6.3 Convolution of Discrete-Time Signals

The discrete-time convolution of two signals $f_1[k]$ and $f_2[k]$ is defined in Chapter 2 as the following infinite sum

$$f[k] = f_1[k] * f_2[k] = \sum_{m=-\infty}^{m=\infty} f_1[m] f_2[k-m], \quad -\infty < k < \infty$$

where k is an integer parameter and m is a dummy variable of summation.

The properties of the discrete-time convolution are:

1) *Commutativity*

$$f_1[k] * f_2[k] = f_2[k] * f_1[k]$$

2) *Distributivity*

$$f_1[k] * \{f_2[k] + f_3[k]\} = f_1[k] * f_2[k] + f_1[k] * f_3[k]$$

3) *Associativity*

$$f_1[k] * \{f_2[k] * f_3[k]\} = \{f_1[k] * f_2[k]\} * f_3[k]$$

4) *Duration*

The duration of a discrete-time signal $f[k]$ is defined by the discrete time instants k_0 and k_f for which for every k outside the interval $[k_0, k_f]$ the discrete-time signal $f[k] = 0$. We use M to denote the discrete-time signal duration. It follows that $M = k_f - k_0$.

Let the signals $f_1[k]$ and $f_2[k]$ have durations, respectively given by M_1 and M_2, then the duration of their convolution, $f[k] = f_1[k] * f_2[k]$, is given by $M_1 + M_2$.

The discrete-time convolution duration property can be also expressed in terms of the number of signal samples. Let the number of samples in the signal (signal size) be denoted by L, then $L = M + 1$. Consider two signals $f_1[k]$ and $f_2[k]$ with the number of samples respectively given by L_1 and L_2. The number of samples in their convolution signal is equal to $L_1 + L_2 - 1$, which corresponds to the duration of $L_1 + L_2 - 2 = M_1 + M_2$.

5) *Time Shifting*

Let $f[k] = f_1[k] * f_2[k]$. Then, convolutions of the shifted functions are

$$f_1[k - k_1] * f_2[k] = f[k - k_1]$$

$$f_1[k] * f_2[k - k_2] = f[k - k_2]$$

$$f_1[k - k_1] * f_2[k - k_2] = f[k - k_1 - k_2]$$

The proofs of these properties are similar to the proofs of the corresponding continuous-time convolution properties. For example, in order to establish the commutativity property we have to introduce the change of variables as $k - m = n$

$$f[k] = f_1[k] * f_2[k] = \sum_{m=-\infty}^{m=\infty} f_1[m] f_2[k - m]$$

$$= \sum_{n=-\infty}^{n=\infty} f_1[k - n] f_2[n] = f_2[k] * f_1[k]$$

Example 6.10: The convolution of the discrete-time impulse delta function with a general function $f[k]$ is given by

$$f[k] * \delta[k] = \sum_{m=-\infty}^{m=\infty} f[m]\delta[k-m] = f[k], \quad -\infty < k < \infty$$

Example 6.11: The convolution of two causal exponential functions defined by $f_1[k] = a^k u[k]$ and $f_2[k] = b^k u[k]$ is obtained as follows

$$a^k u[k] * b^k u[k] = \sum_{m=-\infty}^{m=\infty} a^m u[m] b^{k-m} u[k-m]$$

$$= \sum_{m=0}^{m=k} a^m b^{k-m} = b^k \sum_{m=0}^{m=k} \left(\frac{a}{b}\right)^m$$

Using the known summation formula (see Appendix B)

$$\sum_{m=0}^{k} \alpha^m = \frac{1-\alpha^{k+1}}{1-\alpha}$$

we have

$$a^k u[k] * b^k u[k] = b^k \frac{1-(a/b)^{k+1}}{1-(a/b)} = \frac{b^{k+1}-a^{k+1}}{b-a}$$

Example 6.12: The convolution of the unit step signal and any causal signal ($f[k] = 0, k < 0$) produces

$$f[k] * u[k] = \sum_{m=0}^{\infty} f[m]u[k-m] = \sum_{m=0}^{k} f[m] = f[0] + f[1] + \cdots + f[k]$$

Example 6.13: Convolution of two causal signals $f_1[k]$ and $f_2[k]$ is

$$f_1[k] * f_2[k] = \sum_{m=-\infty}^{\infty} f_1[m]f_2[k-m] = \sum_{m=0}^{k} f_1[m]f_2[k-m]$$

$$= f_1[0]f_2[k] + f_1[1]f_2[k-1] + \cdots + f_1[k-1]f_2[1] + f_1[k]f_2[0]$$

$$k = 0, 1, 2, \ldots$$

which represents an easy to remember formula.

6.3.1 Sliding Tape Method

Like in the continuous-time convolution, the discrete-time convolution requires the "flip and slide" steps. For the reason of simplicity, we will explain the method using two causal signals. However, the method is applicable to any two discrete-time signals. Note that by using the discrete-time convolution shifting property, this method can be also applied to noncausal signals. The sliding tape method is presented in the following three steps.

Step 1: The signal values are recorded on two tapes, one tape for the values of the signal $f_1[m]$ and another tape for the values of the signal $f_2[m]$, see Figure 6.17a, done for an example of two causal signals

$$f_1[0],\ f_1[1],\ f_1[2], ..., f_1[M_1 - 1],\ f_1[M_1]$$

$$f_2[0],\ f_2[1],\ f_2[2], ..., f_2[M_2 - 1],\ f_2[M_2]$$

Note that the durations of these signals, which contain $L_1 = M_1 + 1$ and $L_2 = M_2 + 1$ samples (values), are M_1 and M_2.

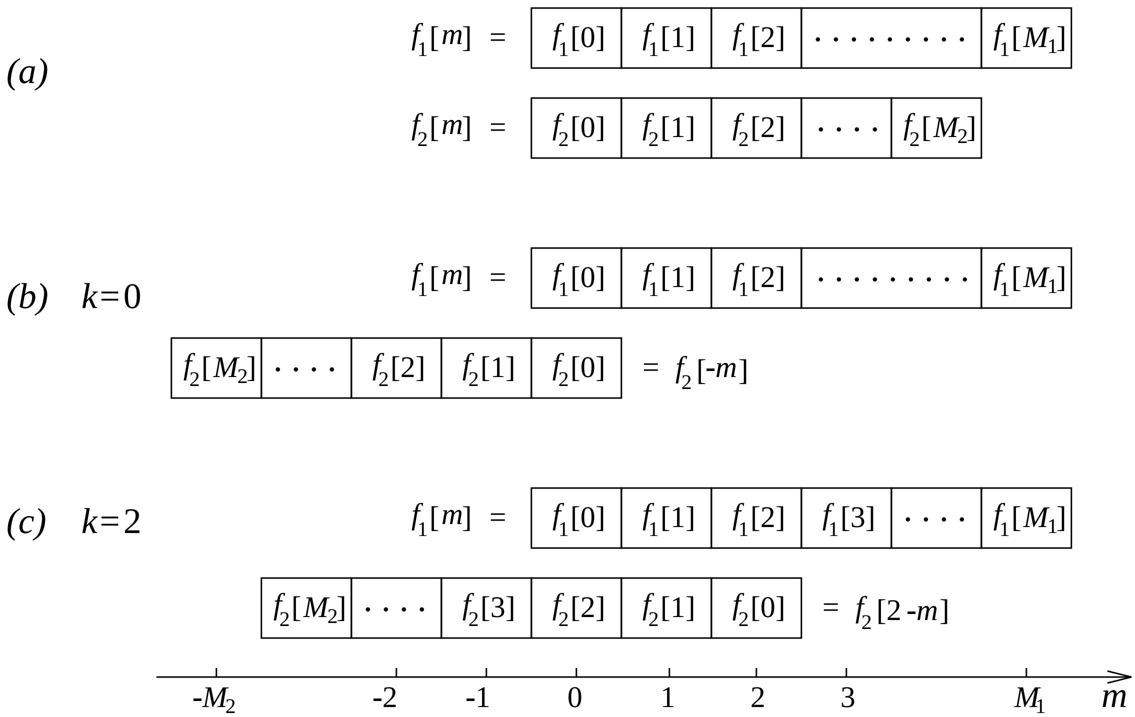

Figure 6.17: Graphical representation for the sliding tape method

Step 2: One of the tapes, say, the second tape, is flipped about its value at $f_2[0]$ to form the signal $f_2[-m]$, see Figure 6.17b. It should be pointed out that the signal $f_2[m]$ is flipped in such a way that the signal value $f_2[0]$ remains in the same position.

Step 3: The second tape is shifted to the left and right, that is, a traveling signal $f_2[k-m]$ is formed. The parameter k is an integer that theoretically takes all values from $-\infty$ to ∞. Practically, we have to shift the second signal only for those values of k for which the convolution sum is different from zero. In that respect, the duration property of the discrete-time convolution plays an important role. After we shift the second tape for the given value of k, we evaluate the products of the corresponding overlapping signal values on the tapes. The sum of all products gives the convolution value for the chosen value of the parameter k, see Figure 6.17c. This procedure is repeated for all values of k for which the convolution sum may be different from zero.

Let $f[k] = f_1[k] * f_2[k]$. From Figure 6.17b, we see that for $k = 0$ only $f_1[0]$

and $f_2[0]$ overlap, hence $f[0] = f_1[0]f_2[0]$. From Figure 6.17c, drawn for $k = 2$, we obtain three pairs of the overlapped signal values, hence the convolution of these two signals for $k = 2$ is given by $f[2] = f_1[0]f_2[2] + f_1[1]f_2[1] + f_1[2]f_2[0]$. Similarly, we evaluate the discrete-time convolution for other values of k. Note that for $k \leq -1$ the signals $f_1[m]$ and $f_2[k-m]$ do not overlap, hence the convolution is equal to zero for $k \leq -1$. Also, no overlapping between the values of $f_1[m]$ and $f_2[k-m]$ exists for $k \geq M_1 + M_2 + 1$, and the corresponding discrete-time convolution is equal to zero in this interval.

Example 6.14: Let the signals be defined as follows

$$f_1[k] = \begin{cases} 1 & k = 0 \\ 2 & k = 1 \end{cases}, \qquad f_2[k] = \begin{cases} -1 & k = 0 \\ 3 & k = 1 \end{cases}$$

The durations of these signals are $M_1 = M_2 = 1$. By the convolution duration property, the convolution sum may be different from zero in the time interval of length $M = M_1 + M_2 = 1 + 1 = 2$. Tapes for $f_1[m]$ and $f_2[-m]$ are shown in Figure 6.18.

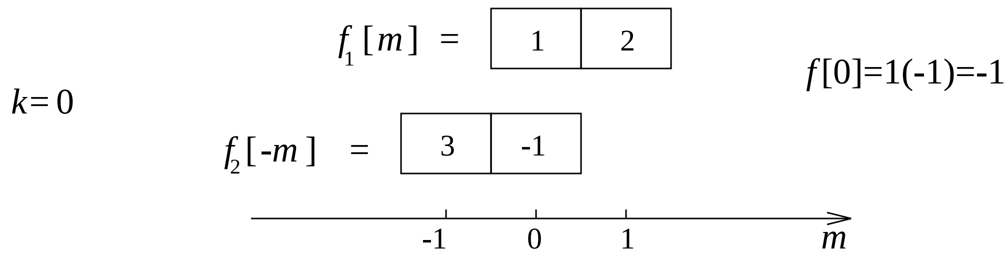

Figure 6.18: The sliding tape method for Example 6.14, $k = 0$

The convolution of these two signals, $f[k] = f_1[k] * f_2[k]$, for $k = 0$, is easily obtained from Figure 6.18 as $f[0] = f_1[0] * f_2[0] = 1 \times (-1) = -1$. If we slide the second tape to the left, which corresponds to $k \leq -1$, we see that the convolution is equal to zero. Sliding the second tape to the right for $k = 1$, we obtain $f_1[1] = 1 \times 3 + 2 \times (-1) = 1$, see Figure 6.19.

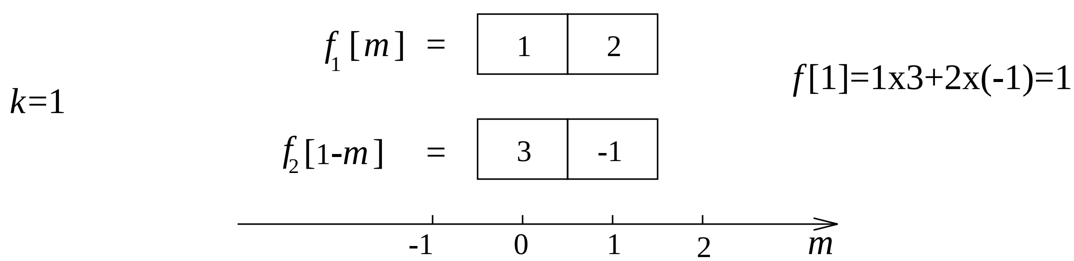

Figure 6.19: The sliding tape method for Example 6.14, $k = 1$

For $k = 2$, according to Figure 6.20, the convolution is given by $f[2] = 2 \times 3 = 6$. For $k \geq 3$, the signals $f_1[m]$ and $f_2[k-m]$ do not overlap, hence, the convolution is equal to zero in this interval.

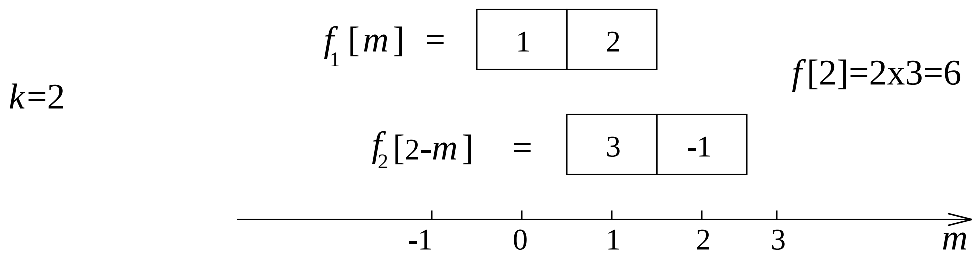

Figure 6.20: The sliding tape method for Example 6.14, $k = 2$

In summary, we have obtained

$$f[k] = f_1[k] * f_2[k] = \begin{cases} 0 & k \leq -1 \\ -1 & k = 0 \\ 1 & k = 1 \\ 6 & k = 2 \\ 0 & k \geq 3 \end{cases}$$

Example 6.15: Let us find the convolution of the following two signals by using the sliding tape method

$$f_1[k] = \begin{cases} -2 & k=-1 \\ 1 & k=0 \\ 3 & k=1 \\ 0 & \text{otherwise} \end{cases}, \quad f_2[k] = \begin{cases} 2 & k=0 \\ 3 & k=1 \\ -1 & k=2 \\ 1 & k=3 \\ 0 & \text{otherwise} \end{cases}$$

Note that the first signal is noncausal. However, the sliding tape procedure to be applied is exactly the same as in the case of causal signals. The durations of these signals are respectively given by $M_1 = 2$ and $M_2 = 3$, hence the duration of their convolution is equal to $M_c = M_1 + M_2 = 5$, which means that at most six ($L_c = M_c + 1$) discrete-time instants the convolution sum may be different from zero.

In Figure 6.21, we present the signals $f_1[m]$ and $f_2[-m]$. It can be seen from this figure that $f[0] = 1 \times 2 + (-2) \times 3 = -4$. It can also be concluded that the convolution is equal to zero for $k \leq -2$ and $k \geq 5$, which is consistent with the discrete-time convolution duration property.

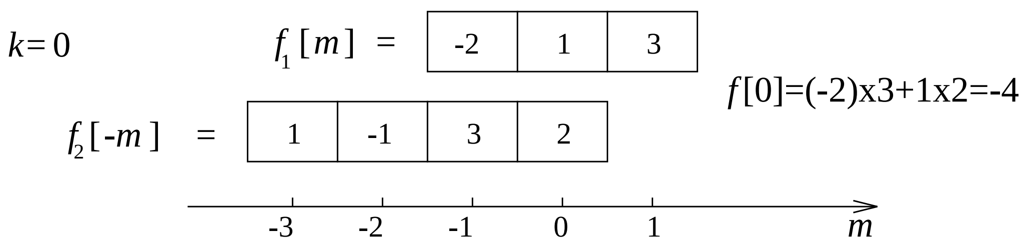

Figure 6.21: The sliding tapes for Example 6.15, $k = 0$

In Figure 6.22, we present the sliding tapes for $k = -1$ and $k = 1$. It can be seen that $f[-1] = (-2) \times 2 = -4$ and $f[1] = (-2) \times (-1) + 1 \times 3 + 3 \times 2 = 11$.

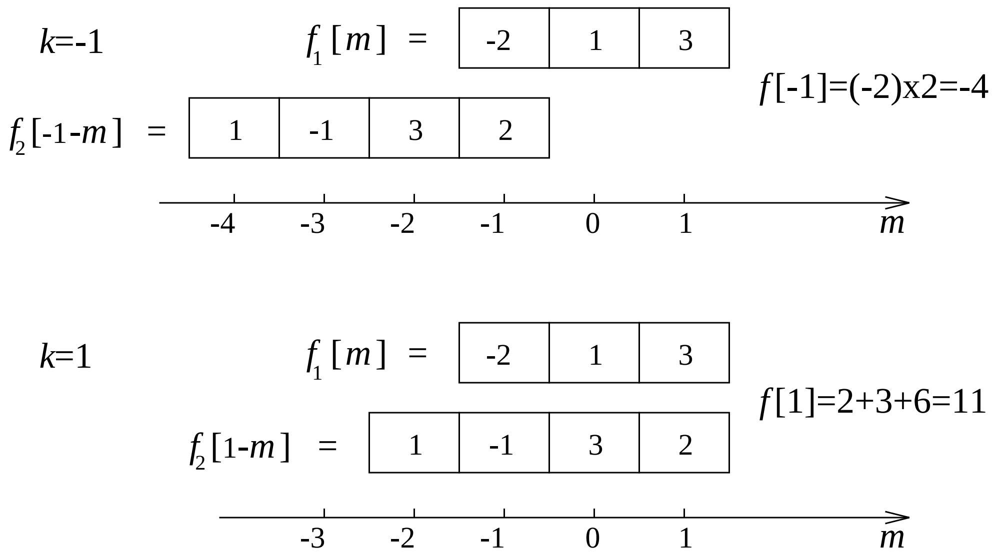

Figure 6.22: The sliding tapes for Example 6.15, $k = -1, 1$

Figure 6.23 presents the situation for $k = 2, 3, 4$. It can be seen from this figure that $f[2] = (-2) \times 1 + 1 \times (-1) + 3 \times 3 = 6$, $f[3] = 1 \times 1 + 3 \times (-1) = -2$, and $f[4] = 3 \times 1 = 3$.

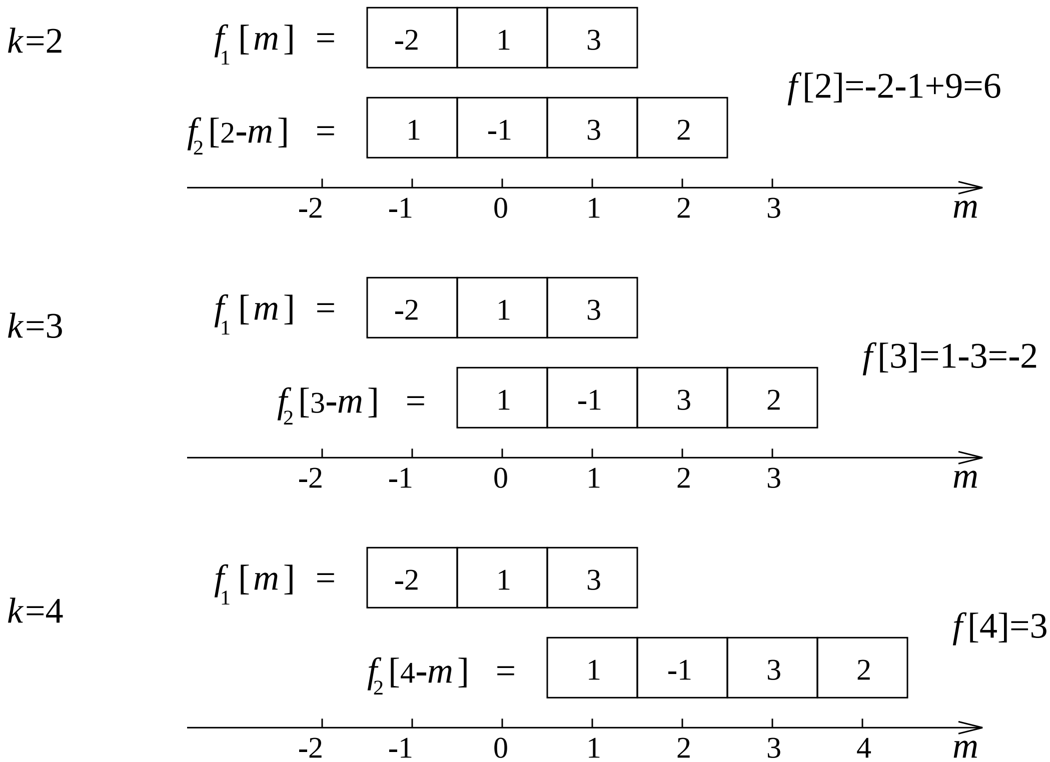

Figure 6.23: The sliding tape method for Example 6.15, $k = 2, 3, 4$

In summary, we have the following values for the convolution of the considered signals

$$f[k] = f_1[k] * f_2[k] = \begin{cases} 0 & k \le -2 \\ -4 & k = -1 \\ -4 & k = 0 \\ 11 & k = 1 \\ 6 & k = 2 \\ -2 & k = 3 \\ 3 & k = 4 \\ 0 & k \ge 5 \end{cases}$$

6.4 Convolution for Linear Discrete-Time Systems

In this subsection we show how to use the discrete-time convolution in order to find the zero-state response of discrete-time linear time invariant systems. We have seen in Chapter 5 that every discrete-time linear time invariant system is uniquely characterized either by its transfer function or by its impulse response. We have also seen that the system transfer function is the $\mathcal{Z}$–transform of the system impulse response, and that the system impulse response is obtained as the inverse $\mathcal{Z}$–transform of the system transfer function. Let us assume that the system initial conditions are set to zero and that the system impulse response is known. This can be symbolically represented as in Figure 6.24, where $I.C. = 0$ stands for zero initial conditions.

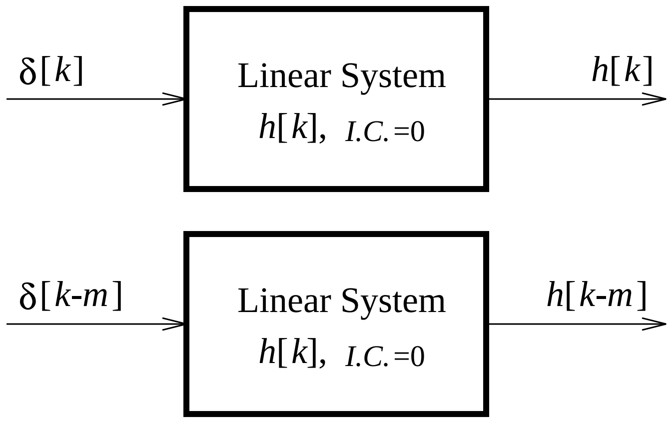

Figure 6.24: Discrete-time system impulse response

By the time invariance principle, $\delta[k-m]$ produces the system output $h[k-m]$. By the linearity principle, a weighted impulse delta function $f[m]\delta[k-m]$, where $f[m]$ is a constant, produces the system output signal $f[m]h[k-m]$, see Figure 6.25. Note that we assume that the system in Figures 6.24 and 6.25 is represented in its integral formulation, for which we have defined the system transfer function and the system impulse response in Chapter 5.

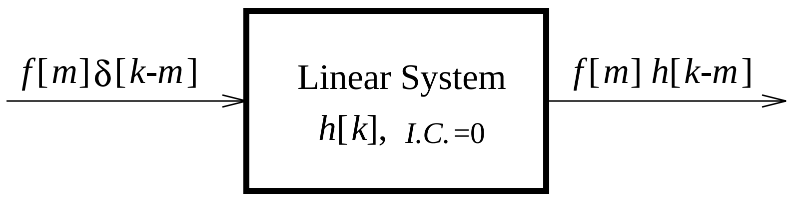

Figure 6.25: A weighted system impulse response

Our goal is to find the system zero-state response due to any input function $f[k]$. In that respect, we have assumed that the system initial conditions are equal to zero, that is

$$y[-n] = y[-(n-1)] = \cdots = y[-2] = y[-1] = 0$$

Let the system input signal be causal ($f[k] = 0, k < 0$) and defined by its values at discrete-time instants

$$f[0],\ f[1], f[2],\ ..., f[k]$$

For any value of k, we can represent $f[k]$ by a weighted sum of delta impulse signals

$$f[k] = \sum_{m=0}^{m=k} f[m]\delta[k-m] = f[k] * \delta[k]$$

By the linearity and time invariance principles, we know that a sum of weighted and shifted delta impulse signals produces on the system output a sum of weighted and shifted system impulse response signals. Hence, using $f[k]$ as the system input, we obtain the system output in the form

$$y^{i}_{zs}[k] = \sum_{m=0}^{m=k} f[m]h[k-m] = f[k] * h[k]$$

which is symbolically presented in Figure 6.26. Note that the upper script i indicates the integral formulation of a discrete-time linear system (see Section 5.3.1) that is consistent with the defined initial conditions.

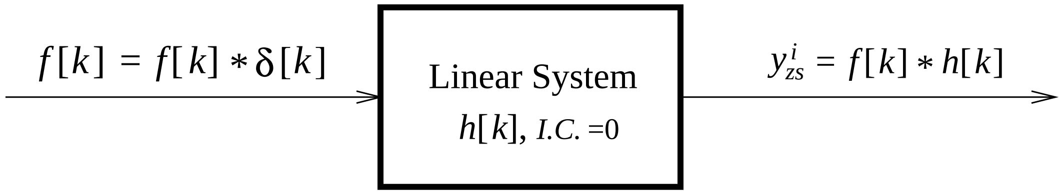

Figure 6.26: Discrete system response as the convolution of a system input and the system impulse response

The preceding derivations establish the most fundamental result of theory of linear discrete-time systems, which is restated in the following theorem.

Theorem 6.2 *The response of a linear discrete-time system at rest (zero initial condition response) due to any input is the convolution of that input and the system impulse response.*

In addition to its importance for linear discrete-time dynamic systems, the discrete-time convolution is also very important for digital signal processing.