1x_2}[k]$$
$$\Rightarrow \quad R_{x_1}[0]R_{x_2}[0] - R^2_{x_1x_2}[k] \geq 0$$

The obtained result indicates that

$$-R_{x_1}[0]R_{x_2}[0] \leq R_{x_1 x_2}[k] \leq R_{x_1}[0]R_{x_2}[0]$$

which establishes the required result.

The crosscorrelation operation is used for detection (estimation) of signals from measured signals that contain the original signal corrupted by an additive noise, that is $x[k] + w[k]$, where $x[k]$ is the original signal and $w[k]$ is noise. The signal that has the highest correlation with the signal $x[k] + w[k]$ is considered as the best estimate of the signal $x[k]$ and denoted by $\hat{x}[k]$.

DTFT of Autocorrelation and Crosscorrelation Functions

The DTFT of the auto- and crosscorrelation functions can be found similarly to the DTFT of the convolution function. Define the following DTFT pairs $x_1[k] \leftrightarrow X_1(j\Omega)$ and $x_2[k] \leftrightarrow X_2(j\Omega)$. Then, the auto- and crosscorrelation functions of these two signals satisfy

$$R_{x_1x_1}[k] \leftrightarrow X_1(j\Omega)X_1^*(j\Omega) = |X_1(j\Omega)|^2, \; R_{x_1x_2}[k] \leftrightarrow X_1(j\Omega)X_2^*(j\Omega)$$

The proof of the property follows the convolution property proof. The quantity $|X_1(j\Omega)|^2$ is called the *energy spectral density* of the signal $x_1[k]$. Hence, the discrete-time signal energy spectral density is the DTFT of the signal autocorrelation function.