Pole Placement Design Technique

8.2 State Feedback and Pole Placement

Consider a linear dynamic system in the state space form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}$$

In some cases one is able to achieve the goal (e.g. stabilizing the system or improving its transient response) by using the full state feedback, which represents a linear combination of the state variables, that is

$$\mathbf{u} = -\mathbf{F}\mathbf{x}$$

so that the closed-loop system, given by

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{A} - \mathbf{B}\mathbf{F})\mathbf{x} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}$$

has the desired specifications.

The main role of state feedback control is to stabilize a given system so that all closed-loop eigenvalues are placed in the left half of the complex plane. The following theorem gives a condition under which is possible to place system poles in the desired locations.

Theorem 8.1 *Assuming that the pair* $(\mathbf{A}, \mathbf{B})$ *is controllable, there exists a feedback matrix* $\mathbf{F}$ *such that the closed-loop system eigenvalues can be placed in arbitrary locations.*

This important theorem will be proved (justified) for *single-input single-output* systems. For the general treatment of the pole placement problem for multi-input multi-output systems, which is much more complicated, the reader is referred to Chen (1984).

If the pair $(\mathbf{A}, \mathbf{b})$ is controllable, the original system can be transformed into the phase variable canonical form, i.e. it exists a nonsingular transformation

$$\mathbf{x} = \mathbf{Q}\mathbf{z}$$

such that

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

where a_i's are coefficients of the characteristic polynomial of $\mathbf{A}$, that is

$$\det(\lambda \mathbf{I} - \mathbf{A}) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0$$

For single-input single-output systems the state feedback is given by

$$u(\mathbf{z}) = -f_1 z_1 - f_2 z_2 - \cdots - f_n z_n = -\mathbf{f}_c \mathbf{z}$$

After closing the feedback loop with $u(\mathbf{z})$, as given by (8.7), we get from (8.5)

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -(a_0 + f_1) & -(a_1 + f_2) & -(a_2 + f_3) & \dots & -(a_{n-1} + f_n) \end{bmatrix} \mathbf{z}$$

If the desired closed-loop eigenvalues are specified by $\lambda_1^d, \lambda_2^d, ..., \lambda_n^d$, then the desired characteristic polynomial will be given by

$$\begin{aligned}\Delta^d(\lambda) &= \left(\lambda - \lambda_1^d\right)\left(\lambda - \lambda_2^d\right)\cdots\left(\lambda - \lambda_n^d\right)\\ &= s^n + a_{n-1}^d s^{n-1} + a_{n-2}^d s^{n-2} + \cdots + a_1^d s + a_0^d\end{aligned}$$

Since the last row in (8.8) contains coefficients of the characteristic polynomial of the original system after the feedback is applied, it follows from (8.8) and (8.9) that the required feedback gains must satisfy

$$\begin{aligned}a_0 + f_1 = a_0^d \ &\Rightarrow\ f_1 = a_0^d - a_0\\ a_1 + f_2 = a_1^d \ &\Rightarrow\ f_2 = a_1^d - a_1\\ &\cdots\\ a_{n-1} + f_n = a_{n-1}^d \ &\Rightarrow\ f_n = a_{n-1}^d - a_{n-1}\end{aligned}$$

The pole placement procedure using the state feedback for a system which is already in phase variable canonical form is demonstrated in the next example.

Example 8.1: Consider the following system given in phase variable canonical form

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -10 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

It is required to find coefficients f_1, f_2, f_3 such that the closed-loop system has the eigenvalues located at $\lambda_{1,2}^d = -1 \pm j1,\ \lambda_3^d = -5.$ The desired characteristic polynomial is obtained from (8.9) as

$$\Delta^d(\lambda) = (\lambda + 5)(\lambda + 1 + j1)(\lambda + 1 - j1) = \lambda^3 + 7\lambda^2 + 12\lambda + 10$$

so that from (8.10) we have

$$f_1 = a_0^d - a_0 = 10 - 2 = 8$$
$$f_2 = a_1^d - a_1 = 12 - 5 = 7$$
$$f_3 = a_2^d - a_2 = 7 - 10 = -3$$

$\diamond$

In general, in order to be able to apply this technique to all controllable single-input single-output systems we need to find a nonsingular transformation which transfers the original system into phase variable canonical form. This transformation can be obtained by using the linearly independent columns of the system controllability matrix

$$\mathcal{C} = \left[\mathbf{b} \vdots \mathbf{A}\mathbf{b} \vdots \mathbf{A}^2\mathbf{b} \vdots \cdots \vdots \mathbf{A}^{n-1}\mathbf{b} \right]$$

It can be shown (Chen, 1984) that the required transformation is given by

$$\mathbf{Q} = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \cdots \quad \mathbf{q}_n]$$

where

$$\begin{aligned}
&\mathbf{q}_n = \mathbf{b} \\
&\mathbf{q}_{n-1} = \mathbf{A}\mathbf{q}_n + a_{n-1}\mathbf{q}_n = \mathbf{A}\mathbf{b} + a_{n-1}\mathbf{b} \\
&\mathbf{q}_{n-2} = \mathbf{A}\mathbf{q}_{n-1} + a_{n-2}\mathbf{q}_n = \mathbf{A}^2\mathbf{b} + a_{n-1}\mathbf{A}\mathbf{b} + a_{n-2}\mathbf{b} \\
&\cdots \\
&\mathbf{q}_1 = \mathbf{A}\mathbf{q}_2 + a_1\mathbf{q}_n = \mathbf{A}^{n-1}\mathbf{b} + a_{n-1}\mathbf{A}^{n-2}\mathbf{b} + \cdots + a_1\mathbf{b}
\end{aligned}$$

where a_i's are coefficients of the characteristic polynomial of matrix $\mathbf{A}$. After the feedback gain has been found for phase variable canonical form, $\mathbf{f}_c$, in the original coordinates it is obtained as (similarity transformation)

$$\mathbf{f} = \mathbf{f}_c\mathbf{Q}^{-1}$$

Example 8.2: Consider the following linear system given by

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -3 & 4 \\ -1 & 1 & -9 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{c} = [1 \quad 0 \quad 1]$$

The characteristic polynomial of this system is

$$\det(\lambda\mathbf{I} - \mathbf{A}) = s^3 + 13s^2 + 33s + 13$$

Its phase variable canonical form can be obtained either from its transfer function (see Section 3.1.2) or by using the nonsingular (similarity) transformation (8.11). The system transfer function is given by

$$\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \frac{46s + 13}{s^3 + 13s^2 + 33s + 13}$$

Using results from Section 3.1.2 we are able to write new matrices in phase variable canonical form representation as

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -13 & -33 & -13 \end{bmatrix}, \quad \mathbf{b}_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{c}_c = [13 \quad 46 \quad 0]$$

The same matrices could have been obtained by using the similarity transformation with

$$\mathbf{A}_c = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}, \quad \mathbf{b}_c = \mathbf{Q}^{-1}\mathbf{b}, \quad \mathbf{c}_c = \mathbf{c}\mathbf{Q}$$

with $\mathbf{Q}$ obtained from (8.11) and (8.12) as

$$\mathbf{Q} = \begin{bmatrix} 51 & 16 & 1 \\ 19 & 17 & 2 \\ -5 & -3 & -1 \end{bmatrix}$$

Assume that we intend to find the feedback gain for the original system such that its closed-loop eigenvalues are located at $-1, -2, -3,$ then

$$\Delta^d(\lambda) = (\lambda+1)(\lambda+2)(\lambda+3) = \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

From equation (8.10) we get expressions for the feedback gains for the system in phase variable canonical form as

$$f_1 = a_0^d - a_0 = 6 - 13 = -7$$
$$f_2 = a_1^d - a_1 = 11 - 33 = -22$$
$$f_3 = a_2^d - a_2 = 6 - 13 = -7$$

In the original coordinates the feedback gain is obtained from (8.13)

$$\mathbf{f} = \mathbf{f}_c\mathbf{Q}^{-1} = [0.8149 \quad -1.0540 \quad 5.7069]$$

Using this gain in order to close the state feedback around the system we get

$$\dot{\mathbf{x}} = \begin{bmatrix} -1.8149 & 3.0540 & -5.7069 \\ -0.6298 & -0.8920 & -7.4139 \\ -0.1851 & -0.0540 & -3.2931 \end{bmatrix} \mathbf{x}$$

It is easy to check by MATLAB that the eigenvalues of this systems are located at $-1, -2, -3.$

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Comment: Exactly the same procedure as the one given in this section can be used for placing the observer poles in the desired locations. The observers have been considered in Section 5.6. Choosing the observer gain $\mathbf{K}$ such that the closed-loop observer matrix $\mathbf{A} - \mathbf{KC}$ has the desired poles corresponds to the problem of choosing the feedback gain $\mathbf{F}$ such that the closed-loop system matrix $\mathbf{A}^T - \mathbf{F}^T\mathbf{B}^T$ has the same poles. Thus, for the observer pole placement problem, matrix $\mathbf{A}$ should be replaced by $\mathbf{A}^T$, $\mathbf{B}$ replaced by $\mathbf{C}^T$ and $\mathbf{F}$ replaced by $\mathbf{K}^T$. In addition, it is known from Chapter 5 that the observability of the pair $(\mathbf{A}, \mathbf{C})$ is equal to the controllability of the pair $\left(\mathbf{A}^T, \mathbf{C}^T\right)$, and hence the controllability condition stated in Theorem 8.1—the pair $(\mathbf{A}, \mathbf{B})$ is controllable, which for observer pole placement requires that the pair $\left(\mathbf{A}^T, \mathbf{C}^T\right)$ be controllable—is satisfied by assuming that the pair $(\mathbf{A}, \mathbf{C})$ is observable.