5.4 Block Diagrams

Using the $\mathcal{Z}$–transform linearity and convolution properties we can easily extend the concept of transfer function to configurations of several connected linear systems. We will find the equivalent transfer functions for cascade and parallel connections of systems, introduce the feedback (closed-loop) configuration, and define the corresponding feedback system transfer function.

We know from the convolution result that for a system at rest, the system input $F(z)$ produces on the system output the signal $Y_{zs}(z)$ given by

$$Y_{zs}(z) = H(z)F(z)$$

which is symbolically represented in Figure 5.11 using a block diagram, where $H(z)$ represents the *open-loop* discrete-time *system transfer function*.

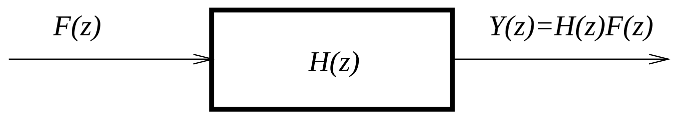
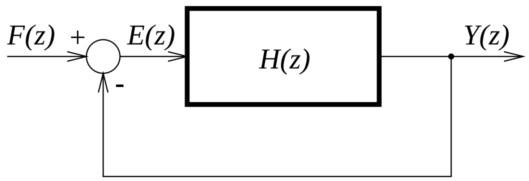


Figure 5.11: An open-loop transfer function

Note that the above block diagram can also be used in the case when the system initial conditions are different from zero. In such a case, an additive component coming from the system initial conditions should be added to the system output. For that reason in all block diagrams presented in this section we will denote the system output by $Y(z) = Y_{zs}(z) + Y_{zi}(z)$.

The open-loop transfer function $H(z)$ is derived from the difference equation that is obtained using known physical laws (mathematical modeling procedure). The accuracy of coefficients that appear in the open-loop system transfer function depends on the accuracy of the system coefficients. These coefficients are not always perfectly known. Furthermore, the coefficients change either due to aging or due to internal and/or external system disturbances. Due to changes in the system coefficients (system parameters), it can happen that the actual system output (in the open-loop system configuration) is pretty different from the one obtained analytically.

It has been indicated in Section 4.4 on the continuous-time block diagrams that a way to cope with the system parameter changes, and a way to reduce the impact of those changes on the system output, is to form the *closed-loop system* configuration, also known as the system feedback configuration. Assuming that it is feasible, we can feed back the system output and form the *closed-loop* around the system as presented in Figure 5.12a. The directed path (pay attention to the arrows) from $F(z)$ to $Y(z)$ is called the *forward path* and the directed path from $Y(z)$ to $E(z)$ is called the *feedback path*. Such a feedback loop is called a unity feedback loop. In general, we can put a dynamic element in the feedback loop $G(z)$



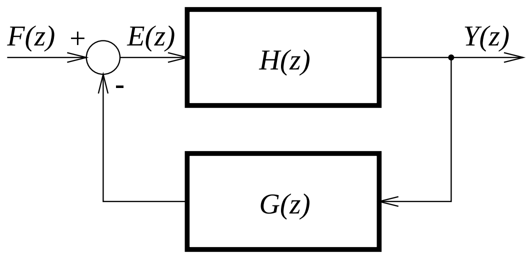


Figure 5.12: Closed-loop system configurations:

(a) unity feedback; (b) nonunity feedback

(another open-loop transfer function) as presented in Figure 5.12b. For notational convenience, we will denote the transfer function in the feedback path by $G(z)$ and the transfer function in the forward path by $H(z)$. It should be pointed out that sometimes in the feedback path we put a static element equal to a constant, that is $G(z) = \mathbf{const}$.

In the feedback configuration presented in Figure 5.12a, the output signal is fed back and compared to the input signal, and the difference of the input and output signals is used as a new input signal to the system. In practice, the feedback signal is taken with the negative sign since, in general, the positive feedback signal causes system instability. Using the convolution property and following signals in the block diagram in the direction of the arrows, we can find the closed-loop system transfer function from $F(z)$ to $Y(z)$, assuming zero initial conditions, as follows

$$Y(z) = H(z)E(z), \quad E(z) = F(z) - Y(z)$$

$$\Rightarrow Y(z) = H(z)(F(z) - Y(z))$$

$$\Rightarrow Y(z) = \frac{H(z)}{1 + H(z)} F(z) \triangleq M(z)F(z)$$

The *closed-loop system transfer function*, for unity feedback, denoted by $M(z)$ is defined by

$$M(z) = \frac{Y(z)}{F(z)}|_{I.C.=0} \triangleq \frac{H(z)}{1+H(z)}$$

The defined closed-loop transfer function is called the closed-loop transfer function with unity feedback. In many applications, in the feedback loop another transfer function is present, see Figure 5.12b. The closed-loop transfer function with non unity feedback is obtained similarly as follows

$$Y(z) = H(z)(F(z) - G(z)Y(z)) \Rightarrow Y(z) = \frac{H(z)}{1+H(z)G(z)}F(z)$$

$$Y(z) \triangleq M(z)F(z), \quad M(z) = \frac{H(z)}{1+H(z)G(z)}$$

We can form more complex configurations of open-loop transfer functions. The *cascade connection* of open-loop transfer functions is presented in Figure 5.13a. The *parallel connection* of open-loop transfer functions is represented in Figure 5.13b.

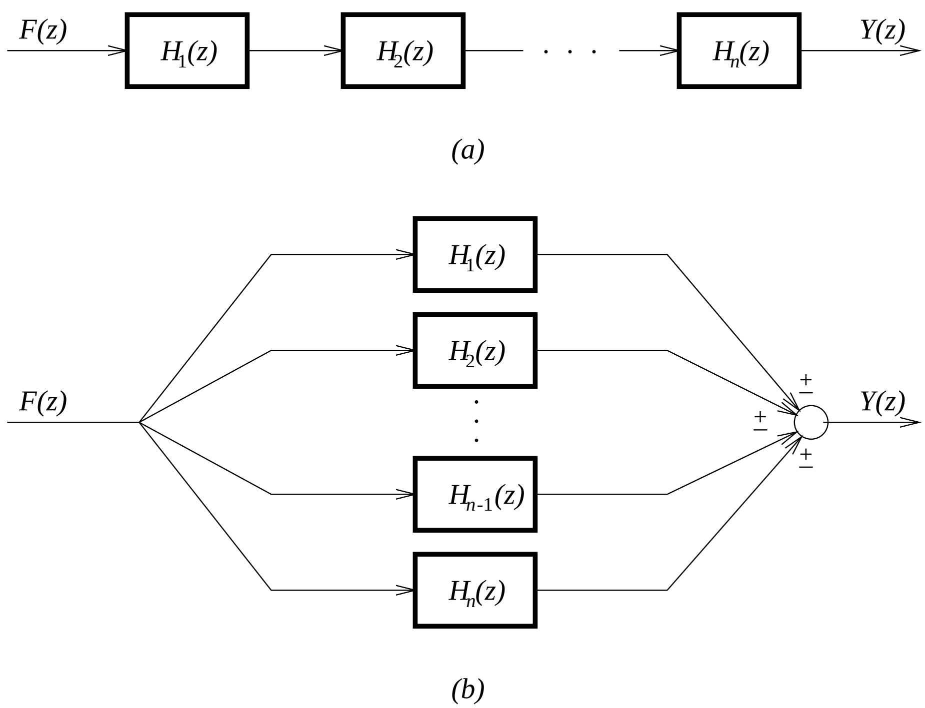


Figure 5.13: Cascade (a) and parallel (b) connections of transfer functions

It is easy to conclude that the equivalent open-loop transfer function for the cascade connection is given by the product of elementary open-loop transfer functions

$$H_{eq}^{cascade}(z) = H_1(z)H_2(z)\cdots H_n(z)$$

This formula can be called the product rule for elementary open-loop transfer functions.

The equivalent open-loop transfer function for the parallel connection is equal to the sum of elementary open-loop transfer functions, that is

$$H_{eq}^{parallel}(z) = H_1(z) \pm H_2(z) \pm \cdots \pm H_n(z)$$

This formula can be called the sum rule for elementary open-loop transfer functions.

Using the basic transfer function rules, we can simplify more complex feedback systems and represent them in the basic feedback form presented in Figure 5.12b. The corresponding procedure is demonstrated in the next example.

Example 5.28: Consider a feedback system given in Figure 5.14.

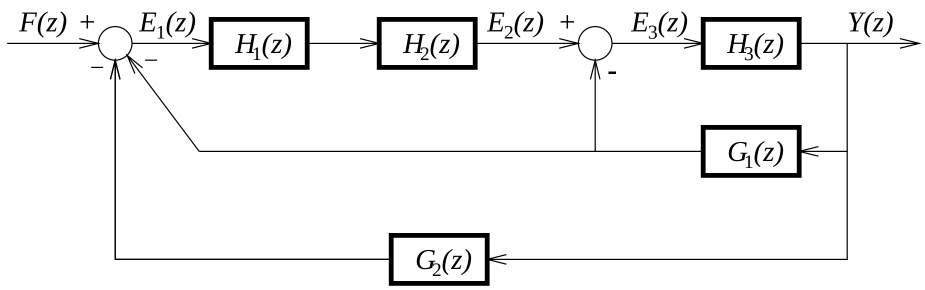


Figure 5.14: A feedback system

We find its closed-loop transfer function as follows. Using the product rule, we have

$$E_2(z) = H_1(z)H_2(z)E_1(z)$$

$$Y(z) = H_3(z)E_3(z)$$

The product rule combined with the sum rule, produces

$$E_1(z) = F(z) - G_1(z)Y(z) - G_2(z)Y(z)$$

$$E_3(z) = E_2(z) - G_1(z)Y(z)$$

Eliminating $E_i(z), i = 1, 2, 3,$ we obtain

$$Y(z) = H_1(z)H_2(z)H_3(z)F(z) - G_1(z)H_3(z)Y(z)$$

$$-(G_1(z) + G_2(z))H_1(z)H_2(z)H_3(z)Y(z)$$

This leads to the closed-loop transfer function of the form

$$M(z) = \frac{Y(z)}{F(z)}$$

$$= \frac{H_1(z)H_2(z)H_3(z)}{1 + G_1(z)H_3(z) + (G_1(z) + G_2(z))H_1(z)H_2(z)H_3(z)}$$

The algebra with transfer functions is simple and convenient for linear feedback systems composed of several loops. In the case of linear systems with many feedback loops, the problem of finding the closed-loop transfer function becomes a very tedious task. In such a case, we can use the so-called Mason's formula obtained using elementary knowledge of graph theory.