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High Accuracy Techniques for Singularly Perturbed Optimal Control and Filtering Problems

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In this talk we present a unified approach for optimal control and filtering of linear continuous-time singularly perturbed linear systems (continuous, discrete, stochastic) that facilitates complete and exact decomposition of optimal control and filtering tasks into pure-slow and pure-fast time scales.

The presented methodology has the following features:

- HIGH ACCURACY

- Elimination of numerical ill-conditioning of the original problems

- Introduction of parallelism into the design procedures

- Independent parallel processing of information in the pure-slow and pure-fast time scales

- Order-reduction of the original problems, which, in general, implies reduction in both off-line and on-line computational requirements.

In the last thirty five years around **one thousand journal papers, more than thirty books, and several overview journal papers**, were published on singularly perturbed systems.

The notion of singular perturbations in mathematics has been used for systems of ordinary differential equations that have some derivatives multiplied by small positive parameters. Such systems of differential equations were extensively studied in the **1950s** and **1960s** by several mathematicians: **Tikhonov, Levin, Levinson, Vasileva, Butuzov, Wasov, Hoppendsteadt, O'Malley, Smith, Chang** (to name those most often cited in the engineering literature)

$$
\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1, x_2, u(x_1, x_2)), \quad x_1(0) = x_{10} \\
\epsilon\frac{dx_2}{dt} &= f_2(x_1, x_2, u(x_1, x_2)), \quad x_2(0) = x_{20}
\end{aligned}
$$

ϵ is a small positive parameter

where $u(x_1(t), x_2(t))$ is the control variable.

The notion of singular perturbations has been extended also to cover ordinary difference equations and partial differential equations.

One of the most important (from the engineering point of view) and most widely used results of **linear** singularly perturbed systems, is the transformation for the exact (at least theoretically) pure-slow and pure-fast decomposition of linear singularly perturbed two-point boundary value problem, known as the **Chang transformation (Chang, 1972)**.

Many real physical systems are singularly perturbed, for example, aircraft, robots, electrical circuits, power systems, nuclear reactors, chemical reactors, dc and induction motors, synchronous machines, distillation columns, flexible structures, automobiles. Since the **middle of the 1960s** singularly perturbed systems have been studied in engineering, primary due to the work of **Petar Kokotovic** and his coworkers.

Singularly perturbed systems are characterized by simultaneous presence of small and large time constants, which introduces clustering of **linear (or linearized) system** eigenvalues into two disjoint groups: (a) eigenvalues corresponding to large time constants located close to the imaginary axis representing slow subsystem state space variables (slow modes), and (b) eigenvalues corresponding to small time constants located far from the imaginary axis representing fast subsystem state space variables (fast modes).

Linear time invariant singularly
perturbed system eigenvalue clusters

The approaches taken in engineering during the **1970s** and **1980s**, in the study of singularly perturbed control systems, were based on expansion methods (power series, asymptotic expansions, Taylor series), the methods developed by previously mentioned mathematicians. The approaches were, in most cases, developed only for an $O(\epsilon)$ accuracy, where ϵ is a small positive singular perturbation parameter.

$O(\epsilon^r)$ stands for $O(\epsilon^r) < c\epsilon^r$, where c is a bounded constant and r is any real number.

Generating higher order expansions for those methods has been analytically pretty cumbersome and numerically pretty inefficient, especially for high-dimensional control systems.

Even more, it has been demonstrated in several publications, that for some applications an $O(\epsilon)$ accuracy is either not sufficient or even more, it does not solve the problem at all

Grodt and Gajic, (1988)
Gajic, Harkara, and Petkovski, (1989)
Skataric and Gajic (1992)
Gajic and Shen (1993)
Mizukami and Suzumura, (1993)
several recent papers by Mizukami, Mukaidani, and Xu

In what follows, the complete solutions to the linear-quadratic optimal control and filtering continuous-time problems will be given in detail. Solutions of the related control and filtering problems will be only outlined.

Pure-Slow and Pure-Fast Decomposition of the Linear-Quadratic Optimal Control Problem

The linear singularly perturbed control system is given by

$$\dot{x}_1(t) = A_1 x_1(t) + A_2 x_2(t) + B_1 u(t), \; x_1(t_0) = x_{10}$$

$$\epsilon \dot{x}_2(t) = A_3 x_1(t) + A_4 x_2(t) + B_2 u(t), \; x_2(t_0) = x_{20}$$

where $x_i(t) \in \Re^{n_i}, \; i = 1, 2, \; u(t) \in \Re^m$ are state and control variables, respectively, and ϵ is a small positive singular perturbation parameter.

With the above linear system we consider the performance criterion to be minimized

$$J = \min_u \frac{1}{2} \int_{t_0}^{\infty} \left\{ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T Q \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u^T(t) R \, u(t) \right\} dt$$

with positive definite R and positive semidefinite Q.

The **open-loop optimal control** problem has the solution

$$u(t) = -R^{-1} B^T p(t)$$

where $p(t) \in \Re^{n_1+n_2}$ is a costate variable that satisfies

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$$

with

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon}A_3 & \frac{1}{\epsilon}A_4 \end{bmatrix}, \; Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} = \begin{bmatrix} q_1^T q_1 & q_1^T q_2 \\ q_2^T q_1 & q_2^T q_2 \end{bmatrix}$$

$$B = \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix}, \; S = BR^{-1}B^T = \begin{bmatrix} S_1 & \frac{1}{\epsilon}Z \\ \frac{1}{\epsilon}Z^T & \frac{1}{\epsilon^2}S_2 \end{bmatrix}$$

and $x^T(t) = \left[x_1^T(t) \; x_2^T(t) \right]$.

The optimal **closed-loop control law** has the well known form

$$u(x(t)) = -R^{-1}B^T P x(t) = -Fx(t)$$

where P is the positive semidefinite solution of the algebraic Riccati equation given by

$$0 = PA + A^T P + Q - PSP, \quad P = \begin{bmatrix} P_1 & \epsilon P_2 \\ \epsilon P_2^T & \epsilon P_3 \end{bmatrix}$$

The **Riccati equation plays the central role in optimal linear control and filtering**. The positive semidefinite stabilizing solution of the algebraic Riccati equation exists under the standard stabilizability-detectability conditions.

System Stabilizability and Detectability Assumption: The triple $\left(A, B, \sqrt{Q}\right)$ is stabilizable and detectable.

In the following show how to find the solution of the above algebraic Riccati equation in terms of solutions of the reduced-order, pure-slow and pure-fast, algebraic Riccati equations.

Partitioning and appropriately scaling $p(t)$ as $p^T(t) = [p_1^T(t) \quad \epsilon p_2^T(t)]$ with $p_i(t) \in \Re^{n_i}, i = 1, 2,$ and interchanging second and third rows in the state-costate system, we get

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{p}_1(t) \\ \dot{x}_2(t) \\ \dot{p}_2(t) \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ \frac{1}{\epsilon} T_3 & \frac{1}{\epsilon} T_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix}$$

where

$$T_1 = \begin{bmatrix} A_1 & -S_1 \\ -Q_1 & -A_1^T \end{bmatrix}, \qquad T_2 = \begin{bmatrix} A_2 & -Z \\ -Q_2 & -A_3^T \end{bmatrix}$$

$$T_3 = \begin{bmatrix} A_3 & -Z^T \\ -Q_2^T & -A_2^T \end{bmatrix}, \qquad T_4 = \begin{bmatrix} A_4 & -S_2 \\ -Q_3 & -A_4^T \end{bmatrix}$$

It is important to notice that the above system retains the singular perturbation form. Also, the matrix T_4 is the Hamiltonian matrix of the fast subsystem, and it is nonsingular under stabilizability-detectability conditions imposed on the fast subsystem.

Fast Subsystem Stabilizability and Detectability Assumption: The triple (A_4, B_2, q_2) is stabilizable and detectable.

It should be emphasized that the presented procedure is valid for both the so-called **standard** (matrix A_4 is nonsingular) and **nonstandard** (matrix A_4 is singular) singularly perturbed linear control systems. Note that nonstandard singularly perturbed control systems are the recent trend in theory of singularly perturbed linear control systems.

The celebrated transformation of Chang, used for decomposition of linear singularly perturbed systems, is defined by

$$\mathbf{T}_1 = \begin{bmatrix} I_{2n_1} - \epsilon H L & -\epsilon H \\ L & I_{2n_2} \end{bmatrix}$$

where L and H satisfy

$$T_4 L - T_3 - \epsilon L(T_1 - T_2 L) = 0$$

$$-H(T_4 + \epsilon L T_2) + T_2 + \epsilon(T_1 - T_2 L)H = 0$$

The unique solutions of the above algebraic equations exist for sufficiently small values of ϵ under condition that T_4 is nonsingular, that is, under Fast Subsystem Stabilizability-Detectability Assumption. These algebraic equations can be solved efficiently with very high accuracy as linear algebraic equations using either the **fixed-point algorithm** or the **Newton method**.

The corresponding fixed point and Newton algorithms for solving the L-equation are respectively given by

$$L^{(i+1)} = L^{(0)} + \epsilon T_4^{-1} L^{(i)} \left(T_1 - T_2 L^{(i)} \right)$$

$$L^{(0)} = T_4^{-1} T_3, \quad i = 0, 1, 2, \ldots$$

and

$$D_1^{(i)} L^{(i+1)} + L^{(i+1)} D_2^{(i)} = Q^{(i)}$$

$$L^{(0)} = T_4^{-1}T_3, \quad i = 0, 1, 2, ...$$

$$D_1^{(i)} = T_4 + \epsilon L^{(i)}T_2$$

$$D_2^{(i)} = -\epsilon\Big(T_1 - T_2L^{(i)}\Big),\ Q^{(i)} = T_3 + \epsilon L^{(i)}T_2L^{(i)}$$

The Newton method converges quadratically, hence if it converges, it requires in average only four to five iterations. The fixed-point iterations converge linearly and sometimes require a lot of iterations. In addition, the L-equation can be efficiently solved by using the **eigenvector method** of (Kecman, Bingulac, and Gajic, 1999) and the **Taylor series** expansions of (Derbel, Kamoun, and Poloujadoff (1994). Once the solution for the L-equation is obtained, the H-equation can be solved either directly as a linear Sylvester equation or recursively as

$$H^{(i+1)} = T_2(T_4 + \epsilon LT_2)^{-1}$$

$$+\epsilon(T_1 - T_2L)H^{(i)}(T_4 + \epsilon LT_2)^{-1}$$

$$H^{(0)} = T_2T_4^{-1}, \quad i = 0, 1, 2, ...$$

The application of the Chang transformation to state-costate equations produces two completely decoupled subsystems

$$\dot{\eta}(t) = (T_1 - T_2 L)\eta(t)$$

and

$$\epsilon\dot{\xi}(t) = (T_4 + \epsilon L T_2)\xi(t)$$

where

$$\begin{bmatrix} \eta(t) \\ \xi(t) \end{bmatrix} = \mathrm{T}_1 \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix}$$

Let us define the permutation matrix E_1 by

$$\begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\epsilon} I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ p_1(t) \\ \epsilon p_2(t) \end{bmatrix}$$
$$= E_1 \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$$

Note that the inverse of E_1 can be easily obtained analytically, hence, this matrix is not numerically ill-conditioned with respect to the matrix inversion for small values of ϵ.

The relationship between the original and new coordinates is

$$\begin{bmatrix} \eta_1(t) \\ \xi_1(t) \\ \eta_2(t) \\ \xi_2(t) \end{bmatrix} = E_2^T \mathrm{T}_1 E_1 \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$$

$$= \Pi \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$$

where E_2 is a permutation matrix of the form

$$E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix}$$

Since $p(t) = Px(t)$, where P is the solution of the algebraic Riccati equation, it follows that

$$\begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} = (\Pi_1 + \Pi_2 P)x(t), \ \begin{bmatrix} \eta_2(t) \\ \xi_2(t) \end{bmatrix} = (\Pi_3 + \Pi_4 P)x(t)$$

In the original coordinates, the required optimal solution has a closed-loop nature. We have the same attribute for the new systems (pure-slow and pure-fast subsystems) that is

$$\begin{bmatrix} \eta_2(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix}$$

It can be shown that

$$\begin{bmatrix} \boldsymbol{P_s} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P_f} \end{bmatrix} = (\boldsymbol{\Pi}_3 + \boldsymbol{\Pi}_4 \boldsymbol{P})(\boldsymbol{\Pi}_1 + \boldsymbol{\Pi}_2 \boldsymbol{P})^{-1}$$

We can also find $\boldsymbol{P}$ reversely by introducing

$$\boldsymbol{E}_1^{-1} \mathbf{T}_1^{-1} \boldsymbol{E}_2 = \boldsymbol{\Pi}^{-1} = \boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\Omega}_1 & \boldsymbol{\Omega}_2 \\ \boldsymbol{\Omega}_3 & \boldsymbol{\Omega}_4 \end{bmatrix}$$

where

$$\boldsymbol{E}_1^{-1} = \begin{bmatrix} \boldsymbol{I}_{n_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{I}_{n_2} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{n_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\epsilon} \boldsymbol{I}_{n_2} \end{bmatrix}$$

and it yields

$$\boldsymbol{P} = \left(\boldsymbol{\Omega}_3 + \boldsymbol{\Omega}_4 \begin{bmatrix} \boldsymbol{P_s} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P_f} \end{bmatrix} \right) \left(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 \begin{bmatrix} \boldsymbol{P_s} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P_f} \end{bmatrix} \right)^{-1}$$

It was shown in (Su, Gajic, and Shen, 1992) that the above defined matrix inversions exist for sufficiently small values of $\boldsymbol{\epsilon}$.

Partitioning the pure-slow and pure-fast subsystems as follows

$$\begin{bmatrix} \dot{\eta}_1(t) \\ \dot{\eta}_2(t) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = (T_1 - T_2 L) \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix}$$

$$\epsilon \begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = (T_4 + \epsilon L T_2) \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}$$

and using the relations $\eta_2(t) = P_s \eta_1(t)$ and $\xi_2(t) = P_f \xi_1(t)$ yield to two reduced-order, nonsymmetric, pure-slow and pure-fast, algebraic Riccati equations

$$0 = P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s$$

$$0 = P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f$$

with $a_i, b_i, i = 1, 2, 3, 4,$ defined by above.

The pure-fast algebraic Riccati equation is nonsymmetric, but its $O(\epsilon)$ approximation is a symmetric one, that is

$$P_f A_4 + A_4^T P_f + Q_3 - P_f S_2 P_f + O(\epsilon) = 0$$

We can obtain an $O(\epsilon)$ approximation for P_f as

$$P_f^{(0)} A_4 + A_4^T P_f^{(0)} + Q_3 - P_f^{(0)} S_2 P_f^{(0)} = 0$$

The unique positive semidefinite solution of this equation exists under **Fast Subsystem Stabilizability-Detectability Assumption**.

Hence, we have $P_f = P_f^{(0)} + O(\epsilon)$. The pure-slow algebraic Riccati equation is also nonsymmetric. It can be also shown that the pure-slow algebraic Riccati equation is an $O(\epsilon)$ perturbation of the first-order approximate slow algebraic Riccati equation obtained in (Chow and Kokotovic, 1976)

$$P_s^{(0)} A_s + A_s^T P_s^{(0)} + Q_s - P_s^{(0)} S_s P_s^{(0)} = 0$$

with $P_s = P_s^{(0)} + O(\epsilon)$, where A_s, Q_s, and S_s can be found either using the methodology of (Chow and Kokotovic, 1976) or from the results of (Wang and Frank, 1992) as

$$\begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix} = T_1 - T_2 T_4^{-1} T_3$$

Note that we have

$$\begin{bmatrix} a_1^{(0)} & a_2^{(0)} \\ a_3^{(0)} & a_4^{(0)} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + O(\epsilon)$$

$$= T_1 - T_2 L^{(0)} + O(\epsilon) = T_1 - T_2 T_4^{-1} T_3 + O(\epsilon)$$

which implies

$$\begin{bmatrix} a_1^{(0)} & a_2^{(0)} \\ a_3^{(0)} & a_4^{(0)} \end{bmatrix} = \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix}$$

The unique positive semidefinite stabilizing solution of the approximate slow algebraic Riccati equation exists under the following standard assumption.

Slow Subsystem Stabilizability-Detectability Assumption: The triple $\left(A_s, \sqrt{S_s}, \sqrt{Q_s}\right)$ is stabilizable and detectable.

Lemma (Su, Gajic, and Shen, 1992)
Let the above assumptions be satisfied. Then, $\exists \epsilon_0 > 0$ *such that* $\forall \epsilon \leq \epsilon_0$ *the unique solutions of the pure-slow and pure-fast algebraic Riccati equations exist.*

Having obtained a good initial guess from the approximate symmetric pure-fast algebraic Riccati equation, the Newton algorithm can be used very efficiently finding the solution of the nonsymmetric pure-fast algebraic Riccati equation. The Newton algorithm is given by

$$P_f^{(i+1)}\left(b_1 + b_2 P_f^{(i)}\right) - \left(b_4 - P_f^{(i)} b_2\right) P_f^{(i+1)}$$

$$= b_3 + P_f^{(i)} b_2 P_f^{(i)}, \quad i = 0, 1, 2, \ldots$$

Similarly, the pure-slow nonsymmetric algebraic Riccati equation can be solved

$$P_s^{(i+1)}\left(a_1 + a_2 P_s^{(i)}\right) - \left(a_4 - P_s^{(i)} a_2\right) P_s^{(i+1)}$$

$$= a_3 + P_s^{(i)} a_2 P_s^{(i)}, \qquad i = 0, 1, 2, \ldots$$

It is important to notice that the total number of the scalar quadratic algebraic equations in the pure-slow and pure-fast algebraic Riccati equations is $n_1^2 + n_2^2$. The global algebraic Riccati equation contains $\frac{1}{2}(n_1 + n_2)(n_1 + n_2 + 1)$ scalar algebraic equations. Thus, the presented method, in addition of eliminating ill-conditioning, can even reduce the number of algebraic equations if

$$n_1^2 + n_2^2 < \frac{1}{2}(n_1 + n_2)(n_1 + n_2 + 1)$$

or

$$(n_1 - n_2)^2 < n_1 + n_2$$

which is the case when n_1 and n_2 are close to each other.

Using solutions of the pure-slow and pure-fast Riccati equations, we can get completely decoupled slow and fast feedback subsystems as

$$\dot{\eta}_1(t) = (a_1 + a_2 P_s)\eta_1(t)$$
$$\epsilon\dot{\xi}_1(t) = \left(b_1 + b_2 P_f\right)\xi_1(t)$$

The interpretation of the above result is that the optimal processing of information for this class of systems (filtering and/or control) can be completely performed at the local levels (slow and fast subsystems). The global solution in the original coordinates is then obtained at any time instant as

$$x(t) = (\Pi_1 + \Pi_2 P)^{-1} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix}$$

The quadratic performance criterion to be minimized, in the new coordinates becomes

$$J = \frac{1}{2} \int_{t_0}^{+\infty} \Big(x^T(t) Q x(t) + u^T(t) R u(t) \Big) dt$$

$$= \frac{1}{2} \int_{t_0}^{+\infty} x^T(t)(Q + PSP)x(t) dt$$

$$= \frac{1}{2} \int_{t_0}^{+\infty} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix}^T (\Pi_1 + \Pi_2 P)^{-T}$$

$$\times (Q + PSP)(\Pi_1 + \Pi_2 P)^{-1} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} dt$$

$$= \frac{1}{2} \int_{t_0}^{+\infty} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix}^T \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^T & \Theta_3 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} dt$$

The value of the above integral is obtained as

$$J_{opt} = \frac{1}{2}\mathrm{tr}\left\{V\begin{bmatrix}\eta_1(t)\\ \xi_1(t)\end{bmatrix}\begin{bmatrix}\eta_1(t)\\ \xi_1(t)\end{bmatrix}^T\right\}$$

$$= \frac{1}{2}\mathrm{tr}\left\{\begin{bmatrix}V_1 & \epsilon V_2\\ \epsilon V_2^T & \epsilon V_3\end{bmatrix}\begin{bmatrix}\eta_1(t_0)\eta_1^T(t_0) & \eta_1(t_0)\xi_1^T(t_0)\\ \xi_1(t_0)\eta_1^T(t_0) & \xi_1(t_0)\xi_1^T(t_0)\end{bmatrix}\right\}$$

$$J_{opt} = \frac{1}{2}tr\left\{V_1\eta_1(t_0)\eta_1^T(t_0)\right\}$$

$$+\frac{\epsilon}{2}tr\Big(V_2^T\eta_1(t_0)\xi_1^T(t_0) + V_2\xi_1(t_0)\eta_1^T(t_0)+\Big)$$

$$+\frac{\epsilon}{2}tr\Big(V_3\xi_1(t_0)\xi_1^T(t_0)\Big) = J_s + \epsilon J_f$$

where the matrix V satisfies the algebraic Lyapunov equation

$$\begin{bmatrix}(a_1 + a_2P_1) & 0\\ 0 & \frac{1}{\epsilon}(b_1 + b_2P_2)\end{bmatrix}^T V$$

$$+V\begin{bmatrix}(a_1 + a_2P_1) & 0\\ 0 & \frac{1}{\epsilon}(b_1 + b_2P_2)\end{bmatrix} + \begin{bmatrix}\Theta_1 & \Theta_2\\ \Theta_2^T & \Theta_3\end{bmatrix} = 0$$

It can be concluded that the pure-slow component of the performance criterion is $O(1)$ and that the fast subsystem contributes only an $O(\epsilon)$ to the performance criterion of a linear continuous-time deterministic system. However, the proper design of the fast feedback must assure system stability.

Open-Loop Optimal Control Problem

The optimal open-loop control problem is a two-point boundary value problem with the associated state-costate equations forming the Hamiltonian system of linear differential equations. The two-point boundary value problem of linear singularly perturbed systems is transformed into the pure-slow and pure-fast, reduced-order, completely decoupled initial value problems by following the methodology of **Chang (1972)**.

The stiffness (numerical ill-conditioning) of the original singularly perturbed two-point boundary value problem is converted into the problem of an ill-defined linear system of algebraic equations.

Consider the linear singularly perturbed control system and the performance criterion to be minimized over the time period from t_0 to t_f is defined by

$$J = \min_{u(t)} \frac{1}{2}\{\int_{t_0}^{t_f} \left\{ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T Q \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u^T(t) R\, u(t) \right\} dt$$

$$+\frac{1}{2}\begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix}^T Q_f \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix}\}, \qquad Q_f \geq 0$$

where Q_f is the terminal time penalty matrix.

The **open-loop optimal control** problem has the solution

$$u(t) = -R^{-1}B^T p(t)$$

where $p(t) \in \Re^{n_1+n_2}$ costate satisfies

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$$

with the boundary conditions given by

$$M \begin{bmatrix} x(t_0) \\ p(t_0) \end{bmatrix} + N \begin{bmatrix} x(t_f) \\ p(t_f) \end{bmatrix} = c$$

with

$$M = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ -Q_f & I_n \end{bmatrix}, \quad c = \begin{bmatrix} x(t_0) \\ 0 \end{bmatrix}$$

$$n = n_1 + n_2$$

The terminal penalty matrix is appropriately partitioned as

$$Q_f = \begin{bmatrix} Q_{f1} & \epsilon Q_{f2} \\ \epsilon Q_{f2}^T & \epsilon Q_{f3} \end{bmatrix}$$

Let us partition and appropriately scale the co-state vector $p(t)$ as $p^T(t) = [p_1^T(t) \quad \epsilon p_2^T(t)]$ with $p_1(t) \in \Re^{n_1}$ and $p_2(t) \in \Re^{n_2}$. By interchanging second and third rows in the corresponding state co-state equations, we get the singularly perturbed system as in the case of the closed-loop control, that is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{p}_1(t) \\ \dot{x}_2(t) \\ \dot{p}_2(t) \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ \frac{1}{\epsilon} T_3 & \frac{1}{\epsilon} T_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix}$$

with the boundary conditions

$$x(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad p(t_f) = Q_f x(t_f)$$

The original boundary conditions can be written in a compact form as follows

$$M_1 \begin{bmatrix} x_1(t_0) \\ p_1(t_0) \\ x_2(t_0) \\ p_2(t_0) \end{bmatrix} + N_1 \begin{bmatrix} x_1(t_f) \\ p_1(t_f) \\ x_2(t_f) \\ p_2(t_f) \end{bmatrix} = c_1$$

where

$$M_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad c_1 = \begin{bmatrix} x_{10} \\ 0 \\ x_{20} \\ 0 \end{bmatrix}$$

$$N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -Q_{f1} & I_{n_1} & -\epsilon Q_{f2} & 0 \\ 0 & 0 & 0 & 0 \\ -Q_{f2}^T & 0 & -Q_{f3} & I_{n_2} \end{bmatrix}$$

The Chang transformation applied to the corresponding state-costate equations produces two completely decoupled pure-slow and pure-fast subsystems defined earlier. The boundary conditions in the new coordinates corresponding to those systems are given by

$$M_2 \begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} + N_2 \begin{bmatrix} \eta(t_f) \\ \xi(t_f) \end{bmatrix} = c_1$$

where

$$M_2 = M_1 \mathbf{T}_1^{-1}, \quad N_2 = N_1 \mathbf{T}_1^{-1}$$

Since solutions the pure-slow and pure-fast subsystems

$$\dot{\eta}(t) = (T_1 - T_2 L)\eta(t)$$

and

$$\epsilon \dot{\xi}(t) = (T_4 + \epsilon L T_2)\xi(t)$$

are given by

$$\eta(t) = e^{(T_1 - T_2 L)(t - t_0)}\eta(t_0)$$

$$\xi(t) = e^{\frac{1}{\epsilon}(T_4+\epsilon LT_2)(t-t_0)}\xi(t_0)$$

we can eliminate $\eta(t_f)$ and $\xi(t_f)$ from the boundary conditions, which leads to

$$\left\{M_2 + N_2\begin{bmatrix} e^{(T_1-T_2L)(t_f-t_0)} & 0 \\ 0 & e^{\frac{1}{\epsilon}(T_4+\epsilon LT_2)(t_f-t_0)} \end{bmatrix}\right\}\begin{bmatrix}\eta(t_0) \\ \xi(t_0)\end{bmatrix} = c_1$$

The system of linear algebraic equations obtained is of the form

$$\alpha(\epsilon)\begin{bmatrix}\eta(t_0) \\ \xi(t_0)\end{bmatrix} = c_1$$

It can be shown that $\alpha(\epsilon)$ is invertible, hence $\eta(t_0)$ and $\xi(t_0)$ can be obtained.

It should be emphasized that in the case of a large interval $t-t_0$, ill-conditioning of the linear system of algebraic equations occurs since both matrices $T_1 - T_2L$ and $T_4 + \epsilon LT_2$ contain unstable modes. As a matter of fact they are the Hamiltonian matrices of the pure-slow and pure-fast subsystems, and as such have the eigenvalues symmetrically distributed with respect to the imaginary axis.

Kalman Filtering for Systems with Slow and Fast Modes

Consider the linear continuous-time singularly perturbed stochastic system driven by a white noise stochastic process

$$\dot{x}_1(t) = A_1 x_1(t) + A_2 x_2(t) + G_1 w_1(t)$$

$$\epsilon \dot{x}_2 = A_3 x_1(t) + A_4 x_2(t) + G_2 w_1(t)$$

with the corresponding measurements

$$y(t) = C_1 x_1(t) + C_2 x_2(t) + w_2(t)$$

where $x_i(t) \in \Re^{n_i},\ i = 1, 2,$ are state vectors, $w_i(t) \in \Re^{r_i}$ are zero-mean stationary, mutually uncorrelated, white Gaussian noise stochastic processes with intensities $W_i > 0$, and $y(t) \in \Re^{r_2}$ are system measurements. In the following $A_i, G_j, C_j,\ i = 1, 2, 3, 4,\ j = 1, 2$, are constant matrices.

The optimal Kalman filter driven by the **innovation process**, $v(t)$, is given by

$$\begin{aligned}
\dot{\hat{x}}_1(t) &= A_1 \hat{x}_1(t) + A_2 \hat{x}_2(t) + K_1 v(t) \\
\epsilon \dot{\hat{x}}_2(t) &= A_3 \hat{x}_1(t) + A_4 \hat{x}_2(t) + K_2 v(t) \\
v(t) &= y(t) - C_1 \hat{x}_1(t) - C_2 \hat{x}_2(t)
\end{aligned}$$

where the optimal filter gains K_1 and K_2 are obtained from (Khalil and Gajic 1984)

$$K_1 = \left(P_{1F} C_1^T + P_{2F} C_2^T\right) W_2^{-1}$$

$$K_2 = \left(\epsilon P_{2F}^T C_1^T + P_{3F} C_2^T\right) W_2^{-1}$$

with matrices P_{1F}, P_{2F}, and P_{3F} representing the positive semidefinite stabilizing solution matrix of the filter algebraic Riccati equation

$$AP_F + P_F A^T - P_F S P_F + G W_1 G^T = 0$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon} A_3 & \frac{1}{\epsilon} A_4 \end{bmatrix},\ G = \begin{bmatrix} G_1 \\ \frac{1}{\epsilon} G_2 \end{bmatrix}$$

$$S = C^T W_2^{-1} C,\ P_F = \begin{bmatrix} P_{1F} & P_{2F} \\ P_{2F}^T & \frac{1}{\epsilon} P_{3F} \end{bmatrix}$$

For the decomposition and approximation of the singularly perturbed Kalman filter the Chang transformation have been used in (Khalil and Gajic, 1984; Gajic 1986)

$$\begin{bmatrix} \hat{\eta}_1(t) \\ \hat{\eta}_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} - \epsilon HL & -\epsilon H \\ L & I_{n_2} \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}$$

where L and H satisfy algebraic equations

$$A_4 L - A_3 - \epsilon L(A_1 - A_2 L) = 0$$

$$-HA_4 + A_2 - \epsilon HLA_2 + \epsilon(A_1 - A_2 L)H = 0$$

The Chang transformation applied to the optimal Kalman filter produces

$$\dot{\hat{\eta}}_1(t) = (A_1 - A_2L)\hat{\eta}_1(t) + (K_1 - HK_2 - \epsilon HLK_1)v(t)$$
$$\epsilon\dot{\hat{\eta}}_2(t) = (A_4 + \epsilon LA_2)\hat{\eta}_2(t) + (K_2 + \epsilon LK_1)v(t)$$

In the new coordinates the **innovation process** is given by

$$v(t) = y(t) - (C_1 - C_2L)\hat{\eta}_1(t)$$
$$-[C_2 + \epsilon(C_1 - C_2L)H]\hat{\eta}_2(t)$$

The algebraic filter Riccati equation has the unique stabilizing solutions under the following assumptions.

The fast subsystem triple (A_4, C_2, G_2) is stabilizable and detectable.
The slow subsystem triple (A_0, C_0, G_0) is stabilizable and detectable, where $A_0 = A_1 - A_2A_4^{-1}A_3$, $C_0 = C_1 - C_2A_4^{-1}A_3$, and $G_0 = G_1 - A_2A_4^{-1}G_2$.

In the Kalman decomposition procedure given above the slow and fast filters require some additional communication channels necessary to form the innovation process.

In thefollowing, we present the decomposition scheme for the optimal Kalman filter such that the slow and fast filters are completely decoupled and both of them are driven by the system measurements.

We can use the regulator-filter duality to decouple the optimal Kalman filter into the pure-slow and pure-fast reduced-order Kalman filters.

The optimal regulator gain is defined by

$$
\begin{aligned}
F &= [F_1 \quad F_2] \\
&= \left[R^{-1}\left(B_1^T P_1 + B_2^T P_2^T\right) \quad R^{-1}\left(\epsilon B_1^T P_2 + B_2^T P_3\right)\right]
\end{aligned}
$$

The results of interest that we need, which can be deduced from the linear regulator ptoblem, are summarized in the following lemma.

Lemma: Consider the optimal closed-loop linear system

$$
\begin{aligned}
\dot{x}_1(t) &= (A_1 - B_1F_1)x_1(t) + (A_2 - B_1F_2)x_2(t) \\
\epsilon\dot{x}_2(t) &= (A_3 - B_2F_1)x_1(t) + (A_4 - B_2F_2)x_2(t)
\end{aligned}
$$

Under corresponding stabilizability-detectability assumptions there exists a nonsingular transformation $\mathbf{T}$

$$
\begin{bmatrix} \xi_s(t) \\ \xi_f(t) \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

such that

$$
\begin{aligned}
\dot{\xi}_s(t) &= (a_1 + a_2P_s)\xi_s(t) \\
\epsilon\dot{\xi}_f(t) &= \left(b_1 + b_2P_f\right)\xi_f(t)
\end{aligned}
$$

where P_s *and* P_f *are the unique solutions of the exact pure-slow and pure-fast completely decoupled algebraic regulator Riccati* *equations. The nonsingular* transformation $\mathbf{T}$ *is given by*

$$\mathbf{T} = (\mathbf{\Pi}_1 + \mathbf{\Pi}_2 \boldsymbol{P})$$

Even more, the global solution $\boldsymbol{P}$ *can be obtained from the reduced-order exact pure-slow and pure-fast algebraic Riccati equations, that is*

$$\boldsymbol{P} = \left(\boldsymbol{\Omega}_3 + \boldsymbol{\Omega}_4 \begin{bmatrix} \boldsymbol{P}_s & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_f \end{bmatrix}\right)\left(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 \begin{bmatrix} \boldsymbol{P}_s & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_f \end{bmatrix}\right)^{-1}$$

Known matrices $\boldsymbol{\Omega}_i, \ \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, 4$, *and* $\mathbf{\Pi}_1, \ \mathbf{\Pi}_2$ *are given in terms of solutions of the Chang decoupling equations.*

Consider the **optimal** *closed-loop* **Kalman filter driven by the system measurements**, that is

$$\begin{aligned}
\dot{\hat{\boldsymbol{x}}}_1(\boldsymbol{t}) &= (\boldsymbol{A}_1 - \boldsymbol{K}_1\boldsymbol{C}_1)\hat{\boldsymbol{x}}_1(\boldsymbol{t}) + (\boldsymbol{A}_2 - \boldsymbol{K}_1\boldsymbol{C}_2)\hat{\boldsymbol{x}}_2(\boldsymbol{t}) \\
&\quad + \boldsymbol{K}_1\boldsymbol{y}(\boldsymbol{t}) \\
\boldsymbol{\epsilon}\dot{\hat{\boldsymbol{x}}}_2(\boldsymbol{t}) &= (\boldsymbol{A}_3 - \boldsymbol{K}_2\boldsymbol{C}_1)\hat{\boldsymbol{x}}_1(\boldsymbol{t}) + (\boldsymbol{A}_4 - \boldsymbol{K}_2\boldsymbol{C}_2)\hat{\boldsymbol{x}}_2(\boldsymbol{t}) \\
&\quad + \boldsymbol{K}_2\boldsymbol{y}(\boldsymbol{t})
\end{aligned}$$

By duality between the optimal filter and regulator, the algebraic filter Riccati equation can be solved by using the same decomposition method for solving the algebraic regulator Riccati equation with

$$\begin{aligned}
&\boldsymbol{A} \to \boldsymbol{A}^T, \ \ \boldsymbol{Q} \to \boldsymbol{G}\boldsymbol{W}_1\boldsymbol{G}^T, \ \ \boldsymbol{F}^T = \boldsymbol{K} \\
&\boldsymbol{Z} = \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T \to \boldsymbol{S} = \boldsymbol{C}^T\boldsymbol{W}_2^{-1}\boldsymbol{C}
\end{aligned}$$

In that process the following matrices have to be formed

$$T_{1F} = \begin{bmatrix} A_1^T & -C_1^T W_2^{-1} C_1 \\ -G_1 W_1 G_1^T & -A_1 \end{bmatrix}$$
$$T_{2F} = \begin{bmatrix} A_3^T & -C_1^T W_2^{-1} C_2 \\ -G_1 W_1 G_2^T & -A_2 \end{bmatrix}$$
$$T_{3F} = \begin{bmatrix} A_2^T & -C_2^T W_2^{-1} C_1 \\ -G_2 W_1 G_1^T & -A_3 \end{bmatrix}$$
$$T_{4F} = \begin{bmatrix} A_4^T & -C_2^T W_2^{-1} C_2 \\ -G_2 W_1 G_2^T & -A_4 \end{bmatrix}$$

Note that on the contrary to the results from the optimal regulator problem, where the state-costate variables have to be partitioned and scaled as $x^T(t) = \left[x_1^T(t)\; x_2^T(t)\right]$ and $p^T(t) = \left[p_1^T(t)\; \epsilon p_2^T(t)\right]$, in the case of the dual filter variables, we have to use the following partitions and scaling $x^T(t) = \left[x_1^T(t)\; \epsilon x_2^T(t)\right]$ and $p^T(t) = \left[p_1^T(t)\; p_2^T(t)\right]$. Since matrices T_{1F}, T_{2F}, T_{3F}, T_{4F} correspond to the system matrices of a singularly perturbed linear system, the slow-fast decomposition is achieved by using the Chang decoupling equations

$$T_{4F}M - T_{3F} - \epsilon M(T_{1F} - T_{2F}M) = 0$$

$$-N(T_{4F} + \epsilon M T_{2F}) + T_{2F} + \epsilon(T_{1F} - T_{2F}M)N = 0$$

By using the permutation matrices dual to those from the optimal regulator problem

$$E_{1F} = \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ 0 & 0 & I_{n1} & 0 \\ 0 & \frac{1}{\epsilon} I_{n2} & 0 & 0 \\ 0 & 0 & 0 & I_{n2} \end{bmatrix}$$

$$E_{2F} = \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ 0 & 0 & I_{n1} & 0 \\ 0 & I_{n2} & 0 & 0 \\ 0 & 0 & 0 & I_{n2} \end{bmatrix}$$

we can define

$$\Pi_F = \begin{bmatrix} \Pi_{1F} & \Pi_{2F} \\ \Pi_{3F} & \Pi_{4F} \end{bmatrix} = E_{2F}^T \begin{bmatrix} I_{2n_1} - \epsilon NM & -\epsilon N \\ M & I_{2n_2} \end{bmatrix} E_{1F}$$

Then, the desired transformation is given by

$$\mathbf{T_2} = (\Pi_{1F} + \Pi_{2F} P_F)$$

The transformation $\mathbf{T_2}$ applied to the filter variables as

$$\begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} = \mathbf{T}_2^{-\mathrm{T}} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}$$

produces

$$\begin{bmatrix} \dot{\hat{\eta}}_s(t) \\ \dot{\hat{\eta}}_f(t) \end{bmatrix} = \mathbf{T}_2^{-\mathrm{T}} \begin{bmatrix} A_1 - K_1 C_1 & A_2 - K_1 C_2 \\ \frac{1}{\epsilon}(A_3 - K_2 C_1) & \frac{1}{\epsilon}(A_4 - K_2 C_2) \end{bmatrix}$$

$$\times \mathbf{T}_2^{\mathrm{T}} \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} + \mathbf{T}_2^{-\mathrm{T}} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} y(t)$$

such that the complete closed-loop decomposition is achieved, that is

$$\dot{\hat{\eta}}_s(t) = (a_{1F} + a_{2F} P_{sF})^T \hat{\eta}_s(t) + K_s y(t)$$
$$\epsilon \dot{\hat{\eta}}_f(t) = \left(b_{1F} + b_{2F} P_{fF}\right)^T \hat{\eta}_f(t) + K_f y(t)$$

with

$$\begin{bmatrix} a_{1F} & a_{2F} \\ a_{3F} & a_{4F} \end{bmatrix} = (T_{1F} - T_{2F} M)$$
$$\begin{bmatrix} b_{1F} & b_{2F} \\ b_{3F} & b_{4F} \end{bmatrix} = (T_{4F} + \epsilon M T_{2F})$$

$$\begin{bmatrix} K_s \\ \frac{1}{\epsilon} K_f \end{bmatrix} = \mathbf{T}_2^{-\mathrm{T}} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix}$$

$$0 = P_{sF} a_{1F} - a_{4F} P_{sF} - a_{3F} + P_{sF} a_{2F} P_{sF}$$
$$0 = P_{fF} b_{1F} - b_{4F} P_{fF} - b_{3F} + P_{fF} b_{2F} P_{fF}$$

Pure-slow and pure-fast Kalman filters
driven by the system measurements

Optimal Linear-Quadratic Gaussian Control

Consider the singularly perturbed linear stochastic system

$$\dot{x}_1(t) = A_1x_1(t) + A_2x_2(t) + B_1u(t) + G_1w(t)$$

$$\epsilon\dot{x}_2(t) = A_3x_1(t) + A_4x_2(t) + B_2u(t) + G_2w(t)$$

$$y(t) = C_1x_1(t) + C_2x_2(t) + w_2(t)$$

with the performance criterion

$$J = \lim_{t_f\to\infty} \frac{1}{t_f} E\left\{ \int_{t_0}^{t_f} \left[z^T(t)z(t) + u^T(t)Ru(t) \right] dt \right\}$$

where $x_i(t) \in \Re^{n_i}, i = 1, 2,$ comprise slow and fast state vectors, respectively, $u(t) \in \Re^m$ is the control input, $y(t) \in \Re^{r_2}$ is the observed output, $w_i(t) \in \Re^{r_i}$ are zero-mean stationary, mutually uncorrelated, Gaussian white noise processes with intensities $W_1 > 0$ and $W_2 > 0$, respectively, and $z(t) \in \Re^s$, is the controlled output given by

$$z(t) = D_1x_1(t) + D_2x_2(t)$$

All matrices are of appropriate dimensions and assumed to be constant. The optimal control law for the above optimization problem is given by

$$u_{opt}(t) = -F_1\hat{x}_1(t) - F_2\hat{x}_2(t)$$

where $\hat{x}_1(t)$ and $\hat{x}_2(t)$ are the optimal estimates of the state vectors $x_1(t)$ and $x_2(t)$ obtained from the Kalman filter

$$\begin{aligned}
&\dot{\hat{x}}_1(t) = A_1\hat{x}_1(t) + A_2\hat{x}_2(t) + B_1u(t) + K_1v(t)\\
&\epsilon\dot{\hat{x}}_2(t) = A_3\hat{x}_1(t) + A_4\hat{x}_2(t) + B_2u(t) + K_2v(t)\\
&v(t) = y(t) - C_1\hat{x}_1(t) - C_2\hat{x}_2(t)
\end{aligned}$$

The optimal regulator gains F_1, F_2 and filter gains K_1, K_2 are previuosly obtained.

The optimal global Kalman filter can be put in the form in which the filter is driven by the system measurements and optimal control inputs, that is

$$\begin{aligned}
&\dot{\hat{x}}_1(t) = (A_1 - K_1C_1)\hat{x}_1(t) + (A_2 - K_1C_2)\hat{x}_2(t)\\
&\ + B_1u(t) + K_1y(t)\\
&\epsilon\dot{\hat{x}}_2(t) = (A_3 - K_2C_1)\hat{x}_1(t) + (A_4 - K_2C_2)\hat{x}_2(t)\\
&\ + B_2u(t) + K_2y(t)
\end{aligned}$$

We have shown already that there exists a nonsingular transformation such that the optimal Kalman filter is decoupled into pure-slow and pure-fast local filters both driven by system measurements and system control inputs

$$\begin{aligned}
&\dot{\hat{\eta}}_s(t) = (a_{1F} + a_{2F}P_{sF})^T\hat{\eta}_s(t) + B_su(t) + K_sy(t)\\
&\epsilon\dot{\hat{\eta}}_f(t) = \left(b_{1F} + b_{2F}P_{fF}\right)^T\hat{\eta}_f(t) + B_fu(t) + K_fy(t)
\end{aligned}$$

The pure-slow and pure-fast filter gains, K_s, K_f, are previously obtained. The pure-slow and pure-fast system input matrices are given by

$$\begin{bmatrix} B_s \\ \frac{1}{\epsilon} B_f \end{bmatrix} = \mathrm{T}_2^{-T} \begin{bmatrix} B_1 \\ \frac{1}{\epsilon} B_2 \end{bmatrix}$$

The optimal control in the new coordinates is given by, (Lim, 1994)

$$u_{opt}(t) = -F\hat{x}(t) = -F\mathrm{T}_2^T \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix}$$

$$= -[F_s \quad F_f] \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix}$$

where F_s and F_f are obtained from

$$[F_s \quad F_f] = F\mathrm{T}_2^{\mathrm{T}} = R^{-1}B^T P(\Pi_{1F} + \Pi_{F2}P_F)^T$$

The optimal value of J is given by

$$J_{opt} = \mathrm{tr}\Big\{PKW_2K^T + P_F D^T D\Big\}$$

$$= \mathrm{tr}\Big\{PGW_1G^T + P_F F^T RF\Big\}$$

Parallel pure-slow and pure-fast controllers for
linear-quadratic Gaussian stochastic control problem

Discrete-Time Singularly Perturbed Linear Control Systems

The first papers on singularly perturbed discrete-time systems were published in the **1970s** (Butuzov and Vasileva, 1971; Hoppensteadt and Mirankar, 1977). Discrete-time singularly perturbed **control** systems have been the subject of intensive research since the early **1980s**. Several researchers have produced important results on different aspects of **control** problems of deterministic singularly perturbed discrete systems such as Phillips, Blankenship, Mahmoud, Sawan, Khorasani, Naidu, Khalil and their coworkers.

Linear-Quadratic Optimal Control

Consider the singularly perturbed linear time-invariant discrete control system using the **fast time scale** representation (Litkouhi 1983, Litkouhi and Khalil 1984, 1985)

$$x_1(k+1) = (I_{n_1} + \epsilon A_1)x_1(k) + \epsilon A_2 x_2(k) + \epsilon B_1 u(k)$$

$$x_2(k+1) = A_3 x_1(k) + A_4 x_2(k) + B_2 u(k)$$

$$x_1(0) = x_{10}, \qquad x_2(0) = x_{20}$$

with slow variables $x_1 \in \Re^{n_1}$, fast state variables $x_2 \in \Re^{n_2}$, control inputs $u \in \Re^m$, where ϵ represents a small positive singular perturbation parameter. The performance criterion of the corresponding linear-quadratic optimal control problem is defined by

$$J = \frac{\epsilon}{2}\sum_{k=0}^{\infty}\Big[x(k)^T Q x(k) + u(k)^T R u(k)\Big]$$

where

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \geq 0, \quad R > 0$$

The solution of the optimal regulation problem is given by

$$\begin{aligned} u(k) &= -R^{-1}B^T\lambda(k+1) \\ &= -\Big(R + B^T P B\Big)^{-1} B^T P A x(k) \end{aligned}$$

where $\lambda(k)$ is a costate variable and P is the positive semidefinite stabilizing solution of the discrete algebraic Riccati equation (Dorato and Levis 1971, Lewis 1986) given by

$$P = Q + A^T P A - A^T P B\Big[R + B^T P B\Big]^{-1} B^T P A$$

$$P = \begin{bmatrix} \frac{1}{\epsilon}P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}$$

The state-costate difference equations can be written as the forward recursion (Lewis 1986)

$$\begin{bmatrix} x(k+1) \\ \lambda(k+1) \end{bmatrix} = \mathbf{H}\begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix}$$

with

$$\mathbf{H} = \begin{bmatrix} A + BR^{-1}B^T A^{-T} Q & -BR^{-1}B^T A^{-T} \\ -A^{-T} Q & A^{-T} \end{bmatrix}$$

where the Hamiltonian $\mathbf{H}$ is the symplectic matrix, which has the property that the eigenvalues of $\mathbf{H}$ are grouped into two disjoint subsets $\boldsymbol{\Gamma_1}$ and $\boldsymbol{\Gamma_2}$, such that for every $\boldsymbol{\lambda_c} \in \boldsymbol{\Gamma_1}$ there exists $\boldsymbol{\lambda_d} \in \boldsymbol{\Gamma_2}$, which satisfies $\boldsymbol{\lambda_c} \times \boldsymbol{\lambda_d} = \mathbf{1}$, and we can choose either $\boldsymbol{\Gamma_1}$ or $\boldsymbol{\Gamma_2}$ to contain only the stable eigenvalues (Salgado et al. 1988). The corresponding matrices introduced in $\mathbf{H}$ are given by

$$A = \begin{bmatrix} I_{n_1} + \epsilon A_1 & \epsilon A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} \epsilon B_1 \\ B_2 \end{bmatrix}$$

$$S = BR^{-1}B^T = \begin{bmatrix} \epsilon^2 S_1 & \epsilon Z \\ \epsilon Z^T & S_2 \end{bmatrix}$$

$$S_1 = B_1 R^{-1} B_1^T, \quad S_2 = B_2 R^{-1} B_2^T, \quad Z = B_1 R^{-1} B_2^T$$

The assumption that the matrix A is invertible is used, which requires the invertibility of the matrix A_4. In that case

$$A^{-1} = \begin{bmatrix} I_{n_1} & 0 \\ -A_4^{-1} A_3 & A_4^{-1} \end{bmatrix} + O(\epsilon)$$

Hence, the presentation requires the following assumption.

The fast subsystem matrix A_4 is nonsingular.

In the following, we show how to obtain exactly the solution of the discrete-time algebraic Riccati equation of singularly perturbed systems in terms of solutions of two reduced-order **continuous-time**, pure-slow and pure-fast, algebraic Riccati equations.

Partitioning the vector $\boldsymbol{\lambda}(\boldsymbol{k})$ as $\boldsymbol{\lambda}(\boldsymbol{k}) = \left[\boldsymbol{\lambda}_1^T(\boldsymbol{k}) \;\; \boldsymbol{\lambda}_2^T(\boldsymbol{k})\right]^T$ with $\boldsymbol{\lambda}_1(\boldsymbol{k}) \in \Re^{n_1}$ and $\boldsymbol{\lambda}_2(\boldsymbol{k}) \in \Re^{n_2}$, we get

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \lambda_1(k+1) \\ \lambda_2(k+1) \end{bmatrix} = \mathbf{H} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix}$$

It can been shown after some algebra that the Hamiltonian matrix (181) has the following form (Lim 1994a)

$$\mathbf{H} = \begin{bmatrix} I_{n_1} + \epsilon\overline{A_1} & \epsilon\overline{A_2} & \epsilon^2\overline{S_1} & \epsilon\overline{S_2} \\ \overline{A_3} & \overline{A_4} & \epsilon\overline{S_3} & \overline{S_4} \\ \overline{Q_1} & \overline{Q_2} & I_{n_1} + \epsilon\overline{A_{11}^T} & \overline{A_{21}^T} \\ \overline{Q_3} & \overline{Q_4} & \epsilon\overline{A_{12}^T} & \overline{A_{22}^T} \end{bmatrix}$$

Note that in the remaining part of this section there is no need for analytical expressions for the bared matrices. **Those matrices have to be formed by the computer in the process of calculations, which can be done, for example, using MATLAB.**

Interchanging second and third rows and using the following scaling $[p_1(k) \;\; p_2(k)]^T = [\epsilon\lambda_1(k) \;\; \lambda_2(k)]^T$ yields

$$\begin{bmatrix} x_1(k+1) \\ p_1(k+1) \\ x_2(k+1) \\ p_2(k+1) \end{bmatrix} = \begin{bmatrix} I_{n_1} + \epsilon\overline{A_1} & \epsilon\overline{S_1} & \epsilon\overline{A_2} & \epsilon\overline{S_2} \\ \epsilon\overline{Q_1} & I_{n_1} + \epsilon\overline{A_{11}^T} & \epsilon\overline{Q_2} & \epsilon\overline{A_{21}^T} \\ \overline{A_3} & \overline{S_3} & \overline{A_4} & \overline{S_4} \\ \overline{Q_3} & \overline{A_{12}^T} & \overline{Q_4} & \overline{A_{22}^T} \end{bmatrix}$$

$$\times \begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix} = \begin{bmatrix} I_{2n_1} + \epsilon T_1 & \epsilon T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix}$$

where

$$T_1 = \begin{bmatrix} \overline{A_1} & \overline{S_1} \\ \overline{Q_1} & \overline{A_{11}^T} \end{bmatrix}, \;\; T_2 = \begin{bmatrix} \overline{A_2} & \overline{S_2} \\ \overline{Q_2} & \overline{A_{21}^T} \end{bmatrix}$$
$$T_3 = \begin{bmatrix} \overline{A_3} & \overline{S_3} \\ \overline{Q_3} & \overline{A_{12}^T} \end{bmatrix}, \;\; T_4 = \begin{bmatrix} \overline{A_4} & \overline{S_4} \\ \overline{Q_4} & \overline{A_{22}^T} \end{bmatrix}$$

Introducing the notation

$$U(k) = \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix}, \quad V(k) = \begin{bmatrix} x_2(k) \\ p_2(k) \end{bmatrix}$$

we obtain the singularly perturbed discrete-time linear system

$$U(k+1) = (I_{2n_1} + \epsilon T_1)U(k) + \epsilon T_2 V(k)$$

$$V(k+1) = T_3 U(k) + T_4 V(k)$$

Applying the discrete-time version of the Chang transformation (Chang 1972, Shen 1990) defined by

$$\mathbf{T}_5 = \begin{bmatrix} I_{2n_1} - \epsilon HL & -\epsilon H \\ L & I_{2n_2} \end{bmatrix}$$

$$\mathbf{T}_5^{-1} = \begin{bmatrix} I_{2n_1} & \epsilon H \\ -L & I_{2n_2} - \epsilon LH \end{bmatrix}$$

$$\begin{bmatrix} \eta(k) \\ \xi(k) \end{bmatrix} = \mathbf{T}_5 \begin{bmatrix} U(k) \\ V(k) \end{bmatrix}$$

produces in the new coordinates two completely decoupled subsystems

$$\begin{bmatrix} \eta_1(k+1) \\ \eta_2(k+1) \end{bmatrix} = \eta(k+1) = [I_{2n_1} + \epsilon(T_1 - T_2 L)]\eta(k)$$

$$\begin{bmatrix} \xi_1(k+1) \\ \xi_2(k+1) \end{bmatrix} = \xi(k+1) = (T_4 + \epsilon L T_2)\xi(k)$$

The matrices L and H satisfy

$$-L + T_4 L - T_3 - \epsilon L(T_1 - T_2 L) = 0$$

$$H + T_2 - H T_4 + \epsilon(T_1 - T_2 L)H - \epsilon H L T_2 = 0$$

The unique solutions of the above algebraic equations exist, by the implicit function theorem (Ortega and Rheinboldt 1990), under the condition that the matrix $T_4 - I_{2n_2}$ is nonsingular. It can be shown the matrix T_4 is given by

$$T_4 = T_4^{(0)} + O(\epsilon) = \begin{bmatrix} A_4 + S_2 A_4^{-T} Q_3 & -S_2 A_4^{-T} \\ -A_4^{-T} Q_3 & A_4^{-T} \end{bmatrix} + O(\epsilon)$$

We see that $T_4^{(0)}$ represents the Hamiltonian matrix of the fast subsystem. The nonsingularity of $T_4^{(0)} - I_{2n_2}$ requires the following assumption.

The triple $\left(A_4, B_2, \sqrt{Q_3}\right)$ is stabilizable-detectable.

It follows that under this assumption, the matrix $T_4 - I_{2n_2}$ is nonsingular for sufficiently small values of ϵ.

The algebraic L-H equations can be solved using the Newton method, similarly to the solution of the corresponding continuous-time algebraic equations (Grodt and Gajic 1988). The Newton method converges quadratically in the neighborhood of the sought solution, that is, its rate of convergence is $O\left(\epsilon^{2^i}\right)$. The initial guess required for the Newton method is easily obtained with the accuracy of $O(\epsilon)$, by setting $\epsilon = 0$ in the original equation, that is

$$L^{(0)} = (T_4 - I)^{-1} T_3 = L + O(\epsilon)$$

The Newton algorithm can be constructed by setting $L^{(i+1)} = L^{(i)} + \Delta L^{(i)}$ and neglecting $O(\Delta L)^2$ terms. This leads to a Lyapunov-type equation of the form

$$D_1^{(i)} L^{(i+1)} + L^{(i+1)} D_2^{(i)} = Q^{(i)}$$

with

$$D_1^{(i)} = T_4 - I_{2n_1} + \epsilon L^{(i)} T_2,\ D_2^{(i)} = -\epsilon\left(T_1 - T_2 L^{(i)}\right)$$

$$Q^{(i)} = T_3 + \epsilon L^{(i)} T_2 L^{(i)}, \quad i = 0, 1, 2,\ \ldots$$

The Newton sequence will be $O\left(\epsilon^2\right),\ O\left(\epsilon^4\right),\ O\left(\epsilon^8\right), \ldots, O\left(\epsilon^{2^i}\right)$ close to the exact solution, respectively, in each iteration. Having found the solution of the L-equation up to the desired degree of accuracy, one can get the solution of the H-equation by solving directly the algebraic Lyapunov-like (Sylvester) equation of the form

$$H^{(i)} D_1^{(i)} + D_2^{(i)} H^{(i)} = T_2$$

which implies $H^{(i)} = H + O\left(\epsilon^{2^i}\right)$.

The rearrangement and modification of variables is done by using the permutation matrix E_1 of the form

$$\begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & \epsilon I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} = E_1 \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix}$$

We have the following relationship between the original coordinates and the new ones

$$\begin{bmatrix} \eta_1(k) \\ \xi_1(k) \\ \eta_2(k) \\ \xi_2(k) \end{bmatrix} = E_2^T \mathrm{T}_5 E_1 \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix}$$

$$= \Pi \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix}$$

where E_2 is a permutation matrix of the form

$$E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix}$$

Since $\lambda(k) = Px(k)$, where P satisfies the discrete algebraic Riccati equation, it follows from that

$$\begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} = (\Pi_1 + \Pi_2 P)x(k)$$

$$\begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} = (\Pi_3 + \Pi_4 P)x(k)$$

In the original coordinates, the required optimal solution has a closed-loop nature. We have the same characteristic for the new systems that is,

$$\begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} = \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix}$$

which implies

$$\begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1}$$

Following the same logic, we can find P reversely by introducing

$$E_1^{-1} \mathrm{T}_5^{-1} E_2 = \Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = \Pi^{-1}$$

and it yields

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \right) \left(\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \right)^{-1}$$

It can be shown, by estimating the order of quantity for the entries in matrices $\Pi_1, \Pi_2, \Omega_1, \Omega_2$, that the required matrices are invertible.

Partitioning the system matrices as

$$\begin{bmatrix} \eta_1(k+1) \\ \eta_2(k+1) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix}$$

$$= (I_{2n_1} + \epsilon T_1 - \epsilon T_2 L) \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix}$$

$$\begin{bmatrix}\xi_1(k+1)\\ \xi_2(k+1)\end{bmatrix} = \begin{bmatrix} b_1 & b_2\\ b_3 & b_4\end{bmatrix}\begin{bmatrix}\xi_1(k)\\ \xi_2(k)\end{bmatrix}$$

$$= (T_4 + \epsilon L T_2)\begin{bmatrix}\xi_1(k)\\ \xi_2(k)\end{bmatrix}$$

yields to two reduced-order nonsymmetric, pure-slow and pure-fast, algebraic Riccati equations

$$P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s = 0$$

$$P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f = 0$$

It is very interesting that the algebraic Riccati equation of singularly perturbed discrete-time control systems is completely and exactly decomposed into two reduced-order nonsymmetric **continuous-time** algebraic Riccati equations.

The pure-fast algebraic Riccati equation is nonsymmetric, but its $O(\epsilon)$ perturbation is symmetric one. This can be observed from the fact that

$$\begin{bmatrix} b_1 & b_2\\ b_3 & b_4\end{bmatrix} = (T_4 + \epsilon L T_2)$$

$$= \begin{bmatrix} b_1^{(0)} & b_2^{(0)}\\ b_3^{(0)} & b_4^{(0)}\end{bmatrix} + O(\epsilon) = T_4^{(0)} + O(\epsilon)$$

The coefficients of the Hamiltonian matrix $T_4^{(0)}$ imply the following approximate, fast subsystem, discrete-time algebraic Riccati equation

$$P_f^{(0)} = A_4^T P_f^{(0)} A_4 + Q_3$$

$$-A_4^T P_f^{(0)} B_2 \Big(R + B_2^T P_f^{(0)} B_2\Big)^{-1} B_2^T P_f^{(0)} A_4$$

such that $P_f = P_f^{(0)} + O(\epsilon)$. Note that the positive semidefinite stabilizing solution of exists under the given assumption. The last equation is identical to the approximate fast discrete-time algebraic Riccati equation of Litkouhi and Khalil (1984, 1985).

In order to establish that an $O(\epsilon)$ approximation of the pure-slow algebraic Riccati equation is symmetric, we use the following arguments. It follows that

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = I_{2n_1} + \epsilon(T_1 - T_2 L) = I_{2n_1} + \epsilon T_s$$

$$= I_{2n_1} + \epsilon\Big(T_1 - T_2 L^{(0)}\Big) + O(\epsilon) = I_{2n_1} + \epsilon T_s^{(0)} + O(\epsilon)$$

$$= \begin{bmatrix} a_1^{(0)} & a_2^{(0)} \\ a_3^{(0)} & a_4^{(0)} \end{bmatrix} + O(\epsilon) = I_{2n_1}$$

$$+\epsilon\Big(T_1 - T_2 (T_4 - I_{2n_2})^{-1} T_3\Big) + O(\epsilon)$$

On the other hand, the approximate slow continuous-time algebraic Riccati equation can be obtained from

$$\begin{bmatrix} x_1(k+1) \\ p_1(k+1) \\ x_2(k+1) \\ p_2(k+1) \end{bmatrix} = \begin{bmatrix} I_{2n_1} + \epsilon T_1 & \epsilon T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix}$$

by using the methodology of Litkouhi and Khalil (1984, 1985) and assuming that the fast variables $x_2(k)$ and $p_2(k)$ are at the steady state. Using the fact that at the steady state $x_2(k+1) = x_2(k)$ and $p_2(k+1) = p_2(k)$ we obtain

$$\begin{bmatrix} x_2(k) \\ p_2(k) \end{bmatrix} = (I_{2n_2} - T_4)^{-1} \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix}$$

and

$$\begin{aligned} &\begin{bmatrix} x_1(k+1) \\ p_1(k+1) \end{bmatrix} \\ &= \left\{ I_{2n_1} + \epsilon \Big(T_1 - T_2 (T_4 - I_{2n_2})^{-1} T_3 \Big) \right\} \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix} \\ &= \left(I_{2n_1} + \epsilon T_s^{(0)} + O\big(\epsilon^2\big) \right) \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix} \end{aligned}$$

The matrix $T_s^{(0)}$ determines the coefficients for the approximate slow continuous-time algebraic equation of Litkouhi and Khalil

(1984). It can be observed that

$$\begin{bmatrix} a_1^{(0)} & a_2^{(0)} \\ a_3^{(0)} & a_4^{(0)} \end{bmatrix} = I_{2n_1} + \epsilon T_s^{(0)}$$

$$= I_{2n_1} + \epsilon\left(T_1^{(0)} - T_2^{(0)}\left(T_4^{(0)} - I_{2n_3}\right)^{-1} T_3^{(0)}\right)$$

$$= \begin{bmatrix} I_{n_1} + \epsilon A_s & -\epsilon B_s R_s^{-1} B_s \\ -\epsilon Q_s & I_{n_1} - \epsilon A_s^T \end{bmatrix}$$

$$= \begin{bmatrix} I_{n_1} + \epsilon A_s & -\epsilon S_s \\ -\epsilon Q_s & I_{n_1} - \epsilon A_s^T \end{bmatrix}$$

The corresponding approximate continuous-time algebraic Riccati equation is given by

$$P_s^{(0)} A_s + A_s^T P_s^{(0)} + Q_s - P_s^{(0)} S_s P_s^{(0)} = 0$$

such that $P_s = P_s^{(0)} + O(\epsilon)$. The unique positive semidefinite stabilizing solution of the slow approximate continuous-time algebraic Riccati equation of exists under the assumption that the approximate slow subsystem is stabilizable-detectable.

The approximate **slow subsystem determined by $T_s^{(0)}$ is stabilizable-detectable, that is, the triple $\left(A_s, \sqrt{S_s}, \sqrt{Q_s}\right)$ is stabilizable-detectable**.

We have established that $O(\epsilon)$ perturbations of the derived pure-slow and pure-fast algebraic Riccati equations lead to the symmetric reduced-order approximate slow and fast algebraic

Riccati equations obtained in Litkouhi and Khalil (1984). The solutions of these equations, (Litkouhi and Khalil 1984), can be used as very good initial guesses for the Newton method for solving the obtained nonsymmetric Riccati equations.

The Newton algorithm is given by

$$\boldsymbol{P_s^{(i+1)}\Big(a_1 + a_2 P_s^{(i)}\Big) - \Big(a_4 - P_s^{(i)} a_2\Big) P_s^{(i+1)}}$$
$$\boldsymbol{= a_3 + P_s^{(i)} a_2 P_s^{(i)}, \qquad i = 0, 1, 2, \ldots}$$

with the initial guess $\boldsymbol{P_s^{(0)}}$ obtained from the continuous-time approximate slow algebraic Riccati equation. The Newton algorithm for the pure-fast Riccati equation is similarly obtained as

$$\boldsymbol{P_f^{(i+1)}\Big(b_1 + b_2 P_f^{(i)}\Big) - \Big(b_4 - P_f^{(i)} b_2\Big) P_f^{(i+1)}}$$
$$\boldsymbol{= b_3 + P_f^{(i)} b_2 P_f^{(i)}, \qquad i = 0, 1, 2, \ldots}$$

with the initial guess $\boldsymbol{P_f^{(0)}}$ found by solving the discrete-time approximate fast algebraic Riccati equation.

Lemma *Consider the optimal closed-loop linear discrete system*

$$\begin{aligned}
&x_1(k+1) = (I + \epsilon A_1 - \epsilon B_1 F_1) x_1(k) \\
&\ + \epsilon (A_2 - B_1 F_2) x_2(k) \\
&x_2(k+1) = (A_3 - B_2 F_1) x_1(k) + (A_4 - B_2 F_2) x_2(k)
\end{aligned}$$

There exists a nonsingular transformation $\mathbf{T_6}$

$$\begin{bmatrix} \xi_s(k) \\ \xi_f(k) \end{bmatrix} = \mathbf{T}_6 \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

such that

$$\begin{aligned}
\xi_s(k+1) &= (a_1 + a_2 P_s) \xi_s(k) \\
\xi_f(k+1) &= \big(b_1 + b_2 P_f\big) \xi_f(k)
\end{aligned}$$

where P_s *and* P_f *are the unique solutions of the exact pure-slow and pure-fast completely decoupled algebraic regulator Riccati equations. The nonsingular transformation* $\mathbf{T_6}$ *is given by*

$$\mathbf{T_6} = (\mathbf{\Pi}_1 + \mathbf{\Pi}_2 P)$$

Even more, the global solution P *can be obtained from the reduced-order exact pure-slow and pure-fast algebraic regulator Riccati equations, that is*

$$P = \left(\mathbf{\Omega}_3 + \mathbf{\Omega}_4 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \right) \left(\mathbf{\Omega}_1 + \mathbf{\Omega}_2 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \right)^{-1}$$

Known matrices $\mathbf{\Omega}_i,\ i = 1, 2, 3, 4$ *and* $\mathbf{\Pi}_1,\ \mathbf{\Pi}_2$ *are given in terms of solutions of the Chang decoupling equations.*

Complete solutions to the **Kalman filtering** and linear-quadratic optimal **stochastic Gaussian control problem** in the discrete-time domain are presented in (Lim, 1994; Lim, Gajic, and Shen, 1995).

H_∞ optimal controller is considered in (Hsieh and Gajic, 1998) and **H_∞ filtering** in (Lim and Gajic, 2000).

In (Kecman, Bingulac, and Gajic, 1999), an eigenvector approach is developed for **simultaneous solution of the L-H equations and pure-slow and pure-fast algebraic Riccati equations**. The results are available in the continuous-time only.

Conclusions

Many valuable and practically impementable high accurate results were obtained during the past decade for optimal control and filtering of linear singularly perturbed systems. The Hamiltonian approach is simple and elegant, but of limited applicability.

The recursive approach can be applied to more complex problems, especially game-type situations. The invariant manifold approach is a powerful tool for nonlinear singularly perturbed systems.

Many "high accuracy singularly perturbed" open problems remain in the domain of **bilinear and nonlinear** singularly perturbed systems.