4.4 Block Diagrams

Using the Laplace transform linearity and convolution properties we can easily extend the concept of the transfer function to configurations of several connected linear systems. In that way we will find the equivalent transfer functions for cascade and parallel connections of systems, introduce the feedback (closed-loop) configuration, and define the corresponding feedback system transfer function.

In Section 4.3.1 we have defined the transfer function of a linear time invariant continuous-time system. The system transfer function is the ratio of the Laplace transform of the system output and the Laplace transform of the system input under the assumption that the system initial conditions are zero. This transfer function in fact represents the ***open-loop*** continuous-time *system transfer function*.

For the system at rest, the system input $F(s)$ produces on the system output the signal $Y_{zs}(s)$ given by

$$Y_{zs}(s) = H(s)F(s)$$

which is symbolically represented in Figure 4.6 using a block diagram.

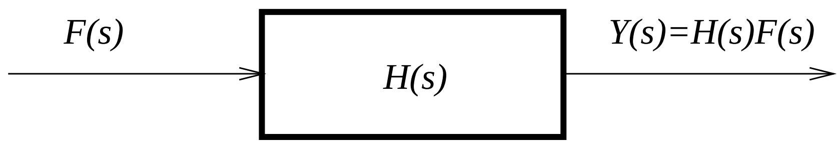
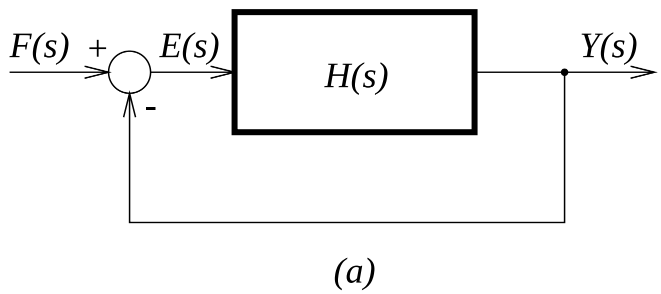


Figure 4.6: An open-loop transfer function

Note that the above block diagram can also be used in the case when the system initial conditions are different from zero. In such a case, an additive component coming from the system initial conditions should be added to the system output. For that reason in all block diagrams presented in this section we will denote the system output by $Y(s) = Y_{zs}(s) + Y_{zi}(s)$.

The open-loop transfer function $H(s)$ is derived from the differential equation that is obtained using known physical laws (mathematical modeling procedure). The accuracy of coefficients that appear in the open-loop system transfer function depends on the accuracy of the system coefficients (inductors, resistors, masses, friction coefficients, and so on). These coefficients are not always perfectly known. Even more, the coefficients change either due to aging or due to internal and/or external system disturbances. Due to changes in the system coefficients (system parameters), it can happen that the actual system output (in the open-loop system configuration) is pretty different from the one obtained analytically.

A way to cope with the system parameter changes, and a way to reduce the impact of those changes on the system output, is to form a *closed-loop system* configuration, also known as the system feedback configuration. We can feed back the system output and form the *closed-loop* around the system as presented in Figure 4.7a. The directed path (as indicated by the arrows) from $F(s)$ to $Y(s)$ is called the *forward path* and the directed path from $Y(s)$ to $E(s)$ is called the *feedback path*. Such a feedback loop is called a unity feedback loop. In general, we can put a dynamic element in the feedback loop $G(s)$ (another open-loop transfer function) as presented in Figure 4.7b. For notational convenience, we will denote the transfer function in the feedback path by $G(s)$ and the transfer function in the forward path by $H(s)$. Sometimes, in the feedback path, we put a static element equal to a constant, that is, $G(s) = \mathbf{const}$.



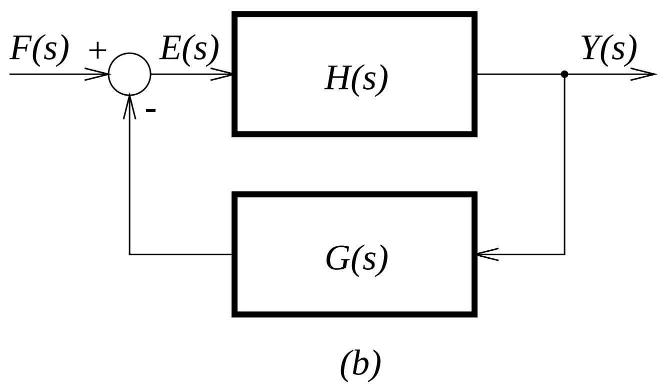


Figure 4.7: Closed-loop system configurations:

(a) unity feedback; (b) nonunity feedback

In the feedback configuration presented in Figure 4.7a, the output signal is fed back and compared to the input signal, and the difference of the input and output signals is used as a new input signal to the system. In practice, the feedback signal is taken with the negative sign since, in general, the positive feedback signal causes system instability. Following signals in the block diagram in the direction of the arrows, we can find the closed-loop system transfer function from $F(s)$ to $Y(s)$, assuming zero initial conditions

$$Y(s) = H(s)E(s), \quad E(s) = F(s) - Y(s)$$

$$\Rightarrow Y(s) = H(s)(F(s) - Y(s))$$

$$\Rightarrow Y(s) = \frac{H(s)}{1 + H(s)}F(s) \triangleq M(s)F(s)$$

The ***closed-loop system transfer function***, for unity feedback, denoted by $M(s)$ is given by

$$M(s) = \frac{Y(s)}{F(s)}\Big|_{I.C.=0} \triangleq \frac{H(s)}{1+H(s)}$$

The defined closed-loop transfer function is called the closed-loop transfer function with unity feedback. In many applications, in the feedback loop another transfer function is present, see Figure 4.7b. The closed-loop transfer function with non unity feedback is obtained similarly as follows

$$Y(s) = H(s)(F(s) - G(s)Y(s)) \Rightarrow Y(s) = \frac{H(s)}{1+H(s)G(s)}F(s)$$

$$Y(s) \triangleq M(s)F(s), \quad M(s) = \frac{H(s)}{1+H(s)G(s)}$$

Definition 4.1: The closed-loop system transfer function for non unity feedback is defined by

$$M(s) \triangleq \frac{Y(s)}{F(s)}|_{I.C.=0} = \frac{H(s)}{1+H(s)G(s)}$$

If we go around the loop of the non unity feedback block diagram presented in Figure 4.7b, we will encounter two transfer function $H(s)$ and $G(s)$. The product is called the loop transfer function. This can be formally stated in the form of a new definition.

Definition 4.2: The *loop transfer function* for nonunity feedback configuration of Figure 4.7b is defined by the product $H(s)G(s)$

We can form more complex configurations of open-loop transfer functions. Open-loop transfer functions can be connected in cascade, parallel, and/or they can form more complex feedback configurations containing several feedback loops.

The *cascade connection* of open-loop transfer functions is shown in Figure 4.8a.

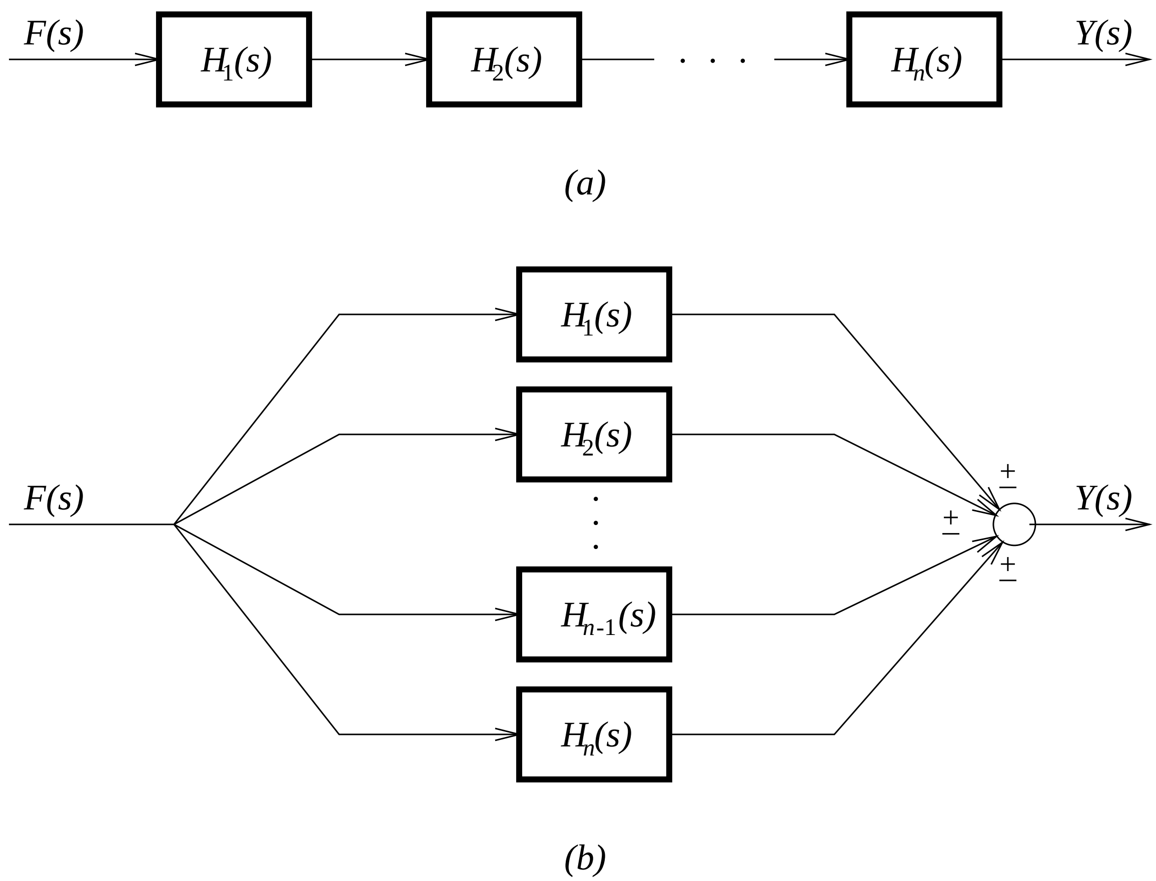


Figure 4.8: Cascade (a) and parallel (b) connections of transfer functions

It is easy to conclude that the equivalent open-loop transfer function is given by the product of elementary open-loop transfer functions

$$H_{eq}^{cascade}(s) = H_1(s)H_2(s)\cdots H_n(s)$$

This formula is called the *product rule* for elementary open-loop transfer functions.

The *parallel connection* of the open-loop transfer functions is represented in Figure 4.8b. Its equivalent open-loop transfer function is equal to the sum of elementary open-loop transfer functions, that is

$$H_{eq}^{parallel}(s) = H_1(s) \pm H_2(s) \pm \cdots \pm H_n(s)$$

The last formula is called the *sum rule* for elementary open-loop transfer functions.

Using the basic transfer function rules, we can simplify complex feedback systems and represent them in the basic feedback form presented in Figure 4.7b.

Example 4.25: Consider a feedback system given in Figure 4.9.

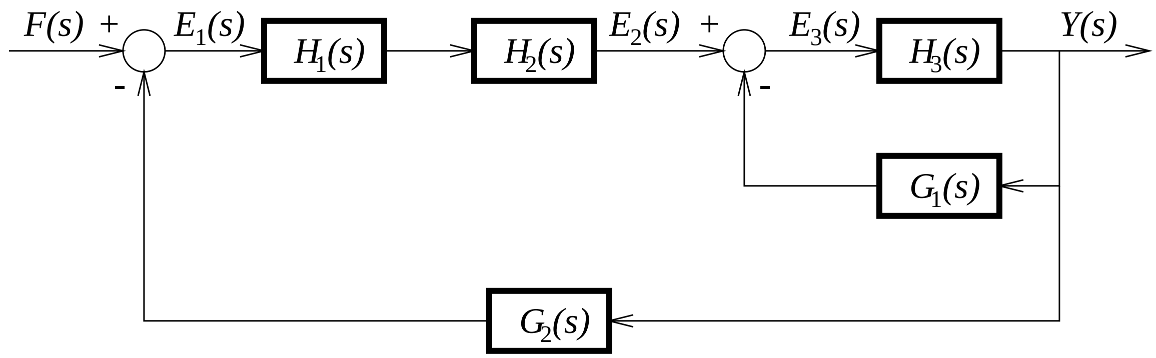


Figure 4.9: A feedback system

Using the product and sum rules, we find the closed-loop transfer function as follows

$$E_2(s) = H_1(s)H_2(s)E_1(s)$$

$$Y(s) = H_3(s)E_3(s)$$

$$E_1(s) = F(s) - G_2(s)Y(s)$$

$$E_3(s) = E_2(s) - G_1(s)Y(s)$$

Eliminating $E_i(s), i = 1, 2, 3,$ we obtain

$$\frac{Y(s)}{H_3(s)} = H_1(s)H_2(s)(F(s) - G_2(s)Y(s)) - G_1(s)Y(s)$$

This leads to the closed-loop transfer function of the form

$$M(s) = \frac{Y(s)}{F(s)} = \frac{H_1(s)H_2(s)H_3(s)}{1 + G_1(s)H_3(s) + G_2(s)H_1(s)H_2(s)H_3(s)}$$

Another way to find the closed-loop transfer function would be to identify the dual elements to those given in the block diagram in Figure 4.7b. It can be seen that

$$G(s) = G_2(s), \qquad H(s) = H_1(s)H_2(s)\frac{H_3(s)}{1 + G_1(s)H_3(s)}$$

Using now the closed-loop transfer function formula we can obtain the identical results for $M(s)$ as the one given above.

