Discrete-Time State Space Analysis

8.3 Discrete-Time Models

Discrete-time systems are either inherently discrete (e.g. models of bank accounts, national economy growth models, population growth models, digital words) or they are obtained as a result of sampling (discretization) of continuous-time systems. In such kinds of systems, inputs, state space variables, and outputs have the discrete form and the system models can be represented in the form of transition tables.

The mathematical model of a discrete-time system can be written in terms of a recursive formula by using linear matrix difference equations as

$$\mathrm{x}((k+1)T) = \mathrm{A}_d\mathrm{x}(kT) + \mathrm{B}_d\mathbf{f}(kT)$$

$$\mathrm{y}(kT) = \mathrm{C}_d\mathrm{x}(kT) + \mathrm{D}_d\mathbf{f}(kT)$$

Here T represents the constant sampling interval, which may be omitted for brevity, that is, we use the following notation

$$\mathrm{x}[k+1] = \mathrm{A}_d\mathrm{x}[k] + \mathrm{B}_d\mathbf{f}[k]$$

$$\mathrm{y}[k] = \mathrm{C}_d\mathrm{x}[k] + \mathrm{D}_d\mathbf{f}[k]$$

Similarly to continuous-time linear systems, discrete state space equations can be derived from difference equations (Section 8.3.1). In Section 8.3.2 we show how to discretize continuous-time linear systems in order to obtain discrete-time linear systems.

8.3.1 Difference Equations and State Space Form

An nth-order difference equation is defined by

$$y[k+n] + a_{n-1}y[k+n-1] + \cdots + a_1 y[k+1] + a_0 y[k]$$

$$= b_n f[k+n] + b_{n-1} f[k+n-1] + \cdots + b_1 f[k+1] + b_0 f[k]$$

The corresponding state space equation can be derived by using the same technique as in the continuous-time case. For phase variable canonical form in discrete-time, we have

$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \\ \vdots \\ \vdots \\ x_n[k+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \\ \vdots \\ \vdots \\ x_n[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} f[k]$$

$$y[k] = \begin{bmatrix} (b_0 - a_0 b_n) & (b_1 - a_1 b_n) & \cdots & (b_{n-1} - a_{n-1} b_n) \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \\ \vdots \\ x_n[k] \end{bmatrix} + b_n f[k]$$

Note that the transformation equations, analogous to the continuous-time case, are given in the discrete-time domain by

$$\xi[k+n] + a_{n-1}\xi[k+n-1] + \cdots + a_1\xi[k+1] + a_0\xi[k] = f[k]$$

$$x_1[k] = \xi[k] \quad \Rightarrow \quad x_1[k+1] = x_2[k]$$

$$x_2[k] = \xi[k+1] \quad \Rightarrow \quad x_2[k+1] = x_3[k]$$

$$x_3[k] = \xi[k+2] \quad \Rightarrow \quad x_3[k+1] = x_4[k]$$

$$\vdots$$

$$x_n[k] = \xi[k+n-1] \quad \Rightarrow \quad x_n[k+1] = \xi[k+n]$$

(which gives the phase variable canonical form)

$$y[k] = b_0\xi[k] + b_1\xi[k+1] + b_2\xi[k+2] + \cdots + b_n\xi[k+n]$$

(which by eliminating $\xi[k+n]$ gives the state space output equation).

8.3.2 Discretization of Continuous-Time Systems

Real physical dynamic systems are continuous in nature. In this section, we show how to obtain discrete-time state space models from continuous-time system models.

Integral Approximation Method

The integral approximation method for discretization of a continuous-time linear system is based on the assumption that the system input is constant during the given sampling period. Namely, the method approximates the input signal by its staircase form, that is

$$\mathbf{f}(t) = \mathbf{f}(kT), \quad kT \le t < (k+1)T, \quad k = 0, 1, 2, \ldots$$

The impact of this approximation to the solution of the state space equations, for $t = T$, is given by

$$\mathrm{x}(T) = e^{\mathbf{A}T}\mathrm{x}(0) + \int_0^T e^{\mathbf{A}(T-\tau)}\mathbf{B}\mathbf{f}(0)d\tau$$

$$= e^{\mathbf{A}T}\mathrm{x}(0) + e^{\mathbf{A}T}\int_0^T e^{-\mathbf{A}\tau}d\tau\mathbf{B}\mathbf{f}(0) = \Phi(T)x(0) + \int_0^T \Phi(T-\tau)d\tau\mathbf{B}\mathbf{f}(0)$$

We can conclude that

$$\mathbf{A}_d = e^{\mathbf{A}T} = \Phi(T)$$

$$\mathbf{B}_d = e^{\mathbf{A}T}\int_0^T e^{-\mathbf{A}\tau}d\tau\mathbf{B} = \int_0^T e^{\mathbf{A}(T-\tau)}d\tau \cdot \mathbf{B} = \int_0^T e^{\mathbf{A}\sigma}d\sigma \cdot \mathbf{B}$$

Note that $\mathbf{A}_d$ and $\mathbf{B}_d$ are obtained for the time interval from 0 to T. It can easily be shown that due to system time invariance the same expressions for $\mathbf{A}_d$ and $\mathbf{B}_d$ are obtained for any time interval. The procedure can be repeated for time intervals $2T, 3T, ...,$ $(k+1)T$ with initial conditions taken as $\mathrm{x}(T), \mathrm{x}(2T), ..., \mathrm{x}(kT)$.

For the time instant $t = (k+1)T$ and for $t_0 = kT$, we have

$$
\begin{aligned}
\mathrm{x}((k+1)T) &= \Phi((k+1)T - kT)\mathrm{x}(kT) \\
&+ \int_{kT}^{(k+1)T} \Phi((k+1)T - \tau)d\tau \mathrm{B}\mathrm{f}(kT) \\
&= \mathrm{A}_d \mathrm{x}(kT) + \mathrm{B}_d \mathrm{f}(kT)
\end{aligned}
$$

From the above equation we see that the matrices A_d and B_d are given by

$$
\mathrm{A}_d = \Phi((k+1)T - kT) = \Phi(T) = e^{\mathrm{A}T}
$$

$$
\mathrm{B}_d = \int_{kT}^{(k+1)T} \Phi((k+1)T - \tau)d\tau \mathrm{B} = \int_0^T \Phi(\sigma)d\sigma \mathrm{B} = \int_0^T e^{\mathrm{A}\sigma} d\sigma \mathrm{B}
$$

The last equality is obtained by using change of variables as $\sigma = (k+1)T - \tau$.

In a similar manner, the formula for the system output at $t = kT$ implies

$$\mathrm{y}(kT) = \mathrm{C}\mathrm{x}(kT) + \mathrm{D}\mathrm{f}(kT)$$

Comparing this equation with the general output equation of linear discrete-time systems, we conclude that

$$\mathbf{C}_d = \mathbf{C}, \quad \mathbf{D}_d = \mathbf{D}$$

In the case of discrete-time linear systems obtained by sampling continuous-time linear systems, the matrix $\mathbf{A}_d$, can be determined from the infinite series

$$\mathbf{A}_d = e^{\mathbf{A}T} = \mathbf{I} + \mathbf{A}T + \frac{1}{2!}\mathbf{A}^2T^2 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i!}\mathbf{A}^iT^i$$

The matrix $\mathbf{A}_d$ can also be obtained either using the Laplace transform method or the method based on the Cayley–Hamilton theorem and setting $t = T$ in $\Phi(t) = e^{\mathbf{A}t}$.

To find $\mathbf{B}_d$, we perform integration (see Appendix A—matrix integrals)

$$\mathbf{B}_d = e^{\mathbf{A}T}\left(-e^{-\mathbf{A}T}\mathbf{A}^{-1} + \mathbf{A}^{-1}\right)\mathbf{B} = (\mathbf{A}_d - \mathbf{I})\mathbf{A}^{-1}\mathbf{B}$$

which is valid under the assumption that **A** is *invertible*.

Example 8.8: The discrete-time state space model of a continuous-time system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)$$

$$y(t) = [1 \quad 0]\mathbf{x}(t)$$

for the sampling period T is equal to 0.1, is obtained as follows.

$$\mathbf{A}_d = \Phi(T) = \begin{bmatrix} 2e^{-T} - e^{-2T} & e^{-T} - e^{-2T} \\ 2e^{-2T} - 2e^{-T} & 2e^{-2T} - e^{-T} \end{bmatrix} = \begin{bmatrix} 0.9909 & 0.0861 \\ -0.1722 & 0.7326 \end{bmatrix}$$

$$\mathbf{B}_d = (\mathbf{A}_d - \mathbf{I})\mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} \frac{1}{2}\left(1 + e^{-2T}\right) - e^{-T} \\ e^{-T} - e^{-2T} \end{bmatrix} = \begin{bmatrix} 0.0045 \\ 0.0861 \end{bmatrix}$$

$$\mathbf{C}_d = \mathbf{C} = [1 \quad 0], \qquad \mathbf{D}_d = \mathbf{D} = 0$$

Euler's Method

Less accurate but simpler than the integral approximation method is Euler's method. It is based on the approximation of the first derivative at $t = kT$

$$\dot{\mathrm{x}}(t) = \frac{d\mathrm{x}(t)}{dt} \approx \frac{1}{T}(\mathrm{x}((k+1)T) - \mathrm{x}(kT))$$

Applying this approximative formula to the state space system equation, we have

$$\frac{1}{T}(\mathrm{x}((k+1)T) - \mathrm{x}(kT)) \approx \mathrm{A}\mathrm{x}(kT) + \mathrm{B}\mathrm{f}(kT)$$

$$\mathrm{x}((k+1)T) \approx (\mathrm{I} + T\mathrm{A})\mathrm{x}(kT) + T\mathrm{B}\mathrm{f}(kT)$$

or

$$\mathrm{x}[k+1] \approx (\mathrm{I} + T\mathrm{A})\mathrm{x}[k] + T\mathrm{B}\mathrm{f}[k]$$

where

$$\mathrm{A}_d = \mathrm{I} + T \cdot \mathrm{A}, \quad \mathrm{B}_d = T \cdot \mathrm{B}$$

8.3.3 Solution of the Discrete-Time State Equation

We find the solution of the difference state equation for the given initial state $\mathbf{x}[0]$ and the input signal $\mathbf{f}[k]$. From the state equation

$$\mathrm{x}[k+1] = \mathrm{A}_d x[k] + \mathrm{B}_d \mathrm{f}[k]$$

for $k = 0, 1, 2...$, it follows

$$\mathrm{x}[1] = \mathrm{A}_d \mathrm{x}[0] + \mathrm{B}_d \mathrm{f}[0]$$

$$\mathrm{x}[2] = \mathrm{A}_d \mathrm{x}[1] + \mathrm{B}_d \mathrm{f}[1] = \mathrm{A}_d^2 \mathrm{x}[0] + \mathrm{A}_d \mathrm{B}_d \mathrm{f}[0] + \mathrm{B}_d \mathrm{f}[1]$$

$$\mathrm{x}[3] = \mathrm{A}_d \mathrm{x}[2] + \mathrm{B}_d \mathrm{f}[2] = \mathrm{A}_d^3 \mathrm{x}[0] + \mathrm{A}_d^2 \mathrm{B}_d \mathrm{f}[0] + \mathrm{A}_d \mathrm{B}_d \mathrm{f}[1] + \mathrm{B}_d \mathrm{f}[2]$$

$$\vdots$$

$$\mathrm{x}[k] = \mathrm{A}_d \mathrm{x}[k-1] + \mathrm{B}_d \mathrm{f}[k-1] = \mathrm{A}_d^k \mathrm{x}[0] + \sum_{i=0}^{k-1} \mathrm{A}_d^{k-i-1} \mathrm{B}_d \mathrm{f}[i]$$

Using the notion of the *discrete-time state transition matrix* defined by

$$\Phi_d[k] = \mathrm{A}_d^k$$

we get

$$\mathrm{x}[k] = \Phi_d[k]\mathrm{x}[0] + \sum_{i=0}^{k-1} \Phi_d[k-i-1]\mathrm{B}_d\mathbf{f}[i]$$

Note that the discrete-time state transition matrix relates the state of an input-free system at initial time ($k = 0$) to the state of the system at any other time $k > 0$, that is

$$\mathrm{x}[k] = \Phi_d[k]\mathrm{x}[0] = \mathrm{A}_d^k\mathrm{x}[0]$$

It is easy to verify that the discrete-time state transition matrix has the following properties

(a) $\mathbf{\Phi}_d[0] = \mathbf{A}_d^0 = \mathbf{I} \;\Leftarrow\; \mathrm{x}[0] = \mathbf{\Phi}_d[0]\mathrm{x}[0]$

(b) $\mathbf{\Phi}_d[k_2 - k_0] = \mathbf{\Phi}_d[k_2 - k_1]\mathbf{\Phi}_d[k_1 - k_0] = \mathbf{A}_d^{k_2-k_1}\mathbf{A}_d^{k_1-k_0} = \mathbf{A}_d^{k_2-k_0}$

(c) $\mathbf{\Phi}_d^i[k] = \mathbf{\Phi}_d[ik] \;\Leftarrow\; \left(\mathbf{A}_d^k\right)^i = \mathbf{A}_d^{ik}$

(d) $\mathbf{\Phi}_d[k+1] = \mathbf{A}_d\mathbf{\Phi}_d[k], \quad \mathbf{\Phi}_d[0] = \mathbf{I}$

The last property follows from

$$\mathrm{x}[k+1] = \mathbf{A}_d\mathrm{x}[k] \;\Rightarrow\; \mathbf{\Phi}_d[k+1]\mathrm{x}[0] = \mathbf{A}_d\mathbf{\Phi}_d[k]\mathrm{x}[0]$$

It is important to point out that the discrete-time state transition matrix may be singular, which follows from the fact that $\mathbf{A}_d^k$ is nonsingular if and only if the matrix $\mathbf{A}_d$ is nonsingular. In the case of inherent discrete-time systems, the matrix $\mathbf{A}_d$ may be singular in general. However, if $\mathbf{A}_d$ is obtained through the discretization procedure of a continuous-time linear system, then

$$(\mathbf{A}_d)^{-1} = \left(e^{\mathbf{A}T}\right)^{-1} = e^{-\mathbf{A}T}$$

The output of the system at sampling instant k is obtained by substituting $\mathrm{x}[k]$ into the output equation, producing

$$\mathrm{y}[k] = \mathrm{C}_d \Phi_d[k]\mathrm{x}[0] + \mathrm{C}_d \sum_{i=0}^{k-1} \Phi[k-i-1]\mathrm{B}_d \mathrm{f}[i] + \mathrm{D}_d f[k]$$

Remark 8.1: If the initial value of the state vector is not $\mathrm{x}[0]$ but $\mathrm{x}[k_0]$, then the solution has to be modified into

$$\mathrm{x}[k_0 + k] = \Phi_d[k]\mathrm{x}[k_0] + \sum_{i=0}^{k-1} \Phi_d[k-i-1]\mathrm{B}_d \mathrm{f}[k_0 + i]$$

Note that for $T \neq 1$, the following modification must be used

$$\mathrm{x}(kT) = \Phi_d(kT)\mathrm{x}(0) + \sum_{i=0}^{k-1} \Phi_d((k-i-1)T)\mathrm{B}_d \mathrm{f}(iT)$$

$$\mathrm{y}(kT) = \mathrm{C}_d \Phi_d(kT)\mathrm{x}(0) + \mathrm{C}_d \sum_{i=0}^{k-1} \Phi((k-i-1)T)\mathrm{B}_d \mathrm{f}(iT) + \mathrm{D}_d \mathrm{f}(kT)$$

Remark 8.2: The discrete-time state transition matrix defined by $\mathbf{A}_d^k$ can be evaluated efficiently for large values of k by using a method based on the Cayley–Hamilton theorem and described in Section 8.5. It can be also evaluated by using the $\mathcal{Z}$-transform method, to be derived in the next subsection.

8.3.4 Solution Using the $\mathcal{Z}$-transform

Applying the $\mathcal{Z}$-transform to the state space equation of a discrete-time linear system

$$\mathbf{x}[k+1] = \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{f}[k]$$

we get

$$z\mathbf{X}(z) - z\mathbf{x}[0] = \mathbf{A}_d\mathbf{X}(z) + \mathbf{B}_d\mathbf{F}(z)$$

The frequency domain state space vector $\mathbf{X}(z)$ can be expressed as

$$\mathbf{X}(z) = (z\mathbf{I} - \mathbf{A}_d)^{-1} z\mathbf{x}[0] + (z\mathbf{I} - \mathbf{A}_d)^{-1}\mathbf{B}_d\mathbf{F}(z)$$

The inverse $\mathcal{Z}$-transform of the last equation gives $\mathbf{x}[k]$, that is

$$\mathbf{x}[k] = \mathcal{Z}^{-1}\Big[(z\mathbf{I} - \mathbf{A}_d)^{-1} z\Big] x[0] + \mathcal{Z}^{-1}\Big[(z\mathbf{I} - \mathbf{A}_d)^{-1}\mathbf{B}_d\mathbf{F}(z)\Big]$$

We conclude that

$$\boldsymbol{\Phi}_d[k] = \mathcal{Z}^{-1}\Big[(z\mathbf{I} - \mathbf{A}_d)^{-1} z\Big] = \mathbf{A}_d^k, \quad k = 1, 2, 3, \ldots$$

and

$$\boldsymbol{\Phi}_d(z) = z(z\mathbf{I} - \mathbf{A}_d)^{-1}$$

The inverse transform of the second term on the right-hand side is obtained directly by the application of the discrete-time convolution, which produces

$$\mathcal{Z}^{-1}\Big\{(z\mathbf{I}-\mathbf{A}_d)^{-1}\mathbf{B}_d\mathbf{F}(z)\Big\} = \sum_{i=0}^{k-1} \mathbf{\Phi}_d[k-i-1]\mathbf{B}_d\mathbf{f}[i]$$

We have the required solution of the discrete-time state space equation as

$$\mathbf{x}[k] = \mathbf{\Phi}_d[k]\mathbf{x}[0] + \sum_{i=0}^{k-1} \mathbf{\Phi}_d[k-i-1]\mathbf{B}_d\mathbf{f}[i]$$

From

$$\mathbf{y}[k] = \mathbf{C}_d\mathbf{x}[k] + \mathbf{D}_d\mathbf{f}[k]$$

the system output response is obtained

$$\mathbf{y}[k] = \mathbf{C}_d\mathbf{\Phi}_d[k]\mathbf{x}[0] + \mathbf{C}_d \sum_{i=0}^{k-1} \mathbf{\Phi}_d[k-i-1]\mathbf{B}_d\mathbf{f}[i] + \mathbf{D}_d\mathbf{f}[k]$$

The frequency domain form of the output vector $\mathbf{Y}(z)$ is obtained if the $\mathcal{Z}$-transform is applied to the output equation, and $\mathbf{X}(z)$ is eliminated, leading to

$$\mathbf{Y}(z) = \mathbf{C}_d(z\mathbf{I} - \mathbf{A}_d)^{-1} z\mathbf{x}[0] + \left[\mathbf{C}_d(z\mathbf{I} - \mathbf{A}_d)^{-1}\mathbf{B}_d + \mathbf{D}_d\right]\mathbf{F}(z)$$

From the above expression, for the zero initial condition, i.e. $\mathbf{x}[0] = \mathbf{0}$, the *discrete matrix transfer function* is defined by

$$\mathbf{H}_d(z) = \mathbf{C}_d(z\mathbf{I} - \mathbf{A}_d)^{-1}\mathbf{B}_d + \mathbf{D}_d$$

Example 8.9: Consider the following discrete-time system

$$\mathbf{A}_d = \begin{bmatrix} 0 & 1 \\ -\frac{1}{6} & -\frac{5}{6} \end{bmatrix}, \quad \mathbf{B}_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C}_d = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D}_d = 0$$

The discrete-time state transition matrix in the frequency domain is obtained as

$$\Phi_d(z) = (z\mathbf{I} - \mathbf{A})^{-1} z = \left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{1}{6} & -\frac{5}{6} \end{bmatrix} \right)^{-1} z = \begin{bmatrix} z & -1 \\ \frac{1}{6} & z + \frac{5}{6} \end{bmatrix}^{-1} z$$

$$= \frac{z}{z\left(z + \frac{5}{6}\right) + \frac{1}{6}} \begin{bmatrix} z + \frac{5}{6} & 1 \\ -\frac{1}{6} & z \end{bmatrix} = \begin{bmatrix} \frac{z\left(z+\frac{5}{6}\right)}{\left(z+\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} & \frac{z}{\left(z+\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \\ \frac{-\frac{1}{6}z}{\left(z+\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} & \frac{z^2}{\left(z+\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \end{bmatrix}$$

The time domain state transition matrix is given by

$$\Phi_d[k] = \mathcal{Z}^{-1}\{\Phi_d(z)\} = \mathcal{Z}^{-1}\left\{ \begin{bmatrix} \frac{-2z}{z+\frac{1}{2}} + \frac{3z}{z+\frac{1}{3}} & \frac{-6z}{z+\frac{1}{2}} + \frac{6z}{z+\frac{1}{3}} \\ \frac{z}{z+\frac{1}{2}} - \frac{z}{z+\frac{1}{3}} & \frac{3z}{z+\frac{1}{2}} - \frac{2z}{z+\frac{1}{3}} \end{bmatrix} \right\}$$

$$= \begin{bmatrix} -2\left(-\frac{1}{2}\right)^k + 3\left(-\frac{1}{3}\right)^k & -6\left(-\frac{1}{2}\right)^k + 6\left(-\frac{1}{3}\right)^k \\ \left(-\frac{1}{2}\right)^k - \left(-\frac{1}{3}\right)^k & 3\left(-\frac{1}{2}\right)^k - 2\left(-\frac{1}{3}\right)^k \end{bmatrix} u[k]$$

Let us find the response of this system due to

$$f[k] = (-1)^k u[k], \quad \mathrm{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since the system state transition matrix is already determined, we can use the derived formula, which produces

$$\mathrm{x}[k] = \Phi_d[k]\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sum_{i=0}^{k-1} \Phi_d[k-i-1]\begin{bmatrix} 0 \\ 1 \end{bmatrix}(-1)^i$$

$$= \begin{bmatrix} -2\left(-\frac{1}{2}\right)^k + 3\left(-\frac{1}{3}\right)^k \\ \left(-\frac{1}{2}\right)^k - \left(-\frac{1}{3}\right)^k \end{bmatrix} + \sum_{i=0}^{k-1} \begin{bmatrix} -6\left(-\frac{1}{2}\right)^{k-i-1} + 6\left(-\frac{1}{3}\right)^{k-i-1} \\ 3\left(-\frac{1}{2}\right)^{k-i-1} - 2\left(-\frac{1}{3}\right)^{k-i-1} \end{bmatrix}(-1)^i$$

This can be accepted as the final result. Note that using known formulas for series summation (Appendix B), the above formula can be further simplified, and eventually a closed form solution might be obtained for $\mathrm{x}[k]$. However, if we find in the frequency domain $\mathbf{X}(z)$, then in most cases, the inverse $\mathcal{Z}$–transform will produce a nice closed formula for $\mathrm{x}[k]$. In this example, we have

$$\mathbf{X}(z) = \frac{z}{\left(z+\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \begin{bmatrix} z+\frac{5}{6} & 1 \\ -\frac{1}{6} & z \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$+ \frac{1}{\left(z+\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \begin{bmatrix} z+\frac{5}{6} & 1 \\ -\frac{1}{6} & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{z}{z+1}$$

$$= \frac{z}{\left(z+\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \begin{bmatrix} z+\frac{5}{6} \\ -\frac{1}{6} \end{bmatrix} + \frac{z}{\left(z+\frac{1}{2}\right)\left(z+\frac{1}{3}\right)(z+1)} \begin{bmatrix} 1 \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2z}{z+\frac{1}{2}} + \frac{3z}{z+\frac{1}{3}} \\ \frac{z}{z+\frac{1}{2}} - \frac{z}{z+\frac{1}{3}} \end{bmatrix} + \begin{bmatrix} \frac{-12z}{z+\frac{1}{2}} + \frac{9z}{z+\frac{1}{3}} + \frac{3z}{z+1} \\ \frac{6z}{z+\frac{1}{2}} + \frac{3z}{z+\frac{1}{3}} - \frac{3z}{z+1} \end{bmatrix}$$

Applying the inverse $\mathcal{Z}$–transform we obtain the state response

$$\mathrm{x}[k] = \begin{bmatrix} -2\left(-\frac{1}{2}\right)^k + 3\left(-\frac{1}{3}\right)^k \\ \left(-\frac{1}{2}\right)^k - \left(-\frac{1}{3}\right)^k \end{bmatrix} + \begin{bmatrix} -12\left(-\frac{1}{2}\right)^k + 9\left(-\frac{1}{3}\right)^k + 3(-1)^k \\ 6\left(-\frac{1}{2}\right)^k + 3\left(-\frac{1}{3}\right)^k - 3(-1)^k \end{bmatrix}$$

$$= \begin{bmatrix} -14\left(-\frac{1}{2}\right)^k + 12\left(-\frac{1}{3}\right)^k + 3(-1)^k \\ 7\left(-\frac{1}{2}\right)^k + 2\left(-\frac{1}{3}\right)^k - 3(-1)^k \end{bmatrix} = \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix}$$

For the system output response we have

$$y[k] = [1 \quad 0]\mathrm{x}[k] = x_1[k] = -14\left(-\frac{1}{2}\right)^k + 12\left(-\frac{1}{3}\right)^k + 3(-1)^k$$

8.3.5 Discrete-Time Impulse and Step Responses

The impulse and step responses of multi-input multi-output discrete-time systems are defined for zero initial conditions, and calculated using the formulas

$$\mathrm{x}[k] = \sum_{i=0}^{k-1} \Phi_d[k-i-1]\mathrm{B}_d\mathrm{f}[i], \qquad \mathrm{y}[k] = \mathrm{C}_d\mathrm{x}[k] + \mathrm{Df}[k]$$

Since the input forcing function is a vector of dimensions $r \times 1$, we can define the impulse and step responses for every input of the system. Introduce the discrete-time system input function whose all components are zero except for the jth component, that is

$$\mathbf{f}^j[k] = [0 \quad \cdots \quad 0 \quad f_j[k] \quad 0 \quad \cdots \quad 0]^T$$

Note that

$$\mathbf{B}_d\mathbf{f}^j[k] = f_j[k]\mathbf{b}_j$$

where $\mathbf{b}_j$ is the jth column of the matrix $\mathbf{B}_d$. The system state and output responses due to the jth component of the input signal are given by

$$\mathbf{x}^j[k] = \sum_{i=0}^{k-1} \Phi_d[k-i-1]\mathbf{b}_j f_j[i], \qquad \mathbf{y}^j[k] = \mathbf{C}_d\mathbf{x}^j[k] + \mathbf{d}_j f_j[k]$$

where $\mathbf{d}_j$ is the jth column of the matrix $\mathbf{D}_d$. For $f_j[k] = \delta[k]$, these formulas produce the *system output impulse response*, $\mathbf{h}^j[k]$, due to the delta impulse function on the jth system input and all other inputs equal to zero, that is

$$\mathrm{x}^j[k] = \sum_{i=0}^{k-1} \Phi_d[k-i-1]\mathrm{b}_j\delta[i] = \mathrm{A}_d^{k-1}\mathrm{b}_j$$

$$\mathrm{y}^j[k] = \mathrm{C}_d\mathrm{x}^j[k] + \mathrm{d}_j\delta[k] = \mathrm{C}_d\mathrm{A}_d^{k-1}\mathrm{b}_j + \mathrm{d}_j\delta[k] = \mathrm{h}^j[k]$$

Similarly, with $f_j[k] = u[k]$, we can define the *system output step response* due to the unit step function on the jth system input and all other inputs set to zero

$$\mathrm{x}_{step}^j[k] = \sum_{i=0}^{k-1} \Phi_d[k-i-1]\mathrm{b}_j u_j[i] = \sum_{i=0}^{k-1} \mathrm{A}_d^{k-i-1}\mathrm{b}_j$$

$$\mathrm{y}_{step}^j[k] = \mathrm{C}_d\left(\sum_{i=0}^{k-1} \mathrm{A}_d^{k-i-1}\mathrm{b}_j\right) + \mathrm{d}_j u[k]$$

It follows that

$$\mathrm{y}_{step}^j[k] = \sum_{i=0}^{k} \mathrm{h}^j[i], \quad \mathrm{h}^j[k] = \mathrm{y}_{step}^j[k] - \mathrm{y}_{step}^j[k-1]$$