4.1 Laplace Transform and Its Properties

4.1.1 Definitions and Existence Condition

The Laplace transform of a continuous-time signal $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = F(s) \triangleq \int_{0^-}^{\infty} f(t) e^{-st} dt$$

In general, the *two-sided* Laplace transform, with the lower limit in the integral equal to $-\infty$, can be defined. For the purpose of this course, it is sufficient to use only the one-sided Laplace transform. The reason for using the lower integration limit at 0^- is the fact that the Laplace transform in linear system theory is used first of all to study the response of linear time invariant systems. We must be able to find the system impulse response, which requires integration from 0^- in order to completely include the impulse delta signal within the integration limits.

The existence condition of the Laplace transform requires

$$\left| \int_0^\infty f(t) e^{-(\sigma + j\omega)t} dt \right| < \infty$$

$$\left| \int_0^\infty f(t) e^{-(\sigma + j\omega)t} dt \right| \leq \int_0^\infty \left| f(t) e^{-(\sigma + j\omega)t} dt \right| \leq \int_0^\infty |f(t)| \left| e^{-(\sigma + j\omega)t} dt \right|$$

$$\leq \int_0^\infty |f(t)| e^{-\sigma t} dt < \infty$$

where σ is a nonnegative real number. The last inequality represents the *existence condition of the Laplace transform*.

The inverse Laplace transform can be obtained from the definition of the inverse Fourier transform using the facts that $j\omega$ has to be replaced by $s = \sigma + j\omega$ and that $jd\omega = ds$. This leads to the following definition of the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \triangleq \frac{1}{2\pi j} \int_{\gamma - j\infty}^{\gamma + j\infty} F(s) e^{st} ds$$

where γ is a real value chosen to the right of all singularities of the function $F(s)$ (most common singularities are the poles of $F(s)$, the values of s at which $F(s) = \infty$). The complete definition of singularities of complex variable functions is outside of the scope of this course.

The inverse Laplace transform represents a complex variable integral, which in general is not easy to calculate. In order to avoid integration of a complex variable function (using the method known as contour integration), the procedure used in this textbook for finding the Laplace inverse combines the method of partial fraction expansion, properties of the Laplace transform to be derived in this section and summarized in Table 4.1, and the table of common Laplace transform pairs, Table 4.2. The use of the partial fraction expansion method is sufficient for the purpose of this course. However, in general, in order to find the Laplace transform of any Laplace transformable function we must learn the complex variable integration and use the definition integral of the inverse Laplace transform.

Example 4.1: The unit step signal has the following Laplace transform

$$\mathcal{L}\{u(t)\} = \int_0^\infty u(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \frac{1}{s}$$

Example 4.2: Laplace transforms of the impulse delta signal $\delta(t)$ and its shifted version $\delta(t - t_0)$ are

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^\infty \delta(t)e^{-st}dt = 1$$

$$\mathcal{L}\{\delta(t - t_0)\} = \int_{0^-}^\infty \delta(t - t_0)e^{-st}dt = e^{-st_0}$$

Example 4.3: The exponential signal, $e^{-at}u(t),\ a > 0$, produces

$$\mathcal{L}\{e^{-at}u(t)\} = \int_0^\infty e^{-at}e^{-st}dt = \frac{1}{s + a}, \qquad a > 0$$

4.1.2 Properties of the Laplace Transform

We state and prove the main properties of the Laplace transform. In order to simplify the proofs we will use the definition formula of the Laplace transform in which the lower limit is 0^- unless explicitly indicated otherwise.

Property #1: Linearity

Let $\mathcal{L}\{f_i(t)\} = F_i(s), \quad i = 1, 2, ..., n,$ be Laplace transform pairs. Then for any constants α_i, $i = 1, 2, ..., n,$ the following holds

$$\mathcal{L}\{\alpha_1 f_1(t) \pm \alpha_2 f_2(t) \pm \cdots \pm \alpha_n f_n(t)\}$$

$$= \alpha_1 \mathcal{L}\{f_1(t)\} \pm \alpha_2 \mathcal{L}\{f_2(t)\} \pm \cdots \pm \alpha_n \mathcal{L}\{f_n(t)\}$$

$$= \alpha_1 F_1(s) \pm \alpha_2 F_2(s) \pm \cdots \pm \alpha_n F_n(s)$$

Proof: This property can be proved by using the known properties of integrals, as follows

$$\mathcal{L}\{\alpha_1 f_1(t) \pm \alpha_2 f_2(t) \pm \cdots \pm \alpha_n f_n(t)\}$$
$$= \int_0^\infty [\alpha_1 f_1(t) \pm \alpha_2 f_2(t) \pm \cdots \pm \alpha_n f_n(t)] e^{-st} dt$$
$$= \alpha_1 \int_0^\infty f_1(t) e^{-st} dt \pm \alpha_2 \int_0^\infty f_2(t) e^{-st} dt \pm \cdots \pm \alpha_n \int_0^\infty f_n(t) e^{-st} dt$$
$$= \alpha_1 F_1(s) \pm \alpha_2 F_2(s) \pm \cdots \pm \alpha_n F_n(s)$$

Hence, the Laplace transform of a linear combination of signals is a linear combination of the Laplace transforms of the signals.

Property #2: Time Shifting

This property states

$$\mathcal{L}\{f(t)u(t)\} = F(s) \;\Rightarrow\; \mathcal{L}\{f(t-t_0)u(t-t_0)\} = e^{-t_0 s}F(s), \quad t_0 > 0$$

where t_0 is the positive time shifting parameter. It should be emphasized that shifting the signal left in time as defined by $f(t+t_0)u(t+t_0),\ t_0 > 0$, in general, violates signal causality so that the one-sided Laplace transform can not be correctly applied. For that reason the stated time shifting property is also called the right shift in time property. Only if the signal remains causal under the left time shifting, we will be able to find the corresponding one-sided Laplace transform. For example the rectangular pulse $p_2(t-3)$ can be shifted to the left by two time units and still remain causal.

Proof: By the definition of the Laplace transform, we have

$$\mathcal{L}\{f(t-t_0)u(t-t_0)\} = \int_{t=0}^{t=\infty} f(t-t_0)u(t-t_0)e^{-st}dt$$

Using the change of variables as $t - t_0 = \tau$, we obtain

$$\int_{\tau=-t_0}^{\tau=\infty} f(\tau)u(\tau)e^{-s(t_0+\tau)}d\tau = e^{-st_0}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-st_0}F(s)$$

We want to emphasize the importance of the proper use of the time shifting since in contrast to the Fourier transform where $t \in (-\infty, \infty)$, here, we deal with the semi-infinite time axis $t \in [0, \infty)$. In Figure 4.1, we represent the original function (a), function properly time-shifted (b), and function which analytically looks like being time shifted (c), however that function is only multiplied by e^{t_0} and not shifted in time.

It can be concluded from Figure 4.1 that the causal function (signal) $f(t)$ is properly shifted by finding $f(t - t_0)$ and multiplying it by $u(t - t_0)$, that is, by forming the signal $f(t - t_0)u(t - t_0)$.

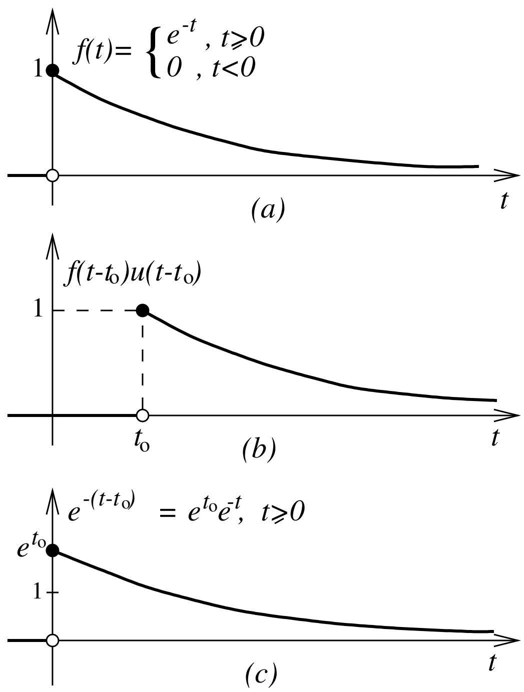

Figure 4.1: Proper shifting of a continuous-time function (signal)

Example 4.4: Using the time shifting property, we find the Laplace transform of the signal $\left(t^2+1\right)u(t-1)$ in the following manner

$$\mathcal{L}\left\{\left(t^2+1\right)u(t-1)\right\} = \mathcal{L}\left\{\left(t^2+2t-2t+1\right)u(t-1)\right\}$$

$$= \mathcal{L}\left\{\left[(t-1)^2+2t\right]u(t-1)\right\}$$

$$= \mathcal{L}\left\{(t-1)^2u(t-1)\right\} + \mathcal{L}\{2tu(t-1)\}$$

$$= e^{-s}\mathcal{L}\left\{t^2u(t)\right\} + \mathcal{L}\{2(t-1+1)u(t-1)\}$$

$$= \frac{2e^{-s}}{s^3} + \mathcal{L}\{2(t-1)u(t-1)\}$$

$$+\mathcal{L}\{2u(t-1)\} = \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{2e^{-s}}{s}$$

Property #3: Time Scaling

The property states

$$\mathcal{L}\{f(t)\} = F(s) \;\Rightarrow\; \mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \qquad a > 0$$

where a is a *positive* time scaling parameter. Note that if the parameter a is negative the original causal signal $f(t)$ is transformed into a noncausal signal $f(-|a|t)$ (time reversal and time scaling) so that the single-sided Laplace transform is not applicable in this case.

Proof: Take the Laplace transform of the signal $f(at)$ and introduce the change of variables as $\sigma = at$, $a > 0$. This leads to

$$\mathcal{L}\{f(at)\} = \int_0^\infty f(at)e^{-st}dt = \frac{1}{a}\int_0^\infty f(\sigma)e^{-\frac{s}{a}\sigma}d\sigma = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0$$

We can also combine both the time scaling and time shifting properties and get

$$\mathcal{L}\{f(a(t-t_0))u(t-t_0)\} = \frac{1}{a}F\left(\frac{s}{a}\right)e^{-t_0 s}, \quad a > 0, \quad t_0 > 0$$

This formula is verified in Problem 4.1.

Problem 4.1

By the definition of the Laplace transform we have

$$\mathcal{L}\{f(a(t-t_0))u(t-t_0)\} = \int_0^\infty f(a(t-t_0))u(t-t_0)e^{-st}dt$$

Using the change of variables as $\sigma = at - t_0,\ a > 0$, we obtain

$$\mathcal{L}\{f(a(t-t_0))u(t-t_0)\} = \int_{-at_0}^{\infty} f(\sigma)u\left(\frac{\sigma}{a}\right)e^{-s\left(t_0+\frac{\sigma}{a}\right)}\frac{d\sigma}{a}$$

$$= \frac{1}{a}e^{-st_0}\int_0^\infty f(\sigma)e^{-\left(\frac{s}{a}\right)\sigma}d\sigma = \frac{1}{a}e^{-st_0}F\left(\frac{s}{a}\right)$$

Property #4: Time Multiplication

It states

$$\mathcal{L}\{f(t)\} = F(s) \;\Rightarrow\; \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$$

where t represents time and s is the complex frequency.

Proof: The proof of this property is as follows

$$F(s) = \int_0^\infty f(t)e^{-st}dt \Rightarrow \frac{dF(s)}{ds} = \int_0^\infty (-t)f(t)e^{-st}dt$$

$$(-1)\frac{dF(s)}{ds} = \int_0^\infty tf(t)e^{-st}dt = \mathcal{L}\{tf(t)\}$$

In general, taking the nth derivative we have

$$\frac{d^n F(s)}{ds^n} = \int_0^\infty (-t)^n f(t)e^{-st}dt$$

which after a multiplication by $(-1)^n$ produces the stated result.

Example 4.5: The Laplace transforms of the signals $t^n u(t),\ n = 1, 2, \ldots,$ are obtained by applying the time multiplication property to $u(t) \leftrightarrow 1/s$

$$\mathcal{L}\{u(t)\} = \frac{1}{s} \ \Rightarrow \ tu(t) \leftrightarrow (-1)\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

$$t^2 u(t) \leftrightarrow (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s}\right) = \frac{2}{s^3}$$

$$\cdots$$

$$t^n u(t) \leftrightarrow (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right) = \frac{n!}{s^{n+1}}$$

Property #5: Frequency Shifting

This property is given by

$$\mathcal{L}\{f(t)\} = F(s) \;\Rightarrow\; \mathcal{L}\left\{f(t)e^{\lambda t}\right\} = F(s-\lambda)$$

where λ represents the frequency shift.

Proof: A simple and a short proof of this property is as follows

$$L\left\{f(t)e^{\lambda t}\right\} = \int_0^\infty f(t)e^{\lambda t}e^{-st}dt = \int_0^\infty f(t)e^{-(s-\lambda)t}dt \triangleq F(s-\lambda)$$

The next property, signal modulation, is a direct consequence of the frequency shifting property.

Property #6: Modulation

This property in the Fourier domain ($j\omega$-domain) is very important for communication systems. In the s-domain, it can be used to find the Laplace transform of some signals. The modulation property is directly derived from the frequency shifting property by using Euler's formula and $\pm j\omega$ frequency shifts of the function $F(s)$. Namely, the relations

$$f(t)e^{j\omega_0 t} = f(t)[\cos(\omega_0 t) + j\sin(\omega_0 t)] \leftrightarrow F(s - j\omega_0)$$

$$f(t)e^{-j\omega_0 t} = f(t)[\cos(\omega_0 t) - j\sin(\omega_0 t)] \leftrightarrow F(s + j\omega_0)$$

imply the modulation property, defined by

$$f(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[F(s + j\omega_0) + F(s - j\omega_0)]$$
$$f(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}[F(s + j\omega_0) - F(s - j\omega_0)]$$

Example 4.6: Laplace transforms of the cosine and sine functions can be found by using the modulation property as follows

$$u(t) \leftrightarrow \frac{1}{s} \;\Rightarrow\; u(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[\frac{1}{s+j\omega_0} + \frac{1}{s-j\omega_0}\right] = \frac{s}{s^2+\omega_0^2}$$

$$u(t) \leftrightarrow \frac{1}{s} \;\Rightarrow\; u(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}\left[\frac{1}{s+j\omega_0} - \frac{1}{s-j\omega_0}\right] = \frac{\omega_0}{s^2+\omega_0^2}$$

Example 4.7: In this example we find the Laplace transform of the signal $te^{-2t}\sin(\pi t)u(t-2)$. In the derivations we use the trigonometric formula

$$\sin(\pi(t-2+2)) = \sin(\pi(t-2))\cos(2\pi) + \cos(\pi(t-2))\sin(2\pi)$$

$$= \sin(\pi(t-2))$$

The derivations are given below

$$\mathcal{L}\{te^{-2t}\sin(\pi t)u(t-2)\}$$

$$= \mathcal{L}\Big\{(t-2+2)e^{-2(t-2+2)}\sin(\pi(t-2+2))u(t-2)\Big\}$$

$$= e^{-4}\mathcal{L}\Big\{(t-2)e^{-2(t-2)}\sin(\pi(t-2))u(t-2)\Big\}$$

$$+2e^{-4}\mathcal{L}\Big\{e^{-2(t-2)}\sin(\pi(t-2))u(t-2)\Big\}$$

Now we can apply the time shifting property, which leads to

$$\mathcal{L}\{te^{-2t}\sin(\pi t)u(t-2)\}$$

$$= e^{-4}e^{-2s}\mathcal{L}\{te^{-2t}\sin(\pi t)u(t)\} + 2e^{-4}e^{-2s}\mathcal{L}\{e^{-2t}\sin(\pi t)u(t)\}$$

$$= e^{-4}e^{-2s}\frac{2\pi(s+2)}{\Big[(s+2)^2+\pi^2\Big]^2} + 2e^{-4}e^{-2s}\frac{\pi}{(s+2)^2+\pi^2}$$

Property #7: Time Derivatives

From the linear system theory point of view, the derivative property of the Laplace transform is one of the most important ones. It helps in converting constant coefficient differential equations into complex coefficient algebraic equations. We can solve these algebraic equations rather easily and find the system output in the frequency domain. Finding the inverse of the Laplace transform we can recover the corresponding time domain system output. The time derivative property states

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f\left(0^-\right)$$

$$\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - sf\left(0^-\right) - f^{(1)}\left(0^-\right)$$

$$\cdots$$

$$\mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f\left(0^-\right) - s^{n-2}f^{(1)}\left(0^-\right) - \cdots - f^{(n-1)}\left(0^-\right)$$

Proof: Starting with the definition of the Laplace transform of the first derivative and performing integration by parts, we obtain

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt = \int_{0^-}^{\infty} e^{-st} df(t)$$

$$= f(\infty)e^{-\infty} - f\left(0^-\right)e^{0} + s\int_{0^-}^{\infty} f(t)e^{-st}dt = sF(s) - f\left(0^-\right)$$

Similarly, using the definition of the Laplace transform for the nth derivative and integrating n times by parts, we can verify the general formula for the Laplace transform of the nth derivative. Another simple proof of the nth derivative property is given in Problem 4.2.

Problem 4.2

Using the fact that $f(t) \leftrightarrow F(s)$ and $f^{(1)}(t) \leftrightarrow sF(s) - f(0^-)$, we have

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = \mathcal{L}\left\{\frac{d}{dt} f^{(1)}(s)\right\} = s\left[sF(s) - f(0^-)\right] - f^{(1)}(0^-)$$
$$= s^2 F(s) - sf(0^-) - f^{(1)}(0^-)$$

and

$$\mathcal{L}\left\{\frac{d^3 f(t)}{dt^3}\right\} = \mathcal{L}\left\{\frac{d}{dt} f^{(2)}(s)\right\}$$
$$= s\left[s^2 F(s) - sf(0^-) - f^{(1)}(0^-)\right] - f^{(2)}(0^-)$$
$$= s^3 F(s) - s^2 f(0^-) - sf^{(1)}(0^-) - f^{(2)}(0^-)$$

Continuing the same procedure, we obtain the required result

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = \mathcal{L}\left\{\frac{d}{dt} f^{(n-1)}(t)\right\}$$

$$= s\left[s^{n-1}F(s) - s^{n-2}f\left(0^-\right) - \cdots - f^{(n-2)}\left(0^-\right)\right] - f^{(n-1)}\left(0^-\right)$$

$$s^n F(s) - s^{n-1}f\left(0^-\right) - \cdots - f^{(n-1)}\left(0^-\right)$$

Property #8: Time Convolution

This property states that if $\mathcal{L}\{f_1(t)\} = F_1(s)$ and $\mathcal{L}\{f_2(t)\} = F_2(s)$ then

$$\mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s)F_2(s)$$

Hence, the Laplace transform of the signal convolution is equal to the product of the Laplace transforms of the signals.

Proof: The convolution of two causal signals is defined by

$$f_1(t) * f_2(t) = \int_0^\infty f_1(t-\tau) f_2(\tau) d\tau$$

The corresponding Laplace transform is

$$\mathcal{L}\{f_1(t) * f_2(t)\} = \int_0^\infty \left[\int_0^\infty f_1(t-\tau) f_2(\tau) d\tau \right] e^{-st} dt$$

$$= \int_0^\infty \left[\int_0^\infty f_1(t-\tau) u(t-\tau) f_2(\tau) d\tau \right] e^{-st} dt$$

We have inserted the shifted unit step function in the above formula in order to make clear the effect of an exchange of integrals.

Interchanging the order of integrals, we have

$$\mathcal{L}\{f_1(t) * f_2(t)\} = \int_0^\infty f_2(\tau)\left[\int_0^\infty f_1(t-\tau)u(t-\tau)e^{-st}dt\right]d\tau$$

$$= \int_0^\infty f_2(\tau)\left[\int_\tau^\infty f_1(t-\tau)e^{-st}dt\right]d\tau$$

Introducing the change of variables as $t - \tau = \sigma$ with $dt = d\sigma$, we obtained the stated result

$$\mathcal{L}\{f_1(t) * f_2(t)\} = \int_0^\infty f_2(\tau)\left[\int_0^\infty f_1(\sigma)e^{-s(\sigma+\tau)}dt\right]d\tau$$

$$= F_1(s)\int_0^\infty f_2(\tau)e^{-s\tau}d\tau = F_1(s)F_2(s)$$

Property #9: Integral

The integral property of the Laplace transform states

$$\mathcal{L}\{f(t)\} = F(s) \;\Rightarrow\; \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

Proof: The proof follows easily by using the convolution property as

$$\int_0^t f(\tau)d\tau = \int_0^t f(\tau)u(t-\tau)d\tau = u(t) * f(t) \leftrightarrow \frac{1}{s}F(s)$$

Property #10: Initial Value Theorem

This theorem helps us to find the initial value of a time signal by finding a limit in the frequency domain. Let the function $f(t)$ be continuous for $t \geq 0^-$. For such a function the initial value theorem states

$$\mathcal{L}\{f(t)\} = F(s) \;\;\Rightarrow\;\; \lim_{t \to 0^-}\{f(t)\} = \lim_{s \to \infty}\{sF(s)\}$$

Proof: The proof of this result is obtained by using the time derivative property and finding the corresponding limits

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt = sF(s) - f(0^-)$$

$$\Rightarrow \lim_{s\to\infty}\left\{\int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt\right\} = \lim_{s\to\infty}\left\{sF(s) - f(0^-)\right\}$$

$$0 = \lim_{s\to\infty}\{sF(s)\} - f(0^-) \Rightarrow f(0^-) = \lim_{t\to 0^-}\{f(t)\} = \lim_{s\to\infty}\{sF(s)\}$$

Let us emphasize that this proof is valid under the assumption that the function $f(t)$ *does not have a jump discontinuity at zero, that is,* $f(0^-) = f(0^+)$. In the case of functions that have jump discontinuities at the origin, say $f_\delta(t)$ has a jump discontinuity at $t = 0$, the rigorous proof of the initial value theorem requires that we split the integral as

$$\int_{0^-}^{\infty} \frac{df_\delta(t)}{dt} e^{-st} dt = \int_{0^-}^{0^+} \frac{df_\delta(t)}{dt} e^{-st} dt + \int_{0^+}^{\infty} \frac{df_\delta(t)}{dt} e^{-st} dt$$

$$= \int_{0^-}^{0^+} \frac{df_\delta(t)}{dt} e^{0} dt + \int_{0^+}^{\infty} \frac{df_\delta(t)}{dt} e^{-st} dt = f_\delta(0^+) - f_\delta(0^-) + \int_{0^+}^{\infty} \frac{df_\delta(t)}{dt} e^{-st} dt$$

$$= sF(s) - f_\delta(0^-)$$

Taking the limit when $s \to \infty$, we obtain

$$\mathcal{L}\{f_\delta(t)\} = F(s) \quad \Rightarrow \quad \lim_{t \to 0^+} \{f_\delta(t)\} = f_\delta(0^+) = \lim_{s \to \infty} \{sF(s)\}$$

Example 4.8: The above distinction is the best demonstrated on the example of the unit step signal $u(t)$. Note that $u(0^+) = 1$ and $u(0^-) = 0$. Since the signal $u(t)$ has the jump discontinuity at zero we have

$$u(0^+) = \lim_{s \to \infty} \{sU(s)\} = \lim_{s \to \infty} \left\{ s\frac{1}{s} \right\} = 1$$

However, the use of the previous formula produces $u(0^-) = 1$, which is not correct.

Property #11: Final Value Theorem

This property is very useful since the signal steady state value in time can be obtained from its Laplace transform without a need to perform first the inverse Laplace transform. The final value theorem states

$$\lim_{t \to \infty} \{f(t)\} = \lim_{s \to 0} \{sF(s)\}$$

Proof: The proof is similar to the proof of the initial value theorem, that is

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt = sF(s) - f\left(0^-\right)$$

$$\Rightarrow \lim_{s\to 0}\left\{\int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt\right\} = \lim_{s\to 0}\left\{sF(s) - f\left(0^-\right)\right\}$$

$$f(\infty) - f\left(0^-\right) = \lim_{s\to 0}\{sF(s)\} - f\left(0^-\right)$$

$$\Rightarrow f(\infty) = \lim_{t\to\infty}\{f(t)\} = \lim_{s\to 0}\{sF(s)\}$$

The final value theorem is very often used in linear control system analysis and design, for example, in order to find the system response steady state errors (Chapter 12). We would like to point out that the final value theorem is applicable only to time functions for which the limit at infinity exist. For example, $\sin(t)$ has no limit at infinity, hence the final value theorem is not applicable to $\sin(t)$.

Note that the Laplace transform of $\sin(t)$ has a pair of complex conjugate poles on the imaginary axis. An easy test in the complex domain whether or not a time function $f(t)$ has a limit at infinity is to examine the poles of its Laplace transform $F(s)$. The test says: *if the function $sF(s)$ has no poles (the values of s at which $F(s) = \infty$) on the imaginary axis and in the right half of the complex plane then the final value theorem of the Laplace transform will be applicable*. Hence, if a function $F(s)$ has all poles in the left half plane and only a *simple (distinct) pole* at the origin, the final value theorem may be applied. This will become more obvious once we introduce the concept of system stability.

Example 4.9: Given the following Laplace transform

$$F(s) = \frac{2s^2 + s + 1}{s^3 + 3s^2 + 3s + 1}$$

By the initial value theorem, we have

$$\lim_{t\to 0^-}\{f(t)\} = \lim_{s\to\infty}\{sF(s)\} = \lim_{s\to\infty}\left\{s\frac{2s^2+s+1}{s^3+3s^2+3s+1}\right\} = 2$$

From the final value theorem, it follows

$$\lim_{t\to\infty}\{f(t)\} = \lim_{s\to 0}\{sF(s)\} = \lim_{s\to 0}\left\{s\frac{2s^2+s+1}{s^3+3s^2+3s+1}\right\} = 0$$

Note that the function $F(s)$ has all poles in the left half of the complex plane, $s_{1,2,3} = -1$, hence the final value theorem is applicable. Thus, without going through the entire procedure of finding the inverse Laplace transform, we are able by using the results of these two theorems to conclude that $f(0^-) = 2$ and $f(\infty) = 0$. Note that we have also assumed that the function $f(t)$ has no jump discontinuities at $t = 0$, which follows from the fact that the degree of the denominator of $F(s)$ is strictly greater than the degree of the numerator of $F(s)$.