Convolution

Convolution is one of the primary concepts of linear system theory. It gives the answer to the problem of finding the system zero-state response due to any input—the most important problem for linear systems. The main convolution theorem states that *the response of a system at rest (zero initial conditions) due to any input is the convolution of that input and the system impulse response*. We have already seen and derived this result in the frequency domain in Chapters 3, 4, and 5, hence, the main convolution theorem is applicable to $j\omega$, s, and z domains, that is, it is applicable to both continuous- and discrete-time linear systems.

In this chapter, we study the convolution concept in the time domain.

6.1 Convolution of Continuous-Time Signals

The continuous-time convolution of two signals $f_1(t)$ and $f_2(t)$ is defined by

$$f(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau, \quad -\infty < t < \infty$$

In this integral τ is a dummy variable of integration, and t is a parameter.

Before we state the convolution properties, we first introduce the notion of the ***signal duration***. The duration of a signal $f_i(t)$ is defined by the time instants t_i and T_i for which for every t outside the interval $[t_i, T_i]$ the signal is equal to zero, that is, $f_i(t) = 0$, $t \notin [t_i, T_i]$. Signals that have finite duration are often called *time-limited signals*. For example, rectangular and triangular pulses are time-limited signals, but $u(t), \sin(t), \cos(t)$ have infinite time durations.

The properties of the convolution integral are:

1) *Commutativity*

$$f_1(t) * f_2(t) = f_2(t) * f_1(t)$$

2) *Distributivity*

$$f_1(t) * \{f_2(t) + f_3(t)\} = f_1(t) * f_2(t) + f_1(t) * f_3(t)$$

3) *Associativity*

$$f_1(t) * \{f_2(t) * f_3(t)\} = \{f_1(t) * f_2(t)\} * f_3(t)$$

4) *Duration*

Let the signals $f_1(t)$ and $f_2(t)$ have the durations, respectively, defined by the time intervals $[t_1, T_1]$ and $[t_2, T_2]$ then

$$f(t) = f_1(t)*f_2(t) = \begin{cases} 0, & t \leq t_1 + t_2 \\ \int\limits_{t_1+t_2}^{T_1+T_2} f_1(\tau) f_2(t-\tau) d\tau, & t_1 + t_2 \leq t \leq T_1 + T_2 \\ 0, & t \geq T_1 + T_2 \end{cases}$$

5) *Time Shifting*

Let $f(t) = f_1(t) * f_2(t)$. Then, convolutions of shifted signals are given by

$$f_1(t-\sigma_1) * f_2(t) = f(t-\sigma_1)$$

$$f_1(t) * f_2(t-\sigma_2) = f(t-\sigma_2)$$

$$f_1(t-\sigma_1) * f_2(t-\sigma_2) = f(t-\sigma_1-\sigma_2)$$

6) *Continuity*

This property simply states that the convolution is a continuous function of the parameter t. The continuity property is useful for plotting convolution graphs and checking obtained convolution results.

Now we give some of the proofs of the stated convolution properties, which are of interest for this class.

Property 1) can be proved by introducing the change of variables in the convolution integral as $\sigma = t - \tau$. This leads to

$$f_1(t) * f_2(t) = \int_{\infty}^{-\infty} f_1(t-\sigma) f_2(\sigma)(-d\sigma)$$

$$= \int_{-\infty}^{\infty} f_2(\sigma) f_1(t-\sigma) d\sigma = f_2(t) * f_1(t)$$

Which of the two forms of the convolution integral should we choose? Definitely, the one that requires less computations. For example, while convolving $e^{-t}u(t)$ and $\sin(t)$ we may use either of the integrals

$$\int_{-\infty}^{\infty} e^{-(t-\tau)} u(t-\tau) \sin(\tau) d\tau = \int_{-\infty}^{\infty} e^{-\tau} u(\tau) \sin(t-\tau) d\tau$$

but the first integral requires less computational effort than the second one. Thus, the better choice is to use the first integral.

The proof of Property 2) follows from the well known integral addition property

$$f_1(t) * \{f_2(t) + f_3(t)\} = \int_{-\infty}^{\infty} f_1(\tau)[f_2(t-\tau) + f_3(t-\tau)]d\tau$$

$$= \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau + \int_{-\infty}^{\infty} f_1(\tau)f_3(t-\tau)d\tau$$

$$= f_1(t) * f_2(t) + f_1(t) * f_3(t)$$

Property 4) can be verified by examining the integration limits for the case when both signals are time-limited. It can be observed that the signals $f_1(\tau)$ and $f_2(t-\tau)$ overlap only in the interval $t \in [t_1 + t_2, T_1 + T_2]$, hence the convolution is equal to zero outside of this time interval.

The proof of Property 5) follows directly from the definition of the convolution integral. This property is used to simplify the graphical convolution procedure.

The proofs of Properties 3) and 6) are omitted.

Example 6.1: Consider the convolution of the delta impulse (singular) signal and any other regular signal $f(t)$

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau, \quad -\infty < t < \infty$$

Based on the sifting property of the delta impulse signal we conclude that

$$f(t) * \delta(t) = f(t)$$

Example 6.2: We have already seen in the context of the integral property of the Fourier transform that the convolution of the unit step signal with a regular function (signal) produces function's integral in the specified limits, that is

$$f(t) * u(t) = \int_{-\infty}^{\infty} f(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} f(\tau)d\tau, \quad -\infty \leq t \leq \infty$$

Note that $u(t-\tau) = 0$ for $\tau > t$.

Example 6.3: Consider the convolution of $e^{-t}u(t)$ and $\sin(t)$

$$e^{-t}u(t) * \sin(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)}u(t-\tau)\sin(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\tau}u(\tau)\sin(t-\tau)d\tau$$

We will evaluate both integrals to show the difference in the computations required.

The first convolution integral produces

$$e^{-t}u(t) * \sin(t) = e^{-t}\int_{-\infty}^{t} e^{\tau}\sin(\tau)d\tau$$

$$= e^{-t}\left[\frac{e^{t}}{2}(\sin(t) - \cos(t)) - 0\right] = \frac{1}{2}(\sin(t) - \cos(t))$$

The evaluation of the second integral requires first an expansion of $\sin(t-\tau)$ term, that is

$$\int_0^\infty e^{-\tau}\sin(t-\tau)d\tau = \int_0^\infty e^{-\tau}[\sin(t)\cos(\tau)-\cos(t)\sin(\tau)]d\tau$$

which gives

$$\sin(t)\int_0^\infty e^{-\tau}\cos(\tau)d\tau - \cos(t)\int_0^\infty e^{-\tau}\sin(\tau)d\tau$$

$$= \sin(t)\left[0-\frac{1}{2}(-1\cos(0)+\sin(0))\right]$$

$$-\cos(t)\left[0-\frac{1}{2}(-\sin(0)-\cos(0))\right]$$

$$= \frac{1}{2}\sin(t)-\frac{1}{2}\cos(t) = e^{-t}u(t)*\sin(t)$$

Thus, both convolution integrals produce the same result, but the first one is obviously less computationally involved.

6.1.1 Graphical Convolution

The graphical presentation of the convolution integral helps in the understanding of every step in the convolution procedure. According to the definition integral, the convolution procedure involves the following steps:

Step 1: Apply the convolution duration property to identify intervals in which the convolution is equal to zero.

Step 2: Flip about the vertical axis one of the signals (the one that has a simpler form (shape) since the commutativity holds), that is, represent one of the signals in the time scale $-\tau$.

Step 3: Vary the parameter t from $-\infty$ to ∞, that is, slide the flipped signal from the left to the right, look for the intervals where it overlaps with the other signal, and evaluate the integral of the product of two signals in the corresponding intervals.

In the above steps one can also incorporate (if applicable) the convolution time shifting property such that all signals start at the origin. In such a case, after the final convolution result is obtained the convolution time shifting formula should be applied appropriately. In addition, the convolution continuity property may be used to check the obtained convolution result, which requires that at the boundaries of adjacent intervals the convolution remains a continuous function of the parameter $\boldsymbol{t}$.

We present several graphical convolution problems starting with the simplest one.

Example 6.4: Consider two rectangular pulses given in Figure 6.1.

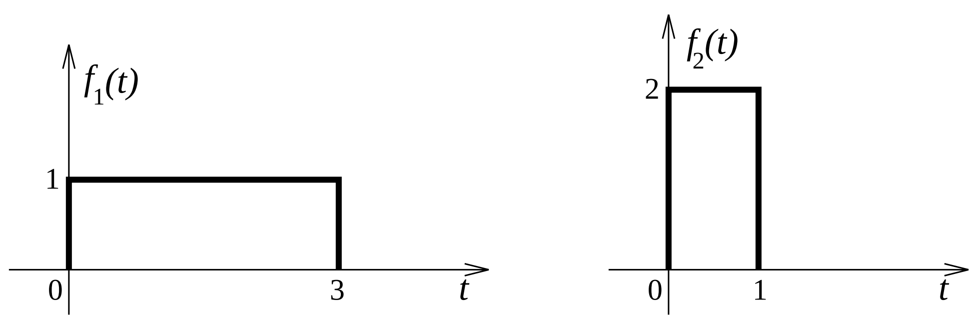


Figure 6.1: Two rectangular signals

Since the durations of the signals $f_1(t)$ and $f_2(t)$ are respectively given by $[t_1, T_1] = [0, 3]$ and $[t_2, T_2] = [0, 1]$, we conclude that the convolution of these two signals is zero in the following intervals (Step 1)

$$f_1(t) * f_2(t) = 0, \quad t \le t_1 + t_2 = 0 + 0 = 0$$

$$f_1(t) * f_2(t) = 0, \quad t \ge T_1 + T_2 = 1 + 3 = 4$$

Thus, we need only to evaluate the convolution integral in the interval $0 \le t \le 4$.

In the second step, we flip about the vertical axis the signal which has a simpler shape. Since in this case both signals are rectangular pulses it is irrelevant which one is flipped. Let us flip $f_2(t)$. Note that the convolution is performed in the time scale τ. In Figure 6.2 we present the signals $f_1(\tau)$ and $f_2(-\tau)$. Figure 6.2 corresponds to the convolution for $t = 0$.

In Step 3, we shift the signal $f_2(-\tau)$ to the left and to the right, that is, we form the signal $f_2(t-\tau)$ for $t \in (-\infty, 0]$ and $t \in [0, +\infty)$. A shift of the signal $f_2(t-\tau)$ to the left $(t < 0)$ produces no overlapping between the signals $f_1(\tau)$ and $f_2(t-\tau)$, thus the convolution integral is equal to zero for $t < 0$ (see Figure 6.3). Note that the same conclusion has been already made in Step 1.

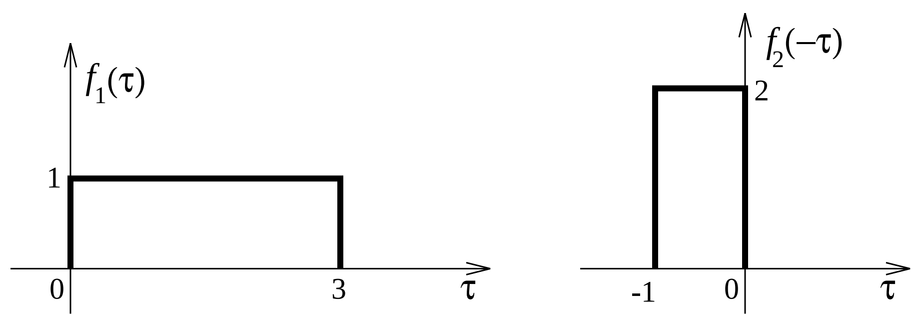


Figure 6.2: Signals $f_1(\tau)$ and $f_2(-\tau)$

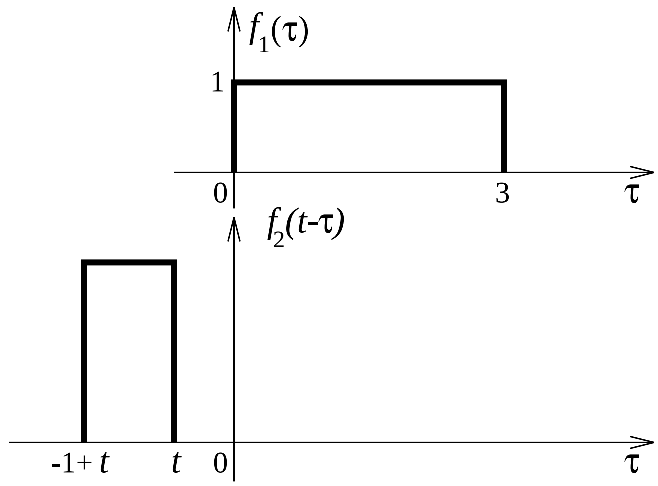


Figure 6.3: Signals $f_1(\tau)$ and $f_2(t-\tau),\ t < 0$

Let us start shifting the signal $f_2(t-\tau)$ to the right ($t > 0$). Consider first the interval $0 \leq t \leq 1$ (see Figure 6.4).

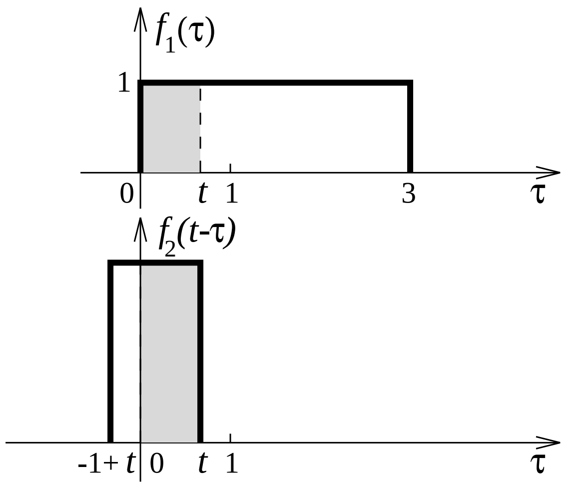


Figure 6.4: Signals $f_1(\tau)$ and $f_2(t-\tau)$, $0 \leq t \leq 1$

It can be seen from Figure 6.4 that in the interval from zero to t the signals overlap, hence their product is different from zero in this interval, which implies that the convolution integral is given by

$$f_1(t) * f_2(t) = \int_0^t 1 \times 2d\tau = 2t, \quad 0 \leq t \leq 1$$

By shifting the signal $f_2(t-\tau)$ further to the right, we get the same "kind of overlap" for $1 \leq t \leq 3$, see Figure 6.5.

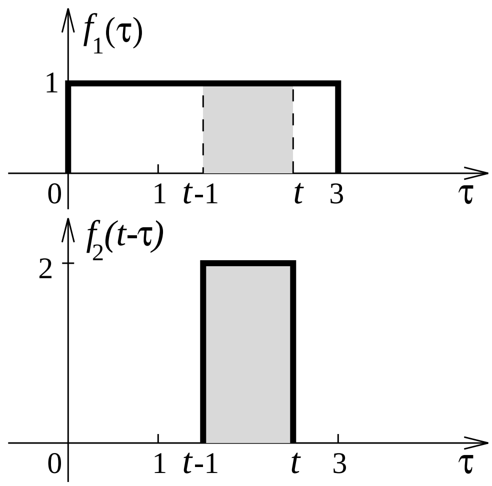


Figure 6.5: Signals $f_1(\tau)$ and $f_2(t-\tau)$, $1 \leq t \leq 3$

From this figure we see that the actual convolution integration limits are from $t-1$ to t, that is

$$f_1(t) * f_2(t) = \int_{t-1}^{t} 1 \times 2 d\tau = 2, \quad 1 \leq t \leq 3$$

By shifting $f_2(t-\tau)$ further to the right, for $3 \leq t \leq 4$, we get the situation presented in Figure 6.6.

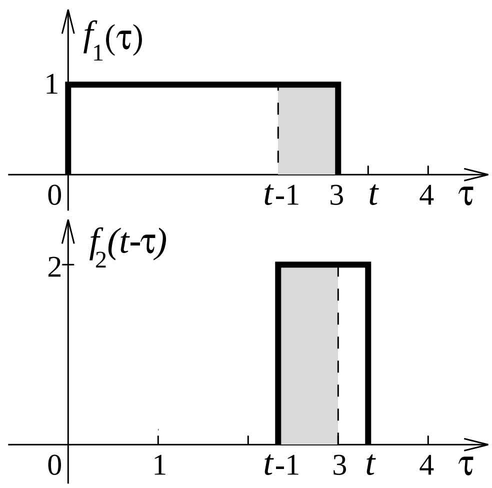


Figure 6.6: Signals $f_1(\tau)$ and $f_2(t-\tau)$, $3 \leq t \leq 4$

In this interval, the convolution integral is given by

$$f_1(t) * f_2(t) = \int_{t-1}^{3} 1 \times 2 d\tau = 8 - 2t, \quad 3 \leq t \leq 4$$

For $t > 4$, the convolution is equal to zero as determined in Step 1. This can be justified by the fact that the signals $f_1(\tau)$ and $f_2(t-\tau)$ do not overlap for

$t > 4$, that is, their product is equal to zero for $t > 4$, which implies that the corresponding integral is equal to zero in the same interval, see Figure 6.7.

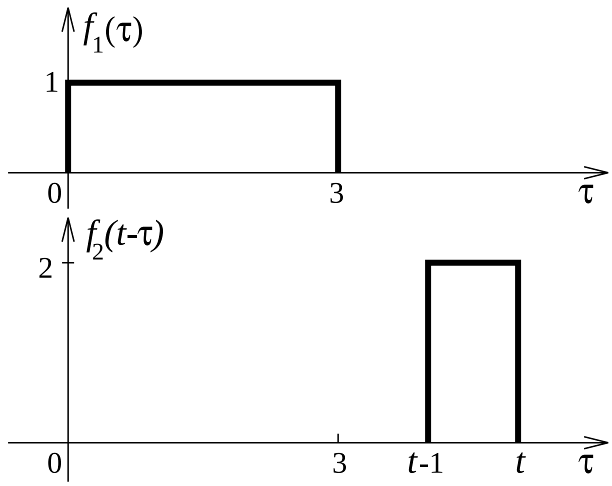


Figure 6.7: Signals $f_1(\tau)$ and $f_2(t-\tau)$, $t > 4$

In summary, the convolution of the considered signals is given by

$$f_1(t) * f_2(t) = \begin{cases} 0 & t \leq 0 \\ 2t & 0 \leq t \leq 1 \\ 2 & 1 \leq t \leq 3 \\ 8 - 2t & 3 \leq t \leq 4 \\ 0 & t \geq 4 \end{cases}$$

Note that from the convolution continuity property, the convolution signal obtained is a continuous function of t. This can be easily checked as follows. For $t = 0$ the expression $2t$ produces zero. At $t = 1$ we see that $2t = 2 \times 1 = 2$, also for $t = 3$ we have $8 - 2t = 8 - 2 \times 3 = 2$, and finally for $t = 4$ we get $8 - 2t = 8 - 2 \times 4 = 0$. Thus the function obtained, $f_1(t) * f_2(t)$, is a continuous function of the parameter t.

Example 6.5: Let us convolve the signals represented in Figure 6.8.

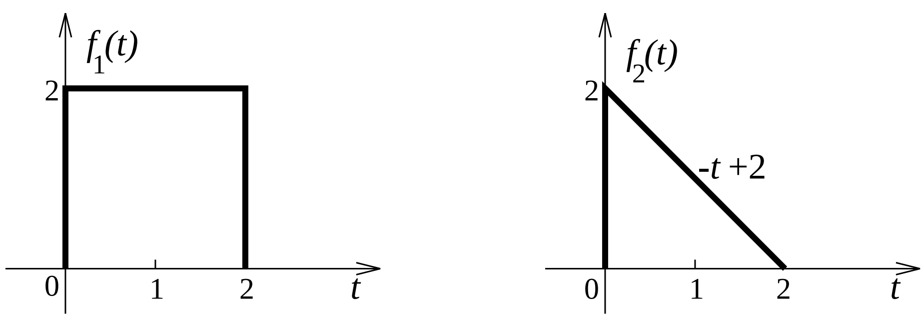


Figure 6.8: Two signals: rectangular and triangular pulses

Since both signals have the duration intervals from zero to two, we conclude that the convolution integral is zero for $t \leq 0$ and $t \geq 4$.

In the next step we flip about the vertical axis the rectangular signal since it apparently has a simpler shape, see Figure 6.9a. In Step 3, we slide the rectangular signal to the right for $\boldsymbol{t} \in [\mathbf{0}, \mathbf{2}]$, Figure 6.9b, and for $\boldsymbol{t} \in [\mathbf{2}, \mathbf{4}]$, Figure 6.9c.

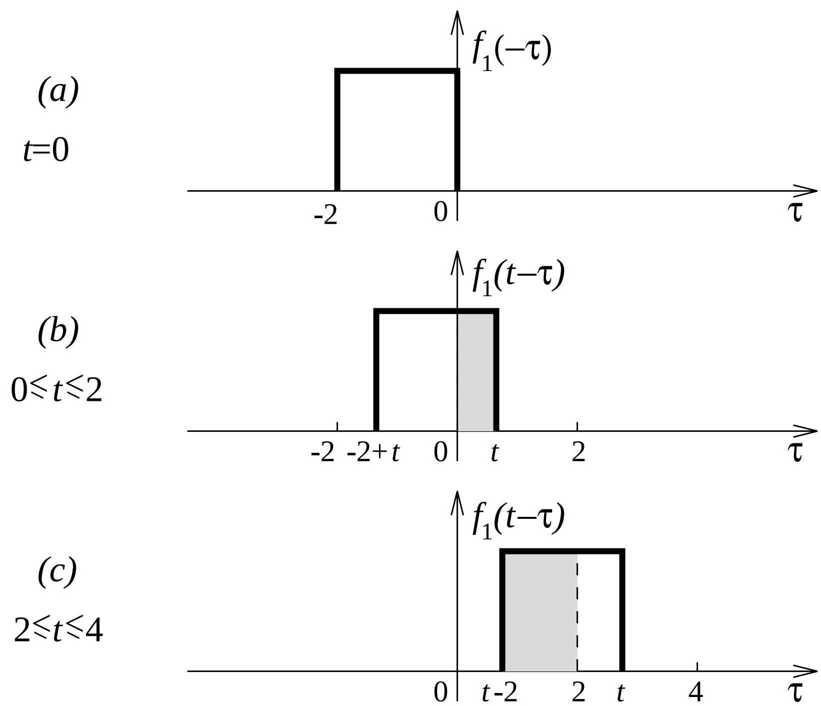


Figure 6.9: Convolution procedure for signals in Example 6.5

The convolution integral in these two intervals, evaluated according to information given in Figures 6.9b and 6.9c, is respectively given by

$$f_1(t) * f_2(t) = \int_0^t 2(-\tau + 2)d\tau = 4t - t^2, \quad 0 \le t \le 2$$

$$f_1(t) * f_2(t) = \int_{t-2}^2 2(-\tau + 2)d\tau = 16 - 8t + t^2, \quad 2 \le t \le 4$$

In summary, we have obtained

$$f_1(t) * f_2(t) = \begin{cases} 0 & t \le 0 \\ 4t - t^2 & 0 \le t \le 2 \\ 16 - 8t + t^2 & 2 \le t \le 4 \\ 0 & t \ge 4 \end{cases}$$

It can be easily checked that the obtained convolution result represents a continuous function of the parameter t.

Example 6.6: We now consider the slightly more difficult problem of convolving two signals with triangular shapes, as presented in Figure 6.10. Note that the signals are represented in the time scale τ. The problem is more difficult in the sense that we must flip about the vertical axis one of these two triangularly shaped signals and find its analytical expression. The remaining part of the problem is the standard convolution technique.

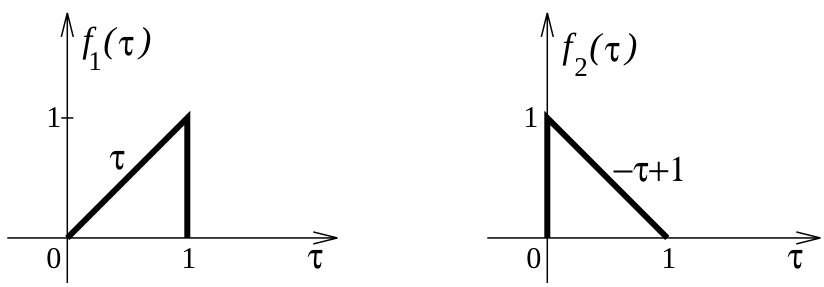


Figure 6.10: Two triangularly shaped signals

Let us flip about the vertical axis the signal $f_2(\tau)$. The flipped signal for $t = 0$ is presented in Figure 6.11a.

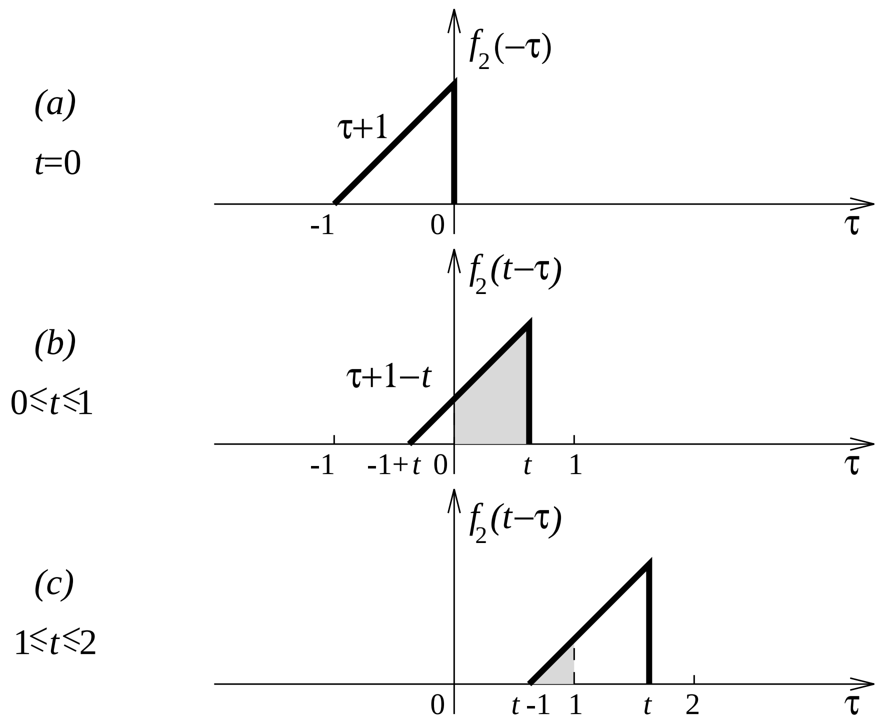


Figure 6.11: Convolution procedure for signals in Example 6.6

Note its new analytical expression, now given by $f_2(-\tau) = \tau + 1$. From the convolution duration property, we conclude that the convolution is equal to zero for $t \leq 0$ and $t \geq 2$. Thus, we have to work only in the interval $0 \leq t \leq 2$.

Consider the interval $0 \le t \le 1$. In this interval, the signal $f_2(t - \tau)$ is given by $f_2(t - \tau) = \tau - t + 1$, see Figure 11.b. Since the signal $f_1(\tau)$ overlaps with the signal $f_2(t - \tau)$ in the interval from zero to t, the convolution is given by

$$f_1(t) * f_2(t) = \int_0^t \tau(\tau - t + 1)d\tau = -\frac{1}{6}t^3 + \frac{1}{2}t^2$$

For $1 \le t \le 2$, the signal $f_2(t - \tau)$, presented in Figure 6.11c, overlaps with the signal $f_1(\tau)$ in the interval from $t - 1$ to 1. Here, the convolution is given by

$$f_1(t) * f_2(t) = \int_{t-1}^{1} \tau(\tau - t + 1)d\tau = \frac{1}{3} - \frac{1}{2}(t-1) + \frac{1}{6}(t-1)^3$$

Example 6.7: Consider the convolution problem that involves shifted signals as presented in Figure 6.12.

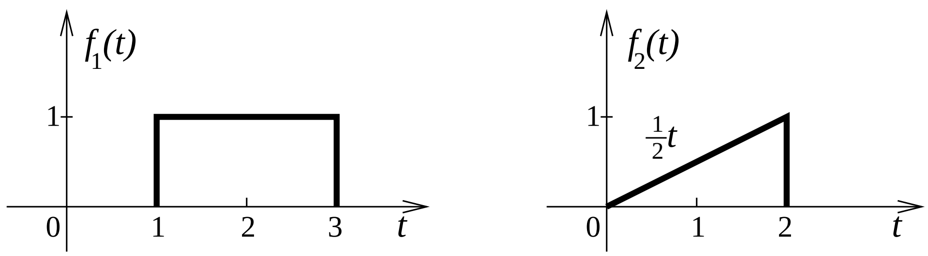


Figure 6.12: Convolution with a shifted signal

According to the convolution time shifting property, we can shift the signal $f_1(t)$ to the origin and find the convolution of the shifted signal $f_1(t+1)$ and the signal $f_2(t)$. Let $f(t+1)$ represent the convolution of $f_1(t+1)$ and $f_2(t)$. In order to find the required original convolution result, the convolution obtained through the regular convolution procedure with $f_1(t+1)$ and $f_2(t)$ has to be shifted backward by one unit.

However, we can convolve these two signal without applying the convolution time shifting property as demonstrated below. Since the time durations of these signals are respectively given by $[t_1, T_1] = [1, 3]$ and $[t_2, T_2] = [0, 2]$, we conclude that the corresponding convolution is equal to zero for $t \le t_1 + t_2 = 1$ and $t \ge T_1 + T_2 = 5$.

Let us flip about the vertical axis the rectangular signal $f_1(t)$, Figure 6.13a.

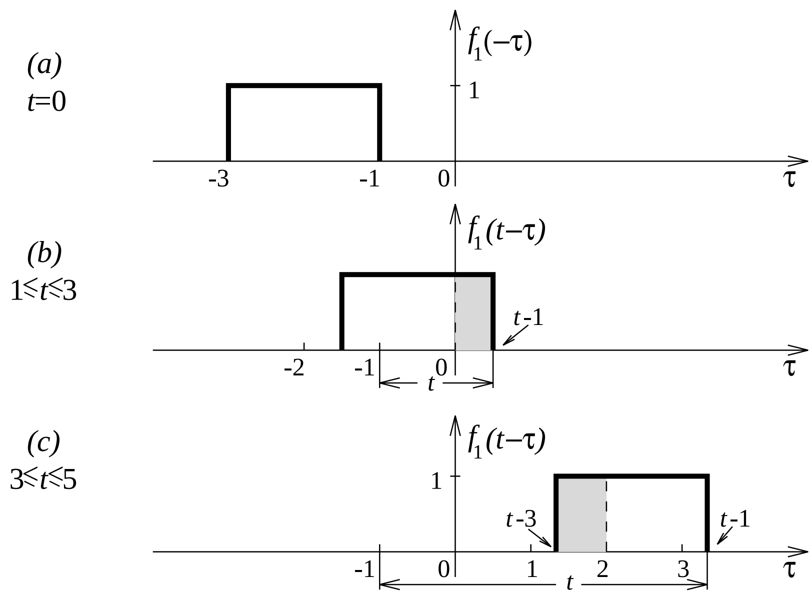


Figure 6.13: Convolution process for signals in Example 6.7

In the interval $1 \le t \le 3$, the convolution is given by (see Figure 6.13b)

$$f(t) = f_1(t) * f_2(t) = \int_0^{t-1} \frac{1}{2}\tau d\tau = \frac{(t-1)^2}{4}$$

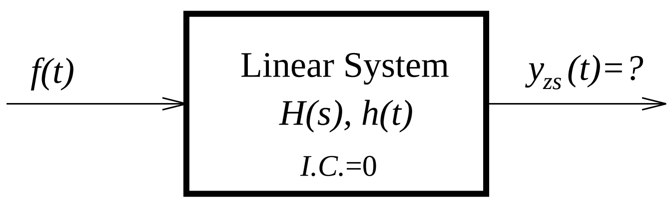
In the interval $3 \le t \le 5$, we have from Figure 6.13c

$$f(t) = f_1(t) * f_2(t) = \int_{t-3}^{2} \frac{1}{2}\tau d\tau = -\frac{1}{4}(t-1)(t-5)$$

In the next section we apply the convolution formula to linear continuous-time invariant systems and show that the system response to any input is given in terms of the convolution integral. To that end, we will use the concepts of system transfer function and system impulse response introduced in Chapters 3 and 4.

6.2 Convolution for Linear Continuous-Time Systems

Consider the problem of finding the response of a system at rest (zero initial conditions) due to any input, say $f(t)$. In the previous chapters on the frequency domain techniques, we have seen that every linear system is characterized by the system transfer function and/or by the system impulse response. The problem that we are faced with is symbolically presented in Figure 6.14.



6.14: System response due to an arbitrary input

Recall that $H(s) = \mathcal{L}\{h(t)\}$. We assume that the system initial conditions are zero (system at rest) and that $f(t)$ is causal ($f(t) = 0, t < 0$). Hence, we are interested in finding the *system zero-state response* for $t \geq 0$ due to the input signal $f(t)$ applied to the system at $t = 0$.

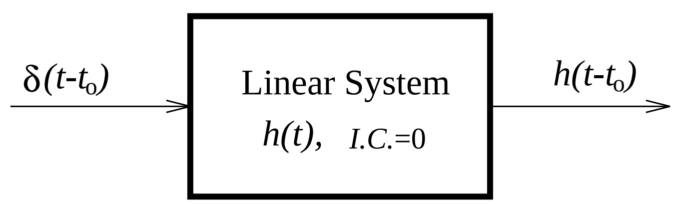
Since the system impulse response is known, we do know the answer to the system impulse response problem, which is symbolically presented in Figure 6.15. If we introduce $\mathbf{L}$ to denote the system action (system operation) on the known input we will have

$$h(t) = \mathrm{L}\{\delta(t)\}$$

or similarly, by assuming time invariance

$$h(t - t_0) = \mathrm{L}\{\delta(t - t_0)\}$$

Note that the linear system action $\mathbf{L}$ is performed in the time scale t.



6.15: System response due to the impulse delta function

We can present any input signal $f(t)$ in terms of the delta impulse signal as (see Example 6.1)

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = f(t) * \delta(t)$$

This follows from the sifting property of the impulse delta function and from the definition of the convolution integral.

Applying the linear system action $\mathbf{L}$ to the input $f(t)$, we get

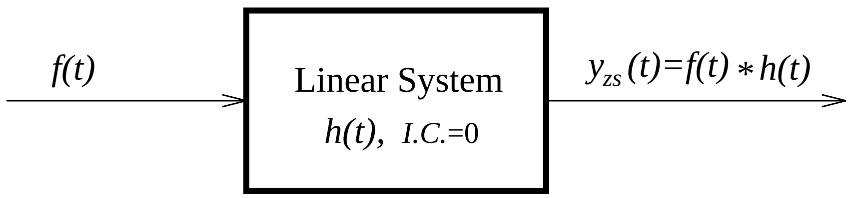
$$y_{zs}(t) = \mathrm{L}\{f(t)\} = \mathrm{L}\left\{ \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau \right\}$$

$$= \int_{-\infty}^{\infty} f(\tau)\mathrm{L}\{\delta(t-\tau)\}d\tau = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = f(t) * h(t)$$

This formula establishes in the time domain the most fundamental result of linear system theory, which can be stated in the following theorem.

Theorem 6.1 *The response of a continuous-time linear system at rest (zero-state response) due to any input is the convolution of that input and the system impulse response.*

The result of the above theorem is symbolically presented in Figure 6.16.



6.16: Zero-state system response is the convolution of the system input and the system impulse response

Note that the obtained zero-state response convolution formula

$$y_{zs}(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

can be represented as a sum of three integrals

$$y_{zs}(t) = \int_{-\infty}^{0} f(\tau)h(t-\tau)d\tau + \int_{0}^{t} f(\tau)h(t-\tau)d\tau + \int_{t}^{\infty} f(\tau)h(t-\tau)d\tau$$

Since $f(t) = 0$ for $t < 0$ (causal input signal), the first integral is equal to zero. In the third integral, the integration is performed in the region $\tau > t$ where $h(t-\tau) = 0$ (causal linear system), hence, the third integral is also equal to zero. Thus, we are left with

$$y_{zs}(t) = \int_{0}^{t} f(\tau)h(t-\tau)d\tau$$

which produces the zero-state system response at time t due to an input signal $f(t)$. By introducing a change of variable as $\sigma = t - \tau$, it can be easily shown that

$$y_{zs}(t) = \int_{0}^{t} f(\tau)h(t-\tau)d\tau = \int_{0}^{t} f(t-\sigma)h(\sigma)d\sigma$$

Example 6.8: Given a linear dynamic system represented by

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{df(t)}{dt} + 2f(t)$$

$$f(t) = e^{-2t}u(t), \quad y(0^-) = \frac{dy(0^-)}{dt} = 0$$

Its transfer function and the impulse response are given by

$$H(s) = \frac{(s+2)}{(s+1)(s+3)} = \frac{1/2}{s+1} + \frac{1/2}{s+3} \;\Rightarrow\; h(t) = \frac{1}{2}\left(e^{-t} + e^{-3t}\right)u(t)$$

The system zero-state response due to $f(t) = e^{-2t}u(t)$ is

$$y_{zs}(t) = \int_0^t h(\tau)f(t-\tau)d\tau = \frac{1}{2}\int_0^t \left(e^{-\tau} + e^{-3\tau}\right)e^{-2(t-\tau)}d\tau$$

$$= \frac{1}{2}e^{-2t}\int_0^t \left(e^{\tau} + e^{-\tau}\right)d\tau = \frac{1}{2}\left(e^{-t} - e^{-3t}\right), \quad t \geq 0$$

Example 6.9: Assume that the same system from Example 6.8 is driven by another external forcing function, for example $f(t) = \sin(t)u(t)$. In this case, we need only evaluate the corresponding convolution integral since the system impulse response is already known

$$y_{zs}(t) = \int_0^t h(t-\tau)f(t)d\tau$$

$$= \int_0^t \frac{1}{2}\left(e^{-(t-\tau)} + e^{-3(t-\tau)}\right)u(t-\tau)\sin(\tau)u(\tau)d\tau$$

$$= \frac{1}{2}e^{-t}\int_0^t e^{\tau}\sin(\tau)d\tau + \frac{1}{2}e^{-3t}\int_0^t e^{3\tau}\sin(\tau)d\tau$$

Using the table of integrals, we obtain

$$y_{zs}(t) = \frac{1}{4}\left(e^{-t} + \frac{1}{5}e^{-3t}\right) + \frac{2}{5}\sin(t) - \frac{3}{10}\cos(t), \quad t \geq 0$$

