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Hamiltonian Approach to Linear Continuous-Time Singularly Perturbed Optimal Control and Filtering Problems

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Abstract* – In this paper we present a unified approach for optimal control and filtering of linear continuous-time singularly perturbed linear systems that facilitates complete and exact decomposition of optimal control and filtering tasks into pure-slow and pure-fast time scales. The presented methodology eliminates numerical ill-conditioning of the original singularly perturbed problems, introduces parallelism into the design procedures, allows independent parallel processing of information in slow and fast time scales, and reduces both off-line and on-line computation requirements. The presentation is done for the classic linear-quadratic open- and closed-loop optimal regulators, Kalman filter, H_∞-optimal linear-quadratic regulator, H_∞-optimal linear filter, and the corresponding linear-quadratic optimal stochastic regulators. In addition, we indicate related control problems solvable by the presented methodology.

***Key words:* Singular perturbations, optimal control, Kalman filtering, H_∞ optimization.**

I. INTRODUCTION

Theory of singular perturbations was introduced to control audience by Kokotovic by the end of the sixties. Due to the fact that many real physical systems are singularly perturbed, for example, aircrafts, robots, electrical circuits, power systems, nuclear reactors, chemical reactors, dc and induction motors, synchronous machines, distillation columns, flexible structures, automobiles, this theory has become very popular in control system engineering, [1]-[7]. Singularly perturbed systems are characterized by simultaneous presence of small and large time constants, which introduces clustering of linear (or linearized) system eigenvalues into two disjoint groups: (a) eigenvalues corresponding to large time constants located close to the imaginary axis representing slow system state space variables (slow modes), and (b) eigenvalues corresponding to small time constants located far from the imaginary axis representing fast system state space variables (fast modes). In the last thirty years almost one thousand journal papers and more than twenty books were published by engineering and mathematics researchers on the subject of singularly perturbed control systems.

The notion of singular perturbations in mathematics stands for systems of differential equations that have some derivatives multiplied by small positive parameters. Such a kind of systems of differential equations was extensively studied in the fifties and sixties by well-known mathematicians such as Tikhonov, Levin, Levinson, Vasileva, Butuzov, Wasov, Hoppendsteadt, O'Malley, Chang. One of the most important and most widely used results of mathematical theory of singular perturbations is the development of the transformation for the exact pure-slow and pure-fast decomposition of linear singularly perturbed systems, known as the Chang transformation, [8].

The approaches taken in engineering during the seventies and eighties, in the study of singularly perturbed control systems, were based on expansion methods (power series, asymptotic expansions, Taylor series), the methods developed by previously mentioned mathematicians. The approaches were, in most cases, accurate only with an $O(\epsilon)$[1] accuracy, where ϵ is a small positive singular perturbation parameter. Generating higher order expansions for those methods has been analytically pretty cumbersome and numerically pretty inefficient, especially for high-dimensional control systems. Even more, it has been demonstrated in several papers, [9]-[12], that for some applications an $O(\epsilon)$ accuracy is either not sufficient or even more, it does not solve the problem at all.

The development of high accuracy efficient techniques for singularly perturbed control systems started in the middle of the eighties along the lines of the slow-fast manifold approach of Sobolev, [13], and the recursive approach based on fixed-point iterations of Gajic, [14]. At the beginning of the nineties, the fixed-point recursive approach culminates in the so-called Hamiltonian approach for the *exact* pure-slow and pure-fast decomposition of singularly perturbed, linear-quadratic, deterministic and stochastic, optimal control and filtering problems. The class of problems solvable by the Hamiltonian approach are steady state linear-quadratic optimal control and filtering problems whose Hamiltonian matrices under appropriate scaling and permutation preserve singularly perturbed forms such that they can be exactly block diagonalized into pure-slow and pure-fast Hamiltonian matrices. That is why, we call this approach the Hamiltonian approach to singularly perturbed

[1] An $O(\epsilon^r)$ stands for $O(\epsilon^r) < c\epsilon^r$, where c is a bounded constant and r is any real number.

optimal linear control systems. Note that the study of singularly perturbed linear-quadratic optimal control systems via the use of the Hamiltonian system of differential equations have been done in the past in different set ups by several researchers, for example, [15]-[20].

The problems presently solvable by the Hamiltonian approach are: linear-quadratic optimal regulator and Kalman filter in continuous- and discrete-time domains, optimal open-loop control of continuous- and discrete-time linear systems, multimodeling estimation and control, H_∞-optimal control and filtering of linear systems, linear-quadratic zero-sum differential games, linear-quadratic high gain, cheap control, and small measurement noise problems, sampled data control systems, and nonstandard linear singularly perturbed optimal control and filtering systems. Some other classes of linear-quadratic type optimal control problems that can be solved by the methodology considered in this paper may emerge in the near future.

The work of [13] based on slow-fast manifold theory resulted also in the exact pure-slow and pure-fast decomposition of the linear-quadratic optimal control problems as demonstrated in [21]-[22]. However, it remains an open question whether or not the integral manifold approach to decomposition of singularly linear-quadratic control problems [21]-[22] leads to the same results as those obtained by the Hamiltonian approach.

This paper represents a comprehensive view of the current state of the knowledge of the Hamiltonian approach to singularly perturbed linear continuous-time optimal control and filtering problems. The presentation is based on the recent research work of the authors and their coworkers. The paper presents a unified theme about the exact pure-slow pure-fast decoupling of the corresponding optimal control and filtering problems owing to the existence of a transformation that exactly decouples the nonlinear algebraic Riccati equation into the pure-slow and pure-fast, reduced-order, algebraic Riccati equations. At the same time, the paper demonstrates the power of the Hamiltonian approach clearly indicating the unified theme that can be used for the most efficient and most accurate solution of variety of optimal control and filtering problems.

A. Exact Decomposition of the Riccati Equation

In this section, it is shown how to exactly decompose the algebraic Riccati equation of singularly perturbed control systems into two reduced-order algebraic Riccati equations corresponding to slow and fast time scales. The reduced-order algebraic Riccati equations obtained are nonsymmetric. The Newton algorithm is very efficient for solving these nonsymmetric algebraic Riccati equations since excellent initial guesses are readily available from the reduced-order, symmetric, algebraic Riccati equations that represent $O(\epsilon)$ perturbations of the nonsymmetric, reduced-order, pure-slow and pure-fast, algebraic Riccati equations. Due to complete and exact decomposition of the Riccati equation, and due to order-reduction, we have an efficient parallel algorithm for solving this equation—the most important equation of the linear-quadratic optimal control and filtering theory.

The procedure used for the time-scale decomposition of the algebraic Riccati equations into the pure-slow and pure-fast algebraic Riccati equations facilitates new insights into optimal filtering and control problems of singularly perturbed linear systems. It is demonstrated in the subsequent sections that corresponding reduced-order linear optimal filters and controllers are completely and exactly decoupled. The slow/fast filters and controllers work in parallel and process information independently in slow and fast time scales with the corresponding sampling rates—the slow ones with the slow sampling rate and the fast ones with the fast sampling rate.

A linear singularly perturbed control system is given by

$$\begin{aligned} \dot{x}_1(t) &= A_1x_1(t) + A_2x_2(t) + B_1u(t),\ x_1(t_0) = x_{10} \\ \epsilon\dot{x}_2(t) &= A_3x_1(t) + A_4x_2(t) + B_2u(t),\ \ x_2(t_0) = x_{20} \end{aligned} \tag{1}$$

where $x_i(t) \in \Re^{n_i},\ i = 1,2,\ u(t) \in \Re^m$ are state and control variables, respectively, and ϵ is a small positive singular perturbation parameter. As a parameter ϵ tends to zero, the solution behaves nonuniformly, producing a so-called singularly perturbed stiff problem (huge slope for the fast state variable at the initial time), which implies numerical ill-conditioning.

With (1), consider the performance criterion to be minimized by the choice of the optimal control strategy

$$J = \min_u \frac{1}{2}\int_{t_0}^{\infty}\left\{\begin{bmatrix} x_1(t)\\ x_2(t)\end{bmatrix}^T Q \begin{bmatrix} x_1(t)\\ x_2(t)\end{bmatrix} + u^T(t)R\,u(t)\right\}dt \tag{2}$$

with positive definite R and positive semidefinite Q. The open-loop optimal control problem of (1)-(2) has the solution

$$u(t) = -R^{-1}B^Tp(t) \tag{3}$$

where $p(t) \in \Re^{n_1+n_2}$ is a costate variable satisfying [23]

$$\begin{bmatrix}\dot{x}(t)\\ \dot{p}(t)\end{bmatrix} = \begin{bmatrix} A & -S\\ -Q & -A^T\end{bmatrix}\begin{bmatrix} x(t)\\ p(t)\end{bmatrix} \tag{4}$$

with

$$A = \begin{bmatrix} A_1 & A_2\\ \frac{1}{\epsilon}A_3 & \frac{1}{\epsilon}A_4\end{bmatrix},\ Q = \begin{bmatrix} Q_1 & Q_2\\ Q_2^T & Q_3\end{bmatrix} = \begin{bmatrix} q_1^Tq_1 & q_1^Tq_2\\ q_2^Tq_1 & q_2^Tq_2\end{bmatrix} \tag{5}$$

$$B = \begin{bmatrix} B_1\\ \frac{1}{\epsilon}B_2\end{bmatrix},\ S = BR^{-1}B^T = \begin{bmatrix} S_1 & \frac{1}{\epsilon}Z\\ \frac{1}{\epsilon}Z^T & \frac{1}{\epsilon^2}S_2\end{bmatrix}$$

and $x^T(t) = \left[x_1^T(t)\ x_2^T(t)\right]$. The optimal closed-loop control law has the very-well known form

$$u(x(t)) = -R^{-1}B^TPx(t) = -Fx(t) \tag{6}$$

where P satisfies the algebraic Riccati equation given by

$$0 = PA + A^T P + Q - PSP, \quad P = \begin{bmatrix} P_1 & \epsilon P_2 \\ \epsilon P_2^T & \epsilon P_3 \end{bmatrix} \quad (7)$$

The positive semidefinite stabilizing solution of the algebraic Riccati equation (7) exists under the standard stabilizability-detectability conditions [23].

Assumption 1.1: The triple $(A, B, \sqrt{Q})$ is stabilizable and detectable.

In the following we follow the results of [24] and show how to find the solution of (7) in terms of solutions of the reduced-order, pure-slow and pure-fast, algebraic Riccati equations. It is well known that the solution of the Riccati equation, can be obtained from the Hamiltonian matrix. It will be shown that for singularly perturbed systems, the Hamiltonian matrix retains the singularly perturbed form by interchanging and appropriately scaling some state and costate variables, hence it can be block diagonalized via the nonsingular transformations of [8], [25].

Partitioning and appropriately scaling $p(t)$ as $p^T(t) = [p_1^T(t) \quad \epsilon p_2^T(t)]$ with $p_i(t) \in \Re^{n_i}, i = 1, 2,$ and interchanging second and third rows in (4), we get

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{p}_1(t) \\ \dot{x}_2(t) \\ \dot{p}_2(t) \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ \frac{1}{\epsilon}T_3 & \frac{1}{\epsilon}T_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} \quad (8)$$

where

$$T_1 = \begin{bmatrix} A_1 & -S_1 \\ -Q_1 & -A_1^T \end{bmatrix}, \qquad T_2 = \begin{bmatrix} A_2 & -Z \\ -Q_2 & -A_3^T \end{bmatrix}$$
$$T_3 = \begin{bmatrix} A_3 & -Z^T \\ -Q_2^T & -A_2^T \end{bmatrix}, \qquad T_4 = \begin{bmatrix} A_4 & -S_2 \\ -Q_3 & -A_4^T \end{bmatrix} \quad (9)$$

It is important to notice that (8) retains the singular perturbation form. Also, the matrix T_4 is the Hamiltonian matrix of the fast subsystem, and it is nonsingular under stabilizability-detectability conditions imposed on the fast subsystem.

Assumption 1.2: The triple (A_4, B_2, q_2) is stabilizable and detectable.

It should be emphasized that the presented procedure is valid for both the so-called standard (matrix A_4 is nonsingular) and nonstandard (matrix A_4 is singular) singularly perturbed linear control systems. Note that nonstandard singularly perturbed control systems are the recent trend in theory of singularly perturbed linear control systems [20], [26]-[28], [39].

The celebrated transformation of [8], used for decomposition of linear singularly perturbed systems, is defined by

$$\mathbf{T}_1 = \begin{bmatrix} I_{2n_1} - \epsilon HL & -\epsilon H \\ L & I_{2n_2} \end{bmatrix} \quad (10)$$

where L and H satisfy

$$T_4 L - T_3 - \epsilon L(T_1 - T_2 L) = 0 \quad (11)$$

$$-H(T_4 + \epsilon L T_2) + T_2 + \epsilon(T_1 - T_2 L)H = 0 \quad (12)$$

The unique solutions of (11) and (12) exist for sufficiently small values of ϵ under condition that T_4 is nonsingular, that is, under Assumption 1.2. These algebraic equations can be solved as linear algebraic equations using either the fixed-point algorithm of [29] or the Newton method of [9]. The corresponding algorithms for solving the L-equation are respectively given by

$$\begin{gathered} L^{(i+1)} = L^{(0)} + \epsilon T_4^{-1} L^{(i)} \left(T_1 - T_2 L^{(i)}\right) \\ L^{(0)} = T_4^{-1} T_3, \quad i = 0, 1, 2, ... \end{gathered} \quad (13)$$

$$\begin{gathered} D_1^{(i)} L^{(i+1)} + L^{(i+1)} D_2^{(i)} = Q^{(i)} \\ L^{(0)} = T_4^{-1} T_3, \quad i = 0, 1, 2, ... \\ D_1^{(i)} = T_4 + \epsilon L^{(i)} T_2 \\ D_2^{(i)} = -\epsilon\left(T_1 - T_2 L^{(i)}\right), \; Q^{(i)} = T_3 + \epsilon L^{(i)} T_2 L^{(i)} \end{gathered} \quad (14)$$

The Newton method converges quadratically, hence if it converges, it requires in average only four to five iterations. The fixed-point iterations converge linearly and sometimes require a lot of iterations. In addition, the L-equation can be efficiently solved by using the eigenvector method of [30] and the Taylor series expansions of [31]. Once the solution for the L-equation is obtained, the H-equation can be solved either directly as a linear Sylvester equation or recursively as

$$\begin{gathered} H^{(i+1)} = T_2 (T_4 + \epsilon L T_2)^{-1} \\ +\epsilon(T_1 - T_2 L) H^{(i)} (T_4 + \epsilon L T_2)^{-1} \\ H^{(0)} = T_2 T_4^{-1}, \quad i = 0, 1, 2, ... \end{gathered} \quad (15)$$

The Chang transformation (10) applied to (8) produces two completely decoupled subsystems

$$\dot{\eta}(t) = (T_1 - T_2 L)\eta(t) \quad (16)$$

and

$$\epsilon\dot{\xi}(t) = (T_4 + \epsilon L T_2)\xi(t) \quad (17)$$

where

$$\begin{bmatrix} \eta(t) \\ \xi(t) \end{bmatrix} = \mathbf{T_1} \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} \quad (18)$$

The rearrangement and modification of the original variables in (8) is done by using the permutation matrix E_1 of the form

$$\begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\epsilon} I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ p_1(t) \\ \epsilon p_2(t) \end{bmatrix} = E_1 \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \quad (19)$$

Note that the inverse of E_1 can be easily obtained analytically, hence, this matrix is not numerically ill-conditioned with respect to the matrix inversion for small values of ϵ.

Combining (18) and (19), we obtain the relationship between the original and new coordinates as

$$\begin{bmatrix}\eta_1(t)\\ \xi_1(t)\\ \eta_2(t)\\ \xi_2(t)\end{bmatrix} = E_2^T \mathbf{T}_1 E_1 \begin{bmatrix}x(t)\\ p(t)\end{bmatrix} = \Pi \begin{bmatrix}x(t)\\ p(t)\end{bmatrix} = \begin{bmatrix}\Pi_1 & \Pi_2\\ \Pi_3 & \Pi_4\end{bmatrix} \begin{bmatrix}x(t)\\ p(t)\end{bmatrix} \tag{20}$$

where E_2 is a permutation matrix in the form

$$E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0\\ 0 & 0 & I_{n_1} & 0\\ 0 & I_{n_2} & 0 & 0\\ 0 & 0 & 0 & I_{n_2}\end{bmatrix} \tag{21}$$

Since $p(t) = Px(t)$, where P satisfies the algebraic Riccati equation (7), it follows that

$$\begin{bmatrix}\eta_1(t)\\ \xi_1(t)\end{bmatrix} = (\Pi_1 + \Pi_2 P)x(t),\ \begin{bmatrix}\eta_2(t)\\ \xi_2(t)\end{bmatrix} = (\Pi_3 + \Pi_4 P)x(t) \tag{22}$$

In the original coordinates, the required optimal solution has a closed-loop nature. We have the same attribute for the new systems (16) and (17); that is

$$\begin{bmatrix}\eta_2(t)\\ \xi_2(t)\end{bmatrix} = \begin{bmatrix}P_s & 0\\ 0 & P_f\end{bmatrix} \begin{bmatrix}\eta_1(t)\\ \xi_1(t)\end{bmatrix} \tag{23}$$

Then, (22) and (23) yield

$$\begin{bmatrix}P_s & 0\\ 0 & P_f\end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1} \tag{24}$$

Following the same logic, we can find P reversely by introducing

$$E_1^{-1}\mathbf{T}_1^{-1}E_2 = \Pi^{-1} = \Omega = \begin{bmatrix}\Omega_1 & \Omega_2\\ \Omega_3 & \Omega_4\end{bmatrix} \tag{25}$$

where

$$E_1^{-1} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0\\ 0 & 0 & I_{n_2} & 0\\ 0 & I_{n_1} & 0 & 0\\ 0 & 0 & 0 & \epsilon I_{n_2}\end{bmatrix} \tag{26}$$

and it yields

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix}P_s & 0\\ 0 & P_f\end{bmatrix}\right)\left(\Omega_1 + \Omega_2 \begin{bmatrix}P_s & 0\\ 0 & P_f\end{bmatrix}\right)^{-1} \tag{27}$$

It is shown in [24] that the matrix inversions in (24) and (27) exist for sufficiently small values of ϵ.

Partitioning (16) and (17) as

$$\begin{bmatrix}\dot{\eta}_1(t)\\ \dot{\eta}_2(t)\end{bmatrix} = \begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix} \begin{bmatrix}\eta_1(t)\\ \eta_2(t)\end{bmatrix} = (T_1 - T_2 L)\begin{bmatrix}\eta_1(t)\\ \eta_2(t)\end{bmatrix} \tag{28}$$

$$\epsilon\begin{bmatrix}\dot{\xi}_1(t)\\ \dot{\xi}_2(t)\end{bmatrix} = \begin{bmatrix}b_1 & b_2\\ b_3 & b_4\end{bmatrix} \begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix} = (T_4 + \epsilon L T_2)\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix} \tag{29}$$

and using (23) yield to two reduced-order, nonsymmetric, pure-slow and pure-fast, algebraic Riccati equations

$$0 = P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s \tag{30}$$

$$0 = P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f \tag{31}$$

with $a_i, b_i, i = 1,2,3,4,$ defined by (28)-(29). Let us point out that the nonsymmetric algebraic Riccati equation was studied by several researchers, see for example [32] and references therein. An algorithm for the solving general nonsymmetric algebraic Riccati equation was derived in [33]—see also [34].

The pure-fast algebraic Riccati equation (31) is nonsymmetric, but its $O(\epsilon)$ approximation is a symmetric one, that is

$$P_f A_4 + A_4^T P_f + Q_3 - P_f S_2 P_f + O(\epsilon) = 0 \tag{32}$$

From (32) one can obtain an $O(\epsilon)$ approximation for P_f as

$$P_f^{(0)} A_4 + A_4^T P_f^{(0)} + Q_3 - P_f^{(0)} S_2 P_f^{(0)} = 0 \tag{33}$$

The unique positive semidefinite stabilizing solution of (33) exists under Assumption 1.2. Hence, we have $P_f = P_f^{(0)} + O(\epsilon)$. The pure-slow algebraic Riccati equation (30) is also nonsymmetric. It can be also shown that (30) is an $O(\epsilon)$ perturbation of the first-order approximate slow algebraic Riccati equation obtained in [35] and [19]

$$P_s^{(0)} A_s + A_s^T P_s^{(0)} + Q_s - P_s^{(0)} S_s P_s^{(0)} = 0 \tag{34}$$

with $P_s = P_s^{(0)} + O(\epsilon)$, where $A_s, Q_s,$ and S_s can be found either using the methodology of [35] or from the results of [19] as

$$\begin{bmatrix}A_s & -S_s\\ -Q_s & -A_s^T\end{bmatrix} = T_1 - T_2 T_4^{-1} T_3 \tag{35}$$

Note that from (11) and (28) we have

$$\begin{bmatrix}a_1^{(0)} & a_2^{(0)}\\ a_3^{(0)} & a_4^{(0)}\end{bmatrix} = \begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix} + O(\epsilon) = T_1 - T_2 L^{(0)} + O(\epsilon) = T_1 - T_2 T_4^{-1} T_3 + O(\epsilon) \tag{36}$$

which implies

$$\begin{bmatrix}a_1^{(0)} & a_2^{(0)}\\ a_3^{(0)} & a_4^{(0)}\end{bmatrix} = \begin{bmatrix}A_s & -S_s\\ -Q_s & -A_s^T\end{bmatrix} \tag{37}$$

The unique positive semidefinite stabilizing solution of the approximate slow algebraic Riccati equation (34) exists under the following assumption.

Assumption 1.3: The triple $\left(A_s, \sqrt{S_s}, \sqrt{Q_s}\right)$ is stabilizable and detectable.

Note that in the case when the matrix A_4 is nonsingular (standard singularly perturbed linear system), Assumption 1.3 can be replaced by a simpler assumption of the form [35].

Assumption 1.3a: The triple (A_0, B_0, q_0) is stabilizable and detectable, with $A_0 = A_1 - A_2 A_4^{-1} A_3$, $B_0 = B_1 - A_2 A_4^{-1} B_2$, $q_0 = q_1 - q_2 A_4^{-1} A_3$.

Assumptions 1.2, 1.3, and 1.3a are the standard assumptions in theory of singularly perturbed linear control systems [4]-[5].

Using the fact that the unique solutions of (33) and (34) exist, then by the implicit function theorem, [36], the existence of the unique solutions of (30) and (31) are guaranteed by the following lemma [24].

Lemma 1.1. Let Assumptions 1.2 and 1.3 be satisfied. Then, $\exists \epsilon_0 > 0$ such that $\forall \epsilon \leq \epsilon_0$ the unique solutions of (30) and (31) exist.

Having obtained a good initial guess, the Newton type algorithm can be used very efficiently for solving (31). The Newton algorithm is given by

$$\begin{aligned} P_f^{(i+1)}\left(b_1 + b_2 P_f^{(i)}\right) - \left(b_4 - P_f^{(i)} b_2\right) P_f^{(i+1)} \\ = b_3 + P_f^{(i)} b_2 P_f^{(i)}, \quad i = 0, 1, 2, \ldots \end{aligned} \tag{38}$$

with an initial guess obtained from (33).

The pure-slow Riccati equation (30) can be solved by using the Newton algorithm also, with an initial guess obtained from (34). The Newton algorithm for (30) is given by

$$\begin{aligned} P_s^{(i+1)}\left(a_1 + a_2 P_s^{(i)}\right) - \left(a_4 - P_s^{(i)} a_2\right) P_s^{(i+1)} \\ = a_3 + P_s^{(i)} a_2 P_s^{(i)}, \quad i = 0, 1, 2, \ldots \end{aligned} \tag{39}$$

It is important to notice that the total number of scalar quadratic algebraic equations in (30) and (31) is $n_1^2 + n_2^2$. On the other hand, the global algebraic Riccati equation (7) contains $\frac{1}{2}(n_1 + n_2)(n_1 + n_2 + 1)$ scalar algebraic equations. Thus, the presented method can even reduce the number of algebraic equations if

$$n_1^2 + n_2^2 < \frac{1}{2}(n_1 + n_2)(n_1 + n_2 + 1) \tag{40}$$

or

$$(n_1 - n_2)^2 < n_1 + n_2 \tag{41}$$

which is the case when n_1 and n_2 are close to each other.

Using solutions of both pure-slow and pure-fast Riccati equations and formulas (23) and (28)-(29), we can get completely decoupled slow and fast subsystems in the form

$$\begin{aligned} \dot{\eta}_1(t) &= (a_1 + a_2 P_s)\eta_1(t) \\ \epsilon\dot{\xi}_1(t) &= (b_1 + b_2 P_f)\xi_1(t) \end{aligned} \tag{42}$$

The interpretation of the result presented by (42) is that the optimal processing of information for this class of systems (filtering and/or control) can be completely performed at the local levels (slow and fast subsystems). The global solution in the original coordinates is then obtained at any time instant by using formula (22), that is

$$x(t) = (\Pi_1 + \Pi_2 P)^{-1} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} \tag{43}$$

where P is obtained from (27). The use of the results given in (42) in optimal filtering (first of all) and control of singularly perturbed linear systems will be presented in the next sections.

The quadratic performance criterion to be minimized, (2), in the new coordinates is given by

$$\begin{aligned} J &= \frac{1}{2}\int_{t_0}^{+\infty} \left(x^T(t) Q x(t) + u^T(t) R u(t)\right) dt \\ &= \frac{1}{2}\int_{t_0}^{+\infty} x^T(t)(Q + PSP)x(t) dt \\ &= \frac{1}{2}\int_{t_0}^{+\infty} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix}^T (\Pi_1 + \Pi_2 P)^{-T} \\ &\times (Q + PSP)(\Pi_1 + \Pi_2 P)^{-1} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} dt \\ &= \frac{1}{2}\int_{t_0}^{+\infty} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix}^T \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^T & \Theta_3 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} dt \end{aligned} \tag{44}$$

The value of the above integral is obtained as

$$\begin{aligned} J_{opt} &= \frac{1}{2}\mathrm{tr}\left\{ V \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix}^T \right\} \\ &= \frac{1}{2}\mathrm{tr}\left\{ \begin{bmatrix} V_1 & \epsilon V_2 \\ \epsilon V_2^T & \epsilon V_3 \end{bmatrix} \begin{bmatrix} \eta_1(t_0)\eta_1^T(t_0) & \eta_1(t_0)\xi_1^T(t_0) \\ \xi_1(t_0)\eta_1^T(t_0) & \xi_1(t_0)\xi_1^T(t_0) \end{bmatrix} \right\} \\ &= \frac{1}{2} tr\left\{V_1 \eta_1(t_0)\eta_1^T(t_0)\right\} \\ &+ \frac{\epsilon}{2} tr\left(V_2^T \eta_1(t_0)\xi_1^T(t_0) + V_2 \xi_1(t_0)\eta_1^T(t_0) +\right) \\ &+ \frac{\epsilon}{2} tr\left(V_3 \xi_1(t_0)\xi_1^T(t_0)\right) = J_s + \epsilon J_f \end{aligned} \tag{45}$$

where the matrix V satisfies the algebraic Lyapunov equation

$$\begin{aligned} &\begin{bmatrix} (a_1 + a_2 P_1) & 0 \\ 0 & \frac{1}{\epsilon}(b_1 + b_2 P_2) \end{bmatrix}^T V \\ &+ V \begin{bmatrix} (a_1 + a_2 P_1) & 0 \\ 0 & \frac{1}{\epsilon}(b_1 + b_2 P_2) \end{bmatrix} + \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^T & \Theta_3 \end{bmatrix} = 0 \end{aligned}$$

which implies three independent, reduced-order, Lyapunov (Sylvester) algebraic equations

$$\begin{aligned} (a_1 + a_2 P_1)^T V_1 + V_1 (a_1 + a_2 P_1) + \Theta_1 = 0 \\ \epsilon (a_1 + a_2 P_1)^T V_2 + V_2 (b_1 + b_2 P_2) + \Theta_2 = 0 \\ (b_1 + b_2 P_2)^T V_3 + V_3 (b_1 + b_2 P_2) + \Theta_3 = 0 \end{aligned} \tag{46}$$

Formula (45) exactly decomposes slow and fast components of the optimal performance criterion. It can be concluded from (45) that the pure-slow component of the performance criterion is $O(1)$ and that the fast subsystem contributes only an $O(\epsilon)$ to the performance criterion of a linear continuous-time deterministic system.

B. Open-Loop Linear Control Problem

The optimal open-loop control problem is a two-point boundary value problem with the associated state-costate equations forming the Hamiltonian system of linear differential equations. In this section, the two-point boundary value problem of linear singularly perturbed systems is transformed into the pure-slow and pure-fast, reduced-order, completely decoupled initial value problems by following methodology of [37]. By doing this, the stiffness (numerical ill-conditioning) of the original singularly perturbed two-point boundary value problem is converted into the problem of an ill-defined linear system of algebraic equations.

Consider the linear singularly perturbed control system (1). The associated performance criterion to be minimized over the time period from t_0 to t_f is defined by

$$J = \min_{u(t)} \frac{1}{2}\{\int_{t_0}^{t_f} \left\{ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T Q \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u^T(t) R\, u(t) \right\} dt + \frac{1}{2} \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix}^T Q_f \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} \}, \qquad Q_f \geq 0 \tag{47}$$

where Q_f is the terminal time penalty matrix. The open-loop optimal control problem of minimizing (47) along trajectories of dynamic system (1) has the solution given by (3)-(4) with boundary conditions given by [23]

$$M \begin{bmatrix} x(t_0) \\ p(t_0) \end{bmatrix} + N \begin{bmatrix} x(t_f) \\ p(t_f) \end{bmatrix} = c \tag{48}$$

with

$$M = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ -Q_f & I_n \end{bmatrix}, \quad c = \begin{bmatrix} x(t_0) \\ 0 \end{bmatrix}$$
$$n = n_1 + n_2 \tag{49}$$

The terminal penalty matrix is appropriately partitioned as

$$Q_f = \begin{bmatrix} Q_{f1} & \epsilon Q_{f2} \\ \epsilon Q_{f2}^T & \epsilon Q_{f3} \end{bmatrix} \tag{50}$$

The approximate optimal solution of the open-loop control for linear singularly perturbed systems has been studied in [18], where the problem order was reduced and the stiff problem was avoided successfully by using the classic approach based on the power-series expansions. The theory developed in [18] was based on the dichotomy transformation of [38], which requires the positive definite and negative definite solutions of the corresponding algebraic Riccati equation. It was concluded in [18] that the developed method is efficient for an $O(\epsilon)$ accuracy only. In this section, the solution to the optimal open-loop control problem of singularly perturbed systems with an arbitrary order of accuracy is presented.

Let us partition and appropriately scale the co-state vector $p(t)$ as $p^T(t) = [p_1^T(t) \quad \epsilon p_2^T(t)]$ with $p_1(t) \in \Re^{n_1}$ and $p_2(t) \in \Re^{n_2}$. By interchanging second and third rows in the corresponding state co-state equations (4), we get the singularly perturbed system (8)-(9) with the boundary conditions

$$x(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad p(t_f) = Q_f x(t_f) \tag{51}$$

The original boundary conditions can be written in a compact form consistent to (8) as follows

$$M_1 \begin{bmatrix} x_1(t_0) \\ p_1(t_0) \\ x_2(t_0) \\ p_2(t_0) \end{bmatrix} + N_1 \begin{bmatrix} x_1(t_f) \\ p_1(t_f) \\ x_2(t_f) \\ p_2(t_f) \end{bmatrix} = c_1 \tag{52}$$

where

$$M_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad c_1 = \begin{bmatrix} x_{10} \\ 0 \\ x_{20} \\ 0 \end{bmatrix}$$
$$N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -Q_{f1} & I_{n_1} & -\epsilon Q_{f2} & 0 \\ 0 & 0 & 0 & 0 \\ -Q_{f2}^T & 0 & -Q_{f3} & In_2 \end{bmatrix} \tag{53}$$

The Chang transformation (10) applied to (8) produces two completely decoupled pure-slow and pure-fast subsystems defined by (16)-(18). The boundary conditions in the new coordinates corresponding to (16)-(18) are given by

$$M_2 \begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} + N_2 \begin{bmatrix} \eta(t_f) \\ \xi(t_f) \end{bmatrix} = c_1 \tag{54}$$

where

$$M_2 = M_1 \mathbf{T}_1^{-1}, \quad N_2 = N_1 \mathbf{T}_1^{-1} \tag{55}$$

Since solutions of (16) and (17) are given by

$$\eta(t) = e^{(T_1 - T_2 L)(t - t_0)} \eta(t_0) \tag{56}$$

$$\xi(t) = e^{\frac{1}{\epsilon}(T_4 + \epsilon L T_2)(t - t_0)} \xi(t_0) \tag{57}$$

we can eliminate $\eta(t_f)$ and $\xi(t_f)$ from (54), which leads to

$$\left\{ M_2 + N_2 \begin{bmatrix} e^{(T_1 - T_2 L)(t_f - t_0)} & 0 \\ 0 & e^{\frac{1}{\epsilon}(T_4 + \epsilon L T_2)(t_f - t_0)} \end{bmatrix} \right\} \times \begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} = c_1 \tag{58}$$

The system of linear algebraic equations obtained, (58), is of the form

$$\alpha(\epsilon) \begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} = c_1 \tag{59}$$

It is proved in [37] that $\alpha(\epsilon)$ is invertible, hence $\eta(t_0)$ and $\xi(t_0)$ can be obtained from (59). The corresponding lemma of [37] is given below.

Lemma 1.2. Under Assumptions 1.2 and 1.3, the matrix $\alpha(\epsilon)$ is invertible.

Now we are able to find $\eta(t)$ and $\xi(t)$ from (56) and (57). Using (18), we can get the values for $p_1(t)$ and $p_2(t)$. The costate variables $p(t)$ and the optimal control law are therefore found.

The only difficulty encountered in the procedure is to compute $\alpha(\epsilon)$ in the case when an ill-defined problem occurs either for ϵ being extremely small or for $(t_f - t_0)$ being very large. Note that the matrix T_4 contains both stable and unstable modes. In that case the $O(\epsilon)$-approximate results of [18] have to be used.

C. Linear Kalman Filtering Problem

In this section a method that facilitates complete decomposition of the optimal global Kalman filter of linear singularly perturbed systems into pure-slow and pure-fast local optimal filters both driven by system measurements is presented. The method is based on the exact decomposition of the global singularly perturbed algebraic filter Riccati equation as presented in Section 1.1 and the duality property that exists between the linear-quadratic optimal filters and regulators.

Filtering problem of linear singularly perturbed continuous-time systems has been well documented in the control literature [40]-[44]. In Haddad [40]-[42] the suboptimal slow and fast Kalman filters were constructed producing an $O(\epsilon)$ accuracy for the estimates of the state trajectories, where a small positive singular perturbation parameter ϵ represents the separation between slow and fast phenomena. In [43]-[44] both the slow and fast (local) Kalman filters were obtained with an arbitrary order of accuracy, that is $O\left(\epsilon^k\right)$, where k stands for either the number of terms of the Taylor series [43] or the number of the fixed-point iterations [44] used to calculate coefficients of the corresponding filters. It is important to point out that the local slow and fast filters of [43]-[44] are driven by the innovation process so that the additional communication channels are required to form the innovation process. In the technique presented in this section, the local filters are driven by the system measurements only. In addition, the optimal filter gains are completely determined in terms of the exact pure-slow and exact pure-fast reduced-order algebraic filter Riccati equations.

Consider the linear continuous-time invariant singularly perturbed stochastic system

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) + G_1 w_1(t) \\ \epsilon \dot{x}_2 &= A_3 x_1(t) + A_4 x_2(t) + G_2 w_1(t) \end{aligned} \tag{60}$$

with the corresponding measurements

$$y(t) = C_1 x_1(t) + C_2 x_2(t) + w_2(t) \tag{61}$$

where $x_i(t) \in \Re^{n_i},\ i = 1, 2$, are state vectors, $w_i(t) \in \Re^{r_i}$ are zero-mean stationary, mutually uncorrelated, white Gaussian noise stochastic processes with intensities $W_i > 0$, and $y(t) \in \Re^{r_2}$ are system measurements. In the following $A_i, G_j, C_j,\ \ i = 1, 2, 3, 4,\ \ j = 1, 2$, are constant matrices. We assume that the system under consideration has the standard singularly perturbed form, [28], that is, the following assumption is satisfied.

Assumption 1.4: The fast subsystem matrix A_4 is nonsingular.

The optimal Kalman filter, corresponding to (60)-(71), driven by the innovation process, $v(t)$, is given by

$$\begin{aligned} \dot{\hat{x}}_1(t) &= A_1 \hat{x}_1(t) + A_2 \hat{x}_2(t) + K_1 v(t) \\ \epsilon \dot{\hat{x}}_2(t) &= A_3 \hat{x}_1(t) + A_4 \hat{x}_2(t) + K_2 v(t) \\ v(t) &= y(t) - C_1 \hat{x}_1(t) - C_2 \hat{x}_2(t) \end{aligned} \tag{62}$$

where the optimal filter gains K_1 and K_2 are obtained from (Khalil and Gajic 1984)

$$\begin{aligned} K_1 &= \left(P_{1F} C_1^T + P_{2F} C_2^T\right) W_2^{-1} \\ K_2 &= \left(\epsilon P_{2F}^T C_1^T + P_{3F} C_2^T\right) W_2^{-1} \end{aligned} \tag{63}$$

with matrices P_{1F}, P_{2F}, and P_{3F} representing the positive semidefinite stabilizing solution matrix of the filter algebraic Riccati equation

$$A P_F + P_F A^T - P_F S P_F + G W_1 G^T = 0 \tag{64}$$

where

$$\begin{aligned} A &= \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon} A_3 & \frac{1}{\epsilon} A_4 \end{bmatrix},\ G = \begin{bmatrix} G_1 \\ \frac{1}{\epsilon} G_2 \end{bmatrix} \\ S &= C^T W_2^{-1} C,\ P_F = \begin{bmatrix} P_{1F} & P_{2F} \\ P_{2F}^T & \frac{1}{\epsilon} P_{3F} \end{bmatrix} \end{aligned} \tag{65}$$

For the decomposition and approximation of the singularly perturbed Kalman filter (62) the Chang transformation have been used in [43]-[44]

$$\begin{bmatrix} \hat{\eta}_1(t) \\ \hat{\eta}_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} - \epsilon H L & -\epsilon H \\ L & I_{n_2} \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \tag{66}$$

where L and H satisfy algebraic equations

$$\begin{aligned} A_4 L - A_3 - \epsilon L(A_1 - A_2 L) &= 0 \\ -H A_4 + A_2 - \epsilon H L A_2 + \epsilon(A_1 - A_2 L) H &= 0 \end{aligned} \tag{67}$$

The Chang transformation, defined by (66) and applied to (62) produces

$$\begin{aligned} \dot{\hat{\eta}}_1(t) &= (A_1 - A_2 L)\hat{\eta}_1(t) + (K_1 - H K_2 - \epsilon H L K_1) v(t) \\ \epsilon \dot{\hat{\eta}}_2(t) &= (A_4 + \epsilon L A_2)\hat{\eta}_2(t) + (K_2 + \epsilon L K_1) v(t) \end{aligned} \tag{68}$$

In the new coordinates the innovation process is given by

$$\begin{aligned} v(t) &= y(t) - (C_1 - C_2 L)\hat{\eta}_1(t) \\ &\quad - [C_2 + \epsilon(C_1 - C_2 L) H]\hat{\eta}_2(t) \end{aligned} \tag{69}$$

Equations (67) are solvable and produce the unique solutions under Assumption 1.4. The algebraic filter Riccati equation (64) has the unique stabilizing solutions under the following assumptions.

Assumption 1.5: The triple (A_4, C_2, G_2) is stabilizable and detectable.

Assumption 1.6: The triple (A_0, C_0, G_0) is stabilizable and detectable, with $A_0 = A_1 - A_2 A_4^{-1} A_3$, $C_0 = C_1 - C_2 A_4^{-1} A_3$, and $G_0 = G_1 - A_2 A_4^{-1} G_2$.

In the decomposition procedure given by (68)-(69) the slow and fast filters (68) require some additional communication channels necessary to form the innovation process (69). In this section, we present a decomposition scheme of [45]–[46] such that the slow and fast filters are completely decoupled and both of them are driven by the system measurements. This method is based on the pure-slow and pure-fast decomposition technique for solving the filter algebraic Riccati equation of singularly perturbed systems—derived by using duality between the optimal filters and regulators and the methodology presented in Section 1.1. In that respect, we give an additional interpretation of the results presented in Section 1.1.

Using (5)-(7), the optimal regulator gain is defined by

$$\begin{aligned} F &= [F_1 \quad F_2] \\ &= \left[R^{-1}\left(B_1^T P_1 + B_2^T P_2^T\right) \quad R^{-1}\left(\epsilon B_1^T P_2 + B_2^T P_3\right)\right] \end{aligned} \tag{70}$$

The results of interest that we need, which can be deduced from Section 1.1, are given in the form of the following lemma.

Lemma 1.3. Consider the optimal closed-loop linear system

$$\begin{aligned} \dot{x}_1(t) &= (A_1 - B_1 F_1)x_1(t) + (A_2 - B_1 F_2)x_2(t) \\ \epsilon \dot{x}_2(t) &= (A_3 - B_2 F_1)x_1(t) + (A_4 - B_2 F_2)x_2(t) \end{aligned} \tag{71}$$

Under Assumptions 1.2 and 1.3 there exists a nonsingular transformation $\mathbf{T}$

$$\begin{bmatrix} \xi_s(t) \\ \xi_f(t) \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{72}$$

such that

$$\begin{aligned} \dot{\xi}_s(t) &= (a_1 + a_2 P_s)\xi_s(t) \\ \epsilon \dot{\xi}_f(t) &= (b_1 + b_2 P_f)\xi_f(t) \end{aligned} \tag{73}$$

where P_s and P_f are the unique solutions of the exact pure-slow and pure-fast completely decoupled algebraic regulator Riccati equations (30)-(31). The nonsingular transformation $\mathbf{T}$ *is given by*

$$\mathbf{T} = (\Pi_1 + \Pi_2 P) \tag{74}$$

Even more, the global solution P can be obtained from the reduced-order exact pure-slow and pure-fast algebraic Riccati equations, that is

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right)\left(\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right)^{-1} \tag{75}$$

Known matrices $\Omega_i,\ i = 1,2,3,4$, and $\Pi_1,\ \Pi_2$ are given in terms of solutions of the Chang decoupling equations, and defined in (20) and (25).

The desired slow-fast decomposition of the Kalman filter (62) will be obtained by producing a dual lemma to Lemma 1.3. Consider the optimal *closed-loop* Kalman filter (62) driven by the system measurements, that is

$$\begin{aligned} \dot{\hat{x}}_1(t) &= (A_1 - K_1 C_1)\hat{x}_1(t) + (A_2 - K_1 C_2)\hat{x}_2(t) \\ &\quad + K_1 y(t) \\ \epsilon \dot{\hat{x}}_2(t) &= (A_3 - K_2 C_1)\hat{x}_1(t) + (A_4 - K_2 C_2)\hat{x}_2(t) \\ &\quad + K_2 y(t) \end{aligned} \tag{76}$$

with the optimal filter gains K_1 and K_2 calculated from (63)-(65). By duality between the optimal filter and regulator, the algebraic filter Riccati equation (64) can be solved by using the same decomposition method for solving the algebraic regulator Riccati equation (7) with

$$\begin{aligned} &A \rightarrow A^T, \ \ Q \rightarrow G W_1 G^T, \ \ F^T = K \\ &Z = B R^{-1} B^T \rightarrow S = C^T W_2^{-1} C \end{aligned} \tag{77}$$

By invoking the results from Section 1.1 and using duality, the following matrices have to be formed

$$\begin{aligned} T_{1F} &= \begin{bmatrix} A_1^T & -C_1^T W_2^{-1} C_1 \\ -G_1 W_1 G_1^T & -A_1 \end{bmatrix} \\ T_{2F} &= \begin{bmatrix} A_3^T & -C_1^T W_2^{-1} C_2 \\ -G_1 W_1 G_2^T & -A_2 \end{bmatrix} \\ T_{3F} &= \begin{bmatrix} A_2^T & -C_2^T W_2^{-1} C_1 \\ -G_2 W_1 G_1^T & -A_3 \end{bmatrix} \\ T_{4F} &= \begin{bmatrix} A_4^T & -C_2^T W_2^{-1} C_2 \\ -G_2 W_1 G_2^T & -A_4 \end{bmatrix} \end{aligned} \tag{78}$$

Note that on the contrary to the results from Section 1.1, where the state-costate variables have to be partitioned and scaled as $x^T(t) = \left[x_1^T(t)\ x_2^T(t)\right]$ and $p^T(t) = \left[p_1^T(t)\ \epsilon p_2^T(t)\right]$, in the case of the dual filter variables, we have to use the following partitions and scaling $x^T(t) = \left[x_1^T(t)\ \epsilon x_2^T(t)\right]$ and $p^T(t) = \left[p_1^T(t)\ p_2^T(t)\right]$. Since matrices $T_{1F},\ T_{2F},\ T_{3F},\ T_{4F}$ correspond to the system matrices of a singularly perturbed linear system, the slow-fast decomposition is achieved by using the Chang decoupling equations

$$\begin{aligned} &T_{4F} M - T_{3F} - \epsilon M (T_{1F} - T_{2F} M) = 0 \\ &-N(T_{4F} + \epsilon M T_{2F}) + T_{2F} + \epsilon (T_{1F} - T_{2F} M) N = 0 \end{aligned} \tag{79}$$

By using the permutation matrices dual to those from Section 1.1 (note E_{1F} is different than the corresponding one from Section 1.1)

$$\begin{aligned} E_{1F} &= \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ 0 & 0 & I_{n1} & 0 \\ 0 & \frac{1}{\epsilon} I_{n2} & 0 & 0 \\ 0 & 0 & 0 & I_{n2} \end{bmatrix} \\ E_{2F} &= \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ 0 & 0 & I_{n1} & 0 \\ 0 & I_{n2} & 0 & 0 \\ 0 & 0 & 0 & I_{n2} \end{bmatrix} \end{aligned} \tag{80}$$

we can define

$$\Pi_F = \begin{bmatrix} \Pi_{1F} & \Pi_{2F} \\ \Pi_{3F} & \Pi_{4F} \end{bmatrix} = E_{2F}^T \begin{bmatrix} I_{2n_1} - \epsilon NM & -\epsilon N \\ M & I_{2n_2} \end{bmatrix} E_{1F} \tag{81}$$

Then, the desired transformation is given by

$$\mathbf{T}_2 = (\Pi_{1F} + \Pi_{2F} P_F) \tag{82}$$

The transformation $\mathbf{T}_2$ applied to the filter variables as

$$\begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} = \mathbf{T}_2^{-\mathrm{T}} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \tag{83}$$

produces

$$\begin{bmatrix} \dot{\hat{\eta}}_s(t) \\ \dot{\hat{\eta}}_f(t) \end{bmatrix} = \mathbf{T}_2^{-\mathrm{T}} \begin{bmatrix} A_1 - K_1 C_1 & A_2 - K_1 C_2 \\ \frac{1}{\epsilon}(A_3 - K_2 C_1) & \frac{1}{\epsilon}(A_4 - K_2 C_2) \end{bmatrix} \times \mathbf{T}_2^{\mathrm{T}} \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} + \mathbf{T}_2^{-\mathrm{T}} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} y(t) \tag{84}$$

such that the complete closed-loop decomposition is achieved, that is

$$\begin{aligned} \dot{\hat{\eta}}_s(t) &= (a_{1F} + a_{2F} P_{sF})^T \hat{\eta}_s(t) + K_s y(t) \\ \epsilon \dot{\hat{\eta}}_f(t) &= (b_{1F} + b_{2F} P_{fF})^T \hat{\eta}_f(t) + K_f y(t) \end{aligned} \tag{85}$$

The matrices in (85) are given by

$$\begin{gathered} \begin{bmatrix} a_{1F} & a_{2F} \\ a_{3F} & a_{4F} \end{bmatrix} = (T_{1F} - T_{2F} M) \\ \begin{bmatrix} b_{1F} & b_{2F} \\ b_{3F} & b_{4F} \end{bmatrix} = (T_{4F} + \epsilon M T_{2F}) \\ \begin{bmatrix} K_s \\ \frac{1}{\epsilon} K_f \end{bmatrix} = \mathbf{T}_2^{-\mathrm{T}} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} \end{gathered} \tag{86}$$

$$\begin{aligned} 0 &= P_{sF} a_{1F} - a_{4F} P_{sF} - a_{3F} + P_{sF} a_{2F} P_{sF} \\ 0 &= P_{fF} b_{1F} - b_{4F} P_{fF} - b_{3F} + P_{fF} b_{2F} P_{fF} \end{aligned} \tag{87}$$

A method for solving nonsymmetric Riccati equations (87) is considered in Section 1.1. Note that the matrices needed for the $O(\epsilon)$ approximate slow filter algebraic Riccati equation dual to (34) and defined by

$$\begin{aligned} &P_{sF}^{(0)} A_{sF}^T + A_{sF} P_{sF}^{(0)} + G_s W_{1s} G_s^T \\ &\quad - P_{sF}^{(0)} C_s^T W_{2s}^{-1} C_s P_{sF}^{(0)} = 0 \end{aligned} \tag{88}$$

can be obtained from [20]

$$\begin{bmatrix} A_{sF}^T & -C_s^T W_{2s}^{-1} C_s \\ -G_s W_{1s} G_s^T & -A_{sF} \end{bmatrix} = T_{1F} - T_{2F} T_{4F}^{-1} T_{3F} \tag{89}$$

Even more, we can obtained the analytical expressions for $A_{sF}, C_s, G_s, W_{1s}, W_{2s}$ using the methodology of [43]. It is important to point out that the matrix P_F in (82) can be obtained in terms of P_{sF} and P_{fF} by using formula (75) with

$$P_s = P_{sF}, \quad P_f = P_{fF} \tag{90}$$

and $\Omega_1, \ \Omega_2, \ \Omega_3, \ \Omega_4$ obtained from

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_{1F}^{-1} \begin{bmatrix} I_{2n_1} & \epsilon N \\ -M & I_{2n_2} - \epsilon MN \end{bmatrix} E_{2F}^{-T} \tag{91}$$

A lemma dual to Lemma 1.3 can be now formulated as follows.

Lemma 1.4: Given the closed-loop optimal Kalman filter (76) of a linear singularly perturbed system. There exists a nonsingular transformation (82), which completely decouples (76) into pure-slow and pure-fast local filters (85) both driven by the system measurements. The decoupling transformation (82) and the filter coefficients given in (86) can be obtained in terms of the exact pure-slow and pure-fast reduced-order completely decoupled algebraic Riccati equations (87).

It can be seen from the previous analysis that the new filtering method allows complete decomposition and parallelism between pure-slow and pure-fast filters.

D. Optimal Linear-Quadratic Gaussian Control

In this section an approach for solving the linear-quadratic optimal Gaussian control problem of singularly perturbed continuous-time stochastic systems is presented. The algorithm proposed is based on the results presented in Sections 1.1 and 1.3. It is shown that the optimal linear-quadratic Gaussian control problem takes the complete decomposition and parallelism between pure-slow and pure-fast filters and controllers.

Singularly perturbed linear-quadratic optimal control problem of stochastic continuous-time systems has been studied in the past by several researchers [41]-[44]. In this section, we present a completely new approach to the stochastic control of linear singularly perturbed systems that is pretty much different than all other methods used so far in the study the same problem by following the results of [45], [47]. The approach is based on a *closed-loop* decomposition technique that guarantees complete decomposition of the optimal filters and regulators and distribution of all required off-line and on-line computations. As a matter of fact, the presented approach combines results presented in Sections 1.1 and 1.3 and uses the separation principle for linear stochastic control [23]. This decomposition allows us to design the linear controllers for slow and fast subsystems completely independently of each other and thus, to achieve the complete and exact separation for the linear-quadratic stochastic regulator problem.

Consider the singularly perturbed linear stochastic system

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) + B_1 u(t) + G_1 w(t) \\ \epsilon \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) + B_2 u(t) + G_2 w(t) \\ y(t) &= C_1 x_1(t) + C_2 x_2(t) + w_2(t) \end{aligned} \tag{92}$$

with the performance criterion

$$J = \lim_{t_f \to \infty} \frac{1}{t_f} E\left\{ \int_{t_0}^{t_f} \left[z^T(t)z(t) + u^T(t)Ru(t) \right] dt \right\} \quad (93)$$

where $x_i(t) \in \Re^{n_i}, i = 1, 2,$ comprise slow and fast state vectors, respectively, $u(t) \in \Re^m$ is the control input, $y(t) \in \Re^{r_2}$ is the observed output, $w_i(t) \in \Re^{r_i}$ are zero-mean stationary, mutually uncorrelated, Gaussian white noise processes with intensities $W_1 > 0$ and $W_2 > 0$, respectively, and $z(t) \in \Re^s$, is the controlled output given by

$$z(t) = D_1 x_1(t) + D_2 x_2(t) \quad (94)$$

All matrices are of appropriate dimensions and assumed to be constant. The optimal control law for (92) with the performance criterion (93) is given by

$$u_{opt}(t) = -F_1 \hat{x}_1(t) - F_2 \hat{x}_2(t) \quad (95)$$

where $\hat{x}_1(t)$ and $\hat{x}_2(t)$ are the optimal estimates of the state vectors $x_1(t)$ and $x_2(t)$ obtained from the Kalman filter

$$\begin{aligned} \dot{\hat{x}}_1(t) &= A_1 \hat{x}_1(t) + A_2 \hat{x}_2(t) + B_1 u(t) + K_1 v(t) \\ \epsilon \dot{\hat{x}}_2(t) &= A_3 \hat{x}_1(t) + A_4 \hat{x}_2(t) + B_2 u(t) + K_2 v(t) \\ v(t) &= y(t) - C_1 \hat{x}_1(t) - C_2 \hat{x}_2(t) \end{aligned} \quad (96)$$

The optimal regulator gains F_1, F_2 and filter gains K_1, K_2 are given, respectively, by (70) and (63). The required positive semidefinite stabilizing solutions of the algebraic regulator and filter Riccati equations (7) and (64) can be obtained in terms of reduced-order, pure-slow and pure-fast, regulator and filter, algebraic Riccati equations, respectively, given by (30)-(31) and (87).

The optimal global Kalman filter (96) can be put in the form in which the filter is driven by the system measurements and optimal control inputs, that is

$$\begin{aligned} \dot{\hat{x}}_1(t) &= (A_1 - K_1 C_1)\hat{x}_1(t) + (A_2 - K_1 C_2)\hat{x}_2(t) \\ &+ B_1 u(t) + K_1 y(t) \\ \epsilon \dot{\hat{x}}_2(t) &= (A_3 - K_2 C_1)\hat{x}_1(t) + (A_4 - K_2 C_2)\hat{x}_2(t) \\ &+ B_2 u(t) + K_2 y(t) \end{aligned} \quad (97)$$

It is known from Section 1.1 that there exists a nonsingular transformation defined by (82) such that (97) is decoupled into pure-slow and pure-fast local filters both driven by system measurements and system control inputs

$$\begin{aligned} \dot{\hat{\eta}}_s(t) &= (a_{1F} + a_{2F} P_{sF})^T \hat{\eta}_s(t) + B_s u(t) + K_s y(t) \\ \epsilon \dot{\hat{\eta}}_f(t) &= (b_{1F} + b_{2F} P_{fF})^T \hat{\eta}_f(t) + B_f u(t) + K_f y(t) \end{aligned} \quad (98)$$

The pure-slow and pure-fast filter gains, K_s, K_f, are defined by (86). The pure-slow and pure-fast system input matrices are given by

$$\begin{bmatrix} B_s \\ \frac{1}{\epsilon} B_f \end{bmatrix} = \mathbf{T}_2^{-T} \begin{bmatrix} B_1 \\ \frac{1}{\epsilon} B_2 \end{bmatrix} \quad (99)$$

As a result, the coefficients of the optimal pure-slow filter are functions of the solution of the pure-slow algebraic Riccati equation only and those of the pure-fast filter are functions of the solution of the pure-fast algebraic Riccati equation only. Thus, these two filters can be implemented independently in the different time scales (slow and fast). It should be noted that the filtering method proposed for singularly perturbed linear stochastic systems allows complete decomposition and parallelism between pure-slow and pure-fast filters.

The optimal control in the new coordinates is given by, [47]

$$\begin{aligned} u_{opt}(t) &= -F\hat{x}(t) = -F\mathbf{T}_2^T \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} \\ &= -\begin{bmatrix} F_s & F_f \end{bmatrix} \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} \end{aligned} \quad (100)$$

where F_s and F_f are obtained from

$$\begin{bmatrix} F_s & F_f \end{bmatrix} = F\mathbf{T}_2^{\mathrm{T}} = R^{-1} B^T P (\Pi_{1F} + \Pi_{F2} P_F)^T \quad (101)$$

The optimal value of J follows from the known formula [23]

$$\begin{aligned} J_{opt} &= \mathrm{tr}\{PKW_2K^T + P_F D^T D\} \\ &= \mathrm{tr}\{PGW_1G^T + P_F F^T RF\} \end{aligned} \quad (102)$$

II. H∞-OPTIMAL CONTROL AND FILTERING

Singularly perturbed H_∞-optimal linear-quadratic control and filtering problems have been studied in the past by several researchers [21]-[22], [48]-[54]. Related problems for singularly perturbed differential games and disturbance attenuation have been considered in [55]-[58].

In this section we study the algebraic Riccati equation of singularly perturbed H_∞-optimal linear-quadratic control problems by generalizing the results of [24] and present an efficient reduced-order algorithm that removes ill-conditioning of the original problem. Another approach to decomposition of the algebraic Riccati equation for the same class of systems, based on a transformation derived in [13], has been considered in [21]-[22]. However, the problem of deriving an algorithm for solving the corresponding algebraic Riccati equation is not addressed in [21]-[22].

It is well known [4]–[5] that the singularly perturbed algebraic Riccati equation is ill-conditioned. In this section, we show how to exactly decouple the algebraic Riccati equation of H_∞-optimal control of singularly perturbed systems in terms of pure-slow and pure-fast, reduced-order, *well-conditioned*, H_∞-algebraic Riccati equations. We also establish conditions that allow such a decomposition, and formulate the corresponding algorithm. Even though, the obtained reduced-order H_∞-algebraic Riccati equations are nonsymmetic, they are efficiently solved in terms of Lyapunov iterations by using the Newton method. The iterative algorithm of [59], also given in terms of Lyapunov iterations, is used to obtain numerical solutions of the corresponding reduced-order, slow and fast, symmetric, H_∞ algebraic Riccati equations, which contain indefinite matrices in quadratic terms, and whose solutions produce excellent initial guesses for the Newton method.

The results presented in this section will facilitate exact and complete time-scale decomposition of H_∞-optimal control and filtering tasks of singularly perturbed systems, and reduced-order parallel processing of all off-line and on-line computational requirements.

A. H_∞–Optimal Linear Control

The linear singularly perturbed control system under disturbances is described by

$$\begin{bmatrix}\dot{x}_1(t)\\ \epsilon\dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix}u(t) + \begin{bmatrix}D_1\\ D_2\end{bmatrix}w(t) \tag{103}$$

where $x_1(t) \in \Re^{n_1}, x_2(t) \in \Re^{n_2}$ are, respectively, system slow and fast state space variables, $u(t) \in \Re^m$ is a control input, $w(t) \in \Re^p$ is a system disturbance, and ϵ is a small positive singular perturbation parameter. The performance criterion to be minimized is given by

$$J = \frac{1}{2}\int_0^\infty \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]dt,\ Q \geq 0,\ R > 0 \tag{104}$$

The H_∞-optimal control problem associated with (103) and (104) has a solution given in terms of solution of the following algebraic Riccati equation [60]-[62]

$$A^TP+PA+Q-P\left(S-\frac{1}{\gamma^2}Z\right)P = 0, P = \begin{bmatrix}P_1 & \epsilon P_2\\ \epsilon P_2^T & \epsilon P_3\end{bmatrix} \tag{105}$$

where

$$A = \begin{bmatrix}A_1 & A_2\\ \frac{1}{\epsilon}A_3 & \frac{1}{\epsilon}A_4\end{bmatrix}, \quad S = \begin{bmatrix}S_1 & \frac{1}{\epsilon}S_2\\ \frac{1}{\epsilon}S_2^T & \frac{1}{\epsilon^2}S_3\end{bmatrix} \geq 0$$

$$Z = \begin{bmatrix}Z_1 & \frac{1}{\epsilon}Z_2\\ \frac{1}{\epsilon}Z_2^T & \frac{1}{\epsilon^2}Z_3\end{bmatrix} \geq 0$$

$$\begin{gathered}S_1 = B_1R^{-1}B_1^T, \quad S_2 = B_1R^{-1}B_2^T, \quad S_3 = B_2R^{-1}B_2^T\\ Z_1 = D_1D_1^T, \quad Z_2 = D_1D_2^T, \quad Z_3 = D_2D_2^T\end{gathered} \tag{106}$$

and γ is a real positive parameter that represents an optimal disturbance attenuation level in the sense

$$\inf_{u(t)}\sup_{w(t)}\left\{\frac{\sqrt{J}}{\|w(t)\|}\right\} = \gamma \tag{107}$$

The optimal controller that guarantees the γ level of optimality is given by

$$u_{opt}(t) = -R^{-1}\begin{bmatrix}B_1\\ \frac{1}{\epsilon}B_2\end{bmatrix}Px(t) \tag{108}$$

The algebraic Riccati equation (105) with an indefinite coefficient matrix in the quadratic term appears also in zero-sum differential games [63], stabilization of uncertain systems [64]-[65], disturbance attenuation problems [66], and decentralized stabilization [67].

B. Singularly Perturbed H_∞–Algebraic Riccati Equation

The Hamiltonian form corresponding to the H_∞ algebraic Riccati equation (105) is used in further analysis. This form is given by

$$\begin{bmatrix}\dot{x}(t)\\ \dot{p}(t)\end{bmatrix} = \begin{bmatrix}A & -\left(S-\frac{1}{\gamma^2}Z\right)\\ -Q & -A^T\end{bmatrix}\begin{bmatrix}x(t)\\ p(t)\end{bmatrix} \tag{109}$$

with

$$p(t) = Px(t) \tag{110}$$

Our goal is to find the solution of (105) in terms of solutions of the reduced-order, pure-slow and pure-fast, H_∞-algebraic Riccati equations by following the methodology of [24] and [68]. In addition, we establish conditions for such a decomposition, and formulate the corresponding algorithm.

By partitioning the costate vector $p(t)$ as $p(t) = [p_1(t) \quad \epsilon p_2(t)]$ with $p_1(t) \in \Re^{n_1}$, $p_2(t) \in \Re^{n_2}$ and interchanging the second and third rows in (109) we get

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{p}_1(t)\\ \dot{x}_2(t)\\ \dot{p}_2(t)\end{bmatrix} = \begin{bmatrix}T_1 & T_2\\ \frac{1}{\epsilon}T_3 & \frac{1}{\epsilon}T_4\end{bmatrix}\begin{bmatrix}x_1(t)\\ p_1(t)\\ x_2(t)\\ p_2(t)\end{bmatrix} \tag{111}$$

with

$$\begin{gathered}T_1 = \begin{bmatrix}A_1 & -\left(S_1-\frac{1}{\gamma^2}Z_1\right)\\ -Q_1 & -A_1^T\end{bmatrix}\\ T_2 = \begin{bmatrix}A_2 & -\left(S_2-\frac{1}{\gamma^2}Z_2\right)\\ -Q_2 & -A_3^T\end{bmatrix}\\ T_3 = \begin{bmatrix}A_3 & -\left(S_2-\frac{1}{\gamma^2}Z_2\right)^T\\ -Q_2^T & -A_2^T\end{bmatrix}\\ T_4 = \begin{bmatrix}A_4 & -\left(S_3-\frac{1}{\gamma^2}Z_3\right)\\ -Q_3 & -A_4^T\end{bmatrix}\end{gathered} \tag{112}$$

It is important to notice that (111) retains the singularly perturbed form. In the following, in order to be able to apply the Chang transformation to (111), we need nonsingularity of the fast subsystem matrix T_4. It is established in [21]-[22] that the matrix T_4 is nonsingular under the following assumption.

Assumption 2.1: The triple $\left(A_4, B_2, \sqrt{Q_3}\right)$ is controllable and observable.

Applying the Chang transformation to (111) we get in the new coordinates two independent pure-slow and pure-fast subsystems

$$\begin{bmatrix}\dot{\eta}_1(t)\\ \dot{\eta}_2(t)\end{bmatrix} = \begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix}\begin{bmatrix}\eta_1(t)\\ \eta_2(t)\end{bmatrix} = (T_1 - T_2L)\begin{bmatrix}\eta_1(t)\\ \eta_2(t)\end{bmatrix} \tag{113}$$

$$\epsilon \begin{bmatrix} \dot{\zeta}_1(t) \\ \dot{\zeta}_2(t) \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} = (T_4 + \epsilon L T_2) \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} \tag{114}$$

where the matrix L is obtained from the Chang transformation equation (11) . Note that one can also apply to (111) the new version of the Chang transformation derived in [25] that produces complete independence between the L and H equations. However, in that case, the H equation is weakly nonlinear. The unique solutions of the corresponding equations (11)-(12) exist for sufficiently small values of ϵ under the assumption that the matrix T_4 is nonsingular (by the Implicit Function Theorem).

The relationship between the new and old state variables is determined by the Chang transformation as

$$\begin{bmatrix} \eta(t) \\ \zeta(t) \end{bmatrix} = \begin{bmatrix} I - \epsilon H L & -\epsilon H \\ L & I \end{bmatrix} \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} = \mathbf{T}_3 \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} \tag{115}$$

The relationship between the original and new coordinates is given by

$$\begin{bmatrix} \eta_1(t) \\ \zeta_1(t) \\ \eta_2(t) \\ \zeta_2(t) \end{bmatrix} = E_2^T \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \\ \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} = E_2^T \mathbf{T}_3 \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ \epsilon p_2(t) \end{bmatrix}$$
$$= E_2^T \mathbf{T}_3 E_1 \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \Pi \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{116}$$

where the permutation matrices E_1 and E_2 are defined in (19) and (21).

Now we can proceed like in Section 1.1, that is, along the lines of (22)-(31). In the new coordinates, the state and costate equations are related by

$$\begin{bmatrix} \eta_2(t) \\ \zeta_2(t) \end{bmatrix} = \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \zeta_1(t) \end{bmatrix} \tag{117}$$

Using (117) in (113)-(114) we get two reduced-order nonsymmetric, pure-slow and pure-fast, H_∞-algebraic Riccati equations, respectively given by

$$\begin{aligned} 0 &= P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s \\ 0 &= P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f \end{aligned} \tag{118}$$

The reduced-order algebraic nonsymmetric Riccati equations can be solved by using the eigenvector method in terms of eigenvectors spanning the stable subspace, [23]. Another approach for solving equations (118), which is more in the spirit of theory of singular perturbations, is given below.

By using the same methodology as in Section 1.1 we get

$$\begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1} \tag{119}$$

Also, we can find P in terms of P_s and P_f as

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \right) \left(\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \right)^{-1} \tag{120}$$

where

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_1^{-1} \mathbf{T}_3^{-1} E_2 = \Pi^{-1} \tag{121}$$

It can be shown that the inversions defined in (119) and (120) exist for sufficiently small values of the singular perturbation parameter ϵ since the corresponding matrices are equal to $I + O(\epsilon)$.

It is interesting to point out that H_∞-algebraic Riccati equations (118) are nonsymmetric, but their $O(\epsilon)$ perturbations are symmetric. Namely, by closely examining the coefficients in (114), the pure-fast H_∞–algebraic Riccati equation is represented by

$$P_f A_4 + A_4^T P_f + Q_3 - P_f \left(S_3 - \frac{1}{\gamma^2} Z_3 \right) P_f + O(\epsilon) = 0 \tag{122}$$

It can be observed from (120) that

$$P = \begin{bmatrix} P_s & 0 \\ 0 & 0 \end{bmatrix} + O(\epsilon) \tag{123}$$

It is known from [49] that the nature of the solution matrix of (105) is

$$P = \begin{bmatrix} P_1 + O(\epsilon) & \epsilon(P_{12} + O(\epsilon)) \\ \epsilon\left(P_{12}^T + O(\epsilon)\right) & \epsilon(P_2 + O(\epsilon)) \end{bmatrix} \tag{124}$$
$$P_1 = P_1^T, \quad P_2 = P_2^T$$

where P_1 satisfies the symmetric slow H_∞-algebraic Riccati equation. It follows from [49] and (123)–(124) that

$$P_s A_s + A_s^T P_s + Q_s - P_s \left(S_s - \frac{1}{\gamma^2} Z_s \right) P_s + O(\epsilon) = 0 \tag{125}$$

From (122) and (125) one can obtain $O(\epsilon)$ approximations for P_s and P_f equations by solving the following H_∞ algebraic Riccati equations

$$P_s^{(0)} A_s + A_s^T P_s^{(0)} + Q_s - P_s^{(0)} \left(S_s - \frac{1}{\gamma^2} Z_s \right) P_s^{(0)} = 0 \tag{126}$$

$$P_f^{(0)} A_4 + A_4^T P_f^{(0)} + Q_3 - P_f^{(0)} \left(S_3 - \frac{1}{\gamma^2} Z_3 \right) P_f^{(0)} = 0 \tag{127}$$

The unique positive semidefinite stabilizing solution of (127) exists under Assumption 2.1, [59]. The unique positive semidefinite stabilizing solution of (126) exists under the following assumption, [59].

Assumption 2.2. The triple $\left(A_s, B_s, \sqrt{Q_s}\right)$ is stabilizable and detectable.

Matrices A_s, B_s, Q_s, can be derived either by using the methodology of [49]-[50] or even in a simpler manner from the results of [20] as

$$\begin{bmatrix} A_s & -\left(S_s - \frac{1}{\gamma^2} Z_s\right) \\ -Q_s & -A_s^T \end{bmatrix} = T_1 - T_2 T_4^{-1} T_3 \tag{128}$$

An important feature of equations (126)-(127), which distinguishes these equations from the standard algebraic Riccati equation of the linear-quadratic optimal control problem is that the quadratic terms in (126)-(127) have indefinite coefficient matrices. The algorithm of [59], given in terms of Lyapunov iterations, converges globally to the positive semidefinite stabilizing solution of (126)-(127) under Assumptions 2.1 and 2.2. It has been demonstrated in [69] that the Lyapunov iterations are very efficient numerical tool for solving many nonlinear algebraic equations arising in optimal control and filtering problems. Using the algorithm of [59], equation (127) is solved by performing the following Lyapunov iterations

$$\begin{aligned} &P_f^{(0)^{(i+1)}}\left(A_4 - S_3 P_f^{(0)^{(i)}}\right) \\ &+\left(A_4 - S_3 P_f^{(0)^{(i)}}\right)^T P_f^{(0)^{(i+1)}} \\ &= -\left(Q_3 + P_f^{(0)^{(i)}} S_3 P_f^{(0)^{(i)}} + \frac{1}{\gamma^2} P_f^{(0)^{(i)}} Z_3 P_f^{(0)^{(i)}}\right) \end{aligned} \tag{129}$$

with the initial condition obtained from the standard algebraic Riccati equation

$$P_f^{(0)^{(0)}} A_4 + A_4^T P_f^{(0)^{(0)}} + Q_3 - P_f^{(0)^{(0)}} S_3 P_f^{(0)^{(0)}} = 0 \tag{130}$$

This choice of the initial condition is an interesting feature of the algorithm of [59], and it is important for the efficiency of the overall algorithm for solving the singularly perturbed H_∞-algebraic Riccati equation. Having obtained an approximate solution $P_f^{(0)} = P_f + O(\epsilon)$, we can implement the Newton method for solving the pure-fast algebraic Riccati equation given in (118) since a good initial guess is available. The Newton method leads to the following Lyapunov-like (Sylvester) iterations

$$\begin{aligned} &P_f^{(i+1)}\left(b_1 + b_2 P_f^{(i)}\right) - \left(b_4 - P_f^{(0)} b_2\right) P_f^{(i+1)} \\ &= b_3 + P_f^{(i)} b_2 P_f^{(i)} \end{aligned} \tag{131}$$

and converges in only few iterations.

Similarly, the algorithm of [59] is applied for solving (126) as

$$\begin{aligned} &P_s^{(0)^{(i+1)}}\left(A_s - S_s P_s^{(0)^{(i)}}\right) \\ &+\left(A_s - S_s P_s^{(0)^{(i)}}\right)^T P_s^{(0)^{(i+1)}} \\ &= -\left(Q_s + P_s^{(0)^{(i)}} S_s P_s^{(0)^{(i)}} + \frac{1}{\gamma^2} P_s^{(0)^{(i)}} Z_s P_s^{(0)^{(i)}}\right) \end{aligned} \tag{132}$$

with the initial condition obtained from the standard slow algebraic Riccati equation

$$P_s^{(0)^{(0)}} A_s + A_s^T P_s^{(0)^{(0)}} + Q_s - P_s^{(0)^{(0)}} S_s P_s^{(0)^{(0)}} = 0 \tag{133}$$

Having obtained an approximate solution $P_s^{(0)} = P_s + O(\epsilon)$, we can implement the Newton method for solving the corresponding pure-slow algebraic Riccati equation defined in (118) since a good initial guess is available. The Newton methods leads to the following Lyapunov-like (Sylvester) iterations

$$\begin{aligned} &P_s^{(i+1)}\left(a_1 + a_2 P_s^{(i)}\right) - \left(a_4 - P_s^{(0)} a_2\right) P_s^{(i+1)} \\ &= a_3 + P_s^{(i)} a_2 P_s^{(i)} \end{aligned} \tag{134}$$

which converge quadratically to the required solution.

C. Singularly Perturbed H_∞–Optimal Linear Filtering

The Kalman filter has been used since 1961 in all areas of control system engineering [70]. It has been also used in signal processing (see for example, [71]–[72] and references therein), image processing, communications, and economics. In this section we present a method that allows complete time-scale separation and parallelism of the H_∞–optimal Kalman filtering problem for linear systems with slow and fast modes (singularly perturbed linear systems). The algebraic Riccati equation of singularly perturbed H_∞–Kalman filtering problem is decoupled into two completely independent, reduced-order, pure-slow and pure-fast, H_∞–algebraic Riccati equations by using the methodology from the previous section. The corresponding H_∞–Kalman filter is decoupled into independent reduced-order, well-defined, pure-slow and pure-fast, Kalman filters driven by system measurements. The proposed exact closed-loop decomposition technique produces a lot of savings in both on-line and off-line computations and allows parallel processing of information with different sampling rates for slow and fast signals.

During the last fifteen years the H_∞–optimization became one of the most interesting and challenging areas of optimal control and filtering theories and their applications. The main advantage of the H_∞–optimization is that such obtained controllers and filters are robust with respect to internal and external disturbances. In the case of Kalman filtering, the additional advantage of the H_∞–Kalman filter over the standard Kalman filter is that the former one does not require knowledge of the system and measurement noise intensity matrices—data hardly exactly known.

It is known that the singularly perturbed Kalman filter is numerically ill-conditioned due to coupling of the slow and fast modes (signals). Hence, the main goal in theory of singular perturbations is to decouple (separate) the slow and fast signals and process them independently. Difficulties encountered with the full-order H_∞–Kalman filter of singularly perturbed linear systems are in the facts that the corresponding algebraic filter Riccati equation is also ill-conditioned and that it contains an indefinite coefficient matrix multiplying the quadratic term (which makes this equation much more difficult for studying than the corresponding one of standard singularly perturbed optimal filtering problems).

In the previous section the algebraic regulator Riccati equation of H_∞ optimal linear-quadratic regulator problem is decomposed into reduced-order, pure-slow and pure-fast, algebraic regulator Riccati equations. In this section, we extend those results to the decomposition of the corresponding algebraic filter Riccati equation and use them to decompose the H_∞ singularly perturbed Kalman filter into independent, well-defined, reduced-order, Kalman filters. The filters obtained are completely independent and can work in parallel. Each of them can process information with different sampling rate—the fast filter requires small sampling period and the slow one can process information with relatively large sampling period.

Consider the linear singularly perturbed system

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) + D_1 w(t) \\ \epsilon \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) + D_2 w(t) \end{aligned} \tag{135}$$

with the corresponding measurements

$$y(t) = C_1 x_1(t) + C_2 x_2(t) + v(t) \tag{136}$$

where $x_1(t) \in \Re^{n_1}$ and $x_2(t) \in \Re^{n_2}$ are slow and fast state variables, respectively, $w(t) \in \Re^{r_1}$ and $v(t) \in \Re^{r_2}$ are system and measurement disturbances, and $y(t) \in \Re^{r_2}$ are system measurements. $A_i, D_j, C_j,\ i = 1,2,3,4,\ j = 1,2,$ are constant matrices of appropriate dimensions. ϵ is a small positive singular perturbation parameter.

In this section we design a filter to estimate system states $x_1(t)$ and $x_2(t)$. The states to be estimated are given by a linear combination

$$z(t) = G_1 x_1(t) + G_2 x_2(t) \tag{137}$$

The estimation problem is to obtain an estimate $\hat{z}(t)$ of $z(t) \in \Re^q$ using the measurements $y(t)$, [72]-[74]. The measure of the infinite horizon estimation problem is defined as a disturbance attenuation function

$$J = \frac{\int_0^\infty \|z(t) - \hat{z}(t)\|_R^2 dt}{\int_0^\infty \left(\|w(t)\|_{W^{-1}}^2 + \|v(t)\|_{V^{-1}}^2 \right) dt} \tag{138}$$

where $R \geq 0, W > 0$ and $V > 0$ are the weighting matrices to be chosen by designers. The H_∞–filter is to ensure that the energy gain from the disturbances to the estimation errors, $z(t) - \hat{z}(t)$, is less than a prespecified level γ^2. That is,

$$\sup_{w,v} J < \gamma^2 \tag{139}$$

where "sup" stands for supremum and γ^2 is a prescribed level of noise attenuation. The H_∞–filter associated with singularly perturbed linear systems, driven by the innovation process, is given by [46], [72]

$$\begin{aligned} \dot{\hat{x}}_1(t) &= A_1 \hat{x}_1(t) + A_2 \hat{x}_2(t) + K_1 \nu(t) \\ \epsilon \dot{\hat{x}}_2(t) &= A_3 \hat{x}_1(t) + A_4 \hat{x}_2(t) + K_2 \nu(t) \\ \nu(t) &= y(t) - C_1 \hat{x}_1(t) - C_2 \hat{x}_2(t) \end{aligned} \tag{140}$$

where the filter gains K_1 and K_2 are obtained from

$$\begin{aligned} K_1 &= \left(P_{1F} C_1^T + P_{2F} C_2^T \right) V^{-1} \\ K_2 &= \left(\epsilon P_{2F}^T C_1^T + P_{3F} C_2^T \right) V^{-1} \end{aligned} \tag{141}$$

with matrices P_{1F}, P_{2F}, and P_F representing the positive semidefinite stabilizing solution of the following algebraic Riccati equation [72]-[73]

$$\begin{aligned} AP_F + P_F A^T - P_F \left(C^T V^{-1} C - \frac{1}{\gamma^2} G^T R G \right) P_F \\ + DWD^T = 0 \end{aligned} \tag{142}$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon} A_3 & \frac{1}{\epsilon} A_4 \end{bmatrix}, D = \begin{bmatrix} D_1 \\ \frac{1}{\epsilon} D_2 \end{bmatrix}, P_F = \begin{bmatrix} P_{1F} & P_{2F} \\ P_{2F}^T & \frac{1}{\epsilon} P_{3F} \end{bmatrix}$$
$$C = [C_1 \quad C_2], \quad G = [G_1 \quad G_2] \tag{143}$$

In order to form the innovation process defined in (140), communications of the filter estimates are required, thus additional communication channels are necessary. In the following, we will achieve the slow-fast H_∞–filter decomposition in which both filters are directly driven by the system measurements and thus, we will eliminate the need for communication of estimates. The problem of solving the H_∞ singularly perturbed algebraic filter Riccati equation (142) will be solved by using duality between the optimal filters and regulators and the algorithm from the previous section.

The desired decomposition of the H_∞ filter (140) will be obtained by first producing dual results to (118), (120). Consider the optimal closed-loop filter (140) driven by the system measurements

$$\begin{aligned} \dot{\hat{x}}_1(t) &= (A_1 - K_1 C_1) \hat{x}_1(t) + (A_2 - K_1 C_2) \hat{x}_2(t) \\ &\quad + K_1 y(t) \\ \epsilon \dot{\hat{x}}_2(t) &= (A_3 - K_2 C_1) \hat{x}_1(t) + (A_4 - K_2 C_2) \hat{x}_2(t) \\ &\quad + K_2 y(t) \end{aligned} \tag{144}$$

with the optimal filter gains K_1 and K_2 calculated from (141)-(143). By duality between the optimal filter and regulator, the filter algebraic Riccati equation (142) can be solved by using the same decomposition method as the one used for solving (105) with

$$A \to A^T, \quad Q \to DWD^T, \quad F^T \to K = \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix}$$
$$S = BR_1^{-1} B^T \to C^T V^{-1} C, \quad DD^T \to G^T R G \tag{145}$$

By invoking results from Sections 1.3 and 2.2 and using duality between the optimal linear-quadratic controllers and optimal Kalman filters, the following matrices have to be formed

$$T_1 = \begin{bmatrix} A_1^T & -\left(C_1^T V^{-1} C_1 - \frac{1}{\gamma^2} G_1^T R G_1\right) \\ -D_1 W D_1^T & -A_1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} A_3^T & -\left(C_1^T V^{-1} C_2 - \frac{1}{\gamma^2} G_1^T R G_2\right) \\ -D_1 W D_2^T & -A_2 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} A_2^T & -\left(C_2^T V^{-1} C_1 - \frac{1}{\gamma^2} G_2^T R G_1\right) \\ -D_2 W D_1^T & -A_3 \end{bmatrix}$$

$$T_4 = \begin{bmatrix} A_4^T & -\left(C_2^T V^{-1} C_2 - \frac{1}{\gamma^2} G_2^T R G_2\right) \\ -D_2 W D_2^T & -A_4 \end{bmatrix} \tag{146}$$

It can be shown after some algebra that matrices (T_1, T_2, T_3, T_4) comprise the system matrix of a standard singularly perturbed system, namely

$$\begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ \frac{1}{\epsilon} T_3 & \frac{1}{\epsilon} T_4 \end{bmatrix} \begin{bmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} \tag{147}$$

Note that in contrast to the results of Section 2.2, where the state-costate variables have to be partitioned as $x^T = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}$ and $p^T = \begin{bmatrix} p_1^T & \epsilon p_2^T \end{bmatrix}$, in the case of the dual filter variables, we have to use the following partitions $x^T = \begin{bmatrix} x_1^T & \epsilon x_2^T \end{bmatrix}$ and $p^T = \begin{bmatrix} p_1^T & p_2^T \end{bmatrix}$. Since matrices $T_1,\ T_2,\ T_3$, and T_4 correspond to the system matrices of a singularly perturbed linear system, the slow-fast decomposition of (147) can be achieved by using the Chang decoupling transformation obtained from algebraic equations dual to (11)-(12). The unique solutions of these equations exist, by the implicit function theorem, for ϵ sufficiently small, under the assumption that the matrix T_4 is nonsingular. Using the results of [21]-[22] and duality between optimal linear-quadratic regulators and Kalman filters given in (145), it follows that this matrix is nonsingular under the following assumption.

Assumption 2.3: The triple (A_4, C_2, D_2) is controllable and observable.

The required Chang decoupling transformation is given by

$$\mathbf{T}_4 = \begin{bmatrix} I - \epsilon N M & -\epsilon N \\ M & I \end{bmatrix} \tag{148}$$

where N and M satisfy algebraic equations (11)-(12). Then, by duality, from Section 2.2, we have

$$P_F = \left(\Omega_{3F} + \Omega_{4F} \begin{bmatrix} P_{sF} & 0 \\ 0 & P_{fF} \end{bmatrix}\right) \times \left(\Omega_{1F} + \Omega_{2F} \begin{bmatrix} P_{sF} & 0 \\ 0 & P_{fF} \end{bmatrix}\right)^{-1} \tag{149}$$

where, the pure-slow and pure-fast, well-conditioned, reduced-order, algebraic H_∞ filter Riccati equations are given by

$$\begin{aligned} P_{sF} a_1 - a_4 P_{sF} - a_3 + P_{sF} a_2 P_{sF} = 0 \\ P_{fF} b_1 - b_4 P_{fF} - b_3 + P_{fF} b_2 P_{fF} = 0 \end{aligned} \tag{150}$$

with

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = (T_1 - T_2 M), \quad \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = (T_4 + \epsilon M T_2) \tag{151}$$

The $\Omega_i, i = 1,2,3,4,$ matrices in (149) are defined by

$$\Omega_F = \begin{bmatrix} \Omega_{1F} & \Omega_{2F} \\ \Omega_{3F} & \Omega_{4F} \end{bmatrix} = E_1^{-1} \mathbf{T}_4^{-1} E_2 = E_1^{-1} \begin{bmatrix} I & \epsilon N \\ -M & I - \epsilon M N \end{bmatrix} E_2 \tag{152}$$

The permutation matrices E_1 and E_2 are defined in (80).

It can be shown that one can obtain $O(\epsilon)$ approximations for P_{sF} and P_{fF} by solving the following H_∞ symmetric algebraic filter Riccati equations

$$\begin{aligned} &P_{sF}^{(0)} A_s^T + A_s P_{sF}^{(0)} + D_s W_s D_s^T \\ &- P_{sF}^{(0)} \left(C_s^T V_s^{-1} C_s - \frac{1}{\gamma^2} G_s^T R_s G_s\right) P_{sF}^{(0)} = 0 \end{aligned} \tag{153}$$

$$\begin{aligned} &P_{fF}^{(0)} A_4^T + A_4 P_{fF}^{(0)} + D_2 W D_2^T \\ &- P_{fF}^{(0)} \left(C_2^T V_s^{-1} C_2 - \frac{1}{\gamma^2} G_2^T R G_2\right) P_{fF}^{(0)} = 0 \end{aligned} \tag{154}$$

The newly defined matrices appearing in (153) are obtained from

$$\begin{bmatrix} A_s^T & -\left(C_s^T V_s^{-1} C_s - \frac{1}{\gamma^2} G_s^T R_s G_s\right) \\ -D_s W_s D_s^T & -A_s \end{bmatrix} = T_1 - T_2 T_4^{-1} T_3 \tag{155}$$

An important feature of equations (153)–(154), which distinguishes these equations from the standard algebraic filter Riccati equation, is that the quadratic terms have indefinite coefficient matrices.

The algorithm of [59] developed for solving the H_∞–algebraic Riccati equations in terms of Lyapunov iterations, converges globally to the positive semidefinite stabilizing solution of (153)-(154) under the following stabilizability-detectability assumptions.

Assumption 2.4: $\left(A_s, \sqrt{C_s^T V_s^{-1} C_s}, \sqrt{D_s W_s D_s^T}\right)$ is stabilizable and detectable.

Assumption 2.5: The triple (A_4, C_2, D_2) is stabilizable and detectable.

Note that Assumption 2.5 is weaker than Assumption 2.3, hence, it is sufficient to use in this section only Assumptions 2.3 and 2.4. Also, Assumption 2.4 can be written in a simpler form requiring that the triple (A_s, C_s, D_s)

is stabilizable-detectable. However, in that case, one has to find C_s and D_s matrices explicitly. This can be done by using the procedure of [49]-[50] for forming the reduced-order slow approximate system.

Using the algorithm of [59], equation (154) is solved by performing the following Lyapunov iterations

$$\begin{aligned} & P_{fF}^{(0)^{(i+1)}}\left(A_4 - C_2^T V^{-1} C_2 P_{fF}^{(0)^{(i)}}\right)^T \\ & +\left(A_4 - C_2^T V^{-1} C_2 P_{fF}^{(0)^{(i)}}\right) P_{fF}^{(0)^{(i+1)}} \\ = & -\left(D_2 W D_2^T + P_{fF}^{(0)^{(i)}} C_2^T V^{-1} C_2 P_{fF}^{(0)^{(i)}}\right) \\ & -\frac{1}{\gamma^2} P_{fF}^{(0)^{(i)}} G_2^T R G_2 P_{fF}^{(0)^{(i)}} \end{aligned} \tag{156}$$

with the initial condition obtained from the standard algebraic filter Riccati equation

$$\begin{aligned} & P_{fF}^{(0)^{(0)}} A_4^T + A_4 P_{fF}^{(0)^{(0)}} + D_2 W D_2^T \\ & -P_{fF}^{(0)^{(0)}} C_2^T V^{-1} C_2 P_{fF}^{(0)^{(0)}} = 0 \end{aligned} \tag{157}$$

This choice of the initial condition is an interesting feature of the algorithm of [59], and it is important for the efficiency of the overall algorithm for solving the singularly perturbed H_∞–algebraic Riccati equation. Having obtained an approximate solution $P_{fF}^{(0)} = P_{fF} + O(\epsilon)$, we can implement the Newton method for solving the pure-fast algebraic Riccati equation given in (150) since a good initial guess is available. The Newton method leads to the following Lyapunov-like (Sylvester) iterations

$$\begin{aligned} & P_{fF}^{(i+1)}\left(b_1 + b_2 P_{fF}^{(i)}\right) - \left(b_4 - P_{fF}^{(i)} b_2\right) P_{fF}^{(i+1)} \\ & = b_3 + P_{fF}^{(i)} b_2 P_{fF}^{(i)} \end{aligned} \tag{158}$$

with $P_{fF}^{(0)}$ obtain in (156), and converges in only few iterations.

Similarly, the algorithm of [59] is applied for solving (153) as

$$\begin{aligned} & P_{sF}^{(0)^{(i+1)}}\left(A_s - C_s^T V_s^{-1} C_s P_{sF}^{(0)^{(i)}}\right)^T \\ & +\left(A_s - C_s^T V_s^{-1} C_s P_{sF}^{(0)^{(i)}}\right) P_{sF}^{(0)^{(i+1)}} \\ = & -\left(D_s W_s D_s^T + P_{sF}^{(0)^{(i)}} C_s^T V_s^{-1} C_s P_{sF}^{(0)^{(i)}}\right) \\ & -\frac{1}{\gamma^2} P_{sF}^{(0)^{(i)}} G_s^T R_s G_s P_{sF}^{(0)^{(i)}} \end{aligned} \tag{159}$$

with the initial condition obtained from the slow algebraic Riccati equation

$$\begin{aligned} & P_{sF}^{(0)^{(0)}} A_s^T + A_s P_{sF}^{(0)^{(0)}} + D_s W_s D_s^T \\ & -P_{sF}^{(0)^{(0)}} C_s^T V_s^{-1} C_s P_{sF}^{(0)^{(0)}} = 0 \end{aligned} \tag{160}$$

Having obtained an approximate solution $P_{sF}^{(0)} = P_{sF} + O(\epsilon)$ we can implement the Newton method for solving the corresponding slow Riccati equation defined in (150) since a good initial guess is available. The Newton methods leads to the following Lyapunov-like (Sylvester) iterations

$$\begin{aligned} & P_{sF}^{(i+1)}\left(a_1 + a_2 P_{sF}^{(i)}\right) - \left(a_4 - P_{sF}^{(i)} a_2\right) P_{sF}^{(i+1)} \\ & = a_3 + P_{sF}^{(i)} a_2 P_{sF}^{(i)} \end{aligned} \tag{161}$$

with $P_{sF}^{(0)}$ obtained from (159). The iterative scheme (161) converges quadratically to the required solution.

C1. Slow-Fast Decoupling of H_∞–Optimal Linear Filter

It is interesting to point out that for the standard (classical) Kalman filtering, the transformation that relates the old and new coordinates defined by

$$\Gamma = (\Pi_{1F} + \Pi_{2F} P_F) \tag{162}$$

where

$$\Pi_F = \begin{bmatrix} \Pi_{1F} & \Pi_{2F} \\ \Pi_{3F} & \Pi_{4F} \end{bmatrix} = E_2^T \begin{bmatrix} I - \epsilon N M & -\epsilon N \\ M & I \end{bmatrix} E_1 \tag{163}$$

is used to decouple both the algebraic filter Riccati equation and the Kalman filter into independent pure-slow and pure-fast components [46]. However, in the case of the H_∞–Kalman filtering the similarity transformation

$$\begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} = \Gamma^{-1} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \tag{164}$$

does not produce in the new coordinates the optimal pure-slow and optimal pure-fast Kalman filters, that is

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\eta}}_s(t) \\ \dot{\hat{\eta}}_f(t) \end{bmatrix} = & \Gamma^{-1} \begin{bmatrix} A_1 - K_1 C_1 & A_2 - K_1 C_2 \\ \frac{1}{\epsilon}(A_3 - K_2 C_1) & \frac{1}{\epsilon}(A_4 - K_2 C_2) \end{bmatrix} \\ & \times \Gamma \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} + \Gamma^{-1} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} y(t) \end{aligned} \tag{165}$$

does not lead to a block diagonal filter matrix in the new coordinates. The reason for this is inconsistency lies in the fact that the "closed-loop H_∞–filtering problem matrix" is

$$\begin{aligned} & A - P_F\left(C^T V^{-1} C - \frac{1}{\gamma^2} G^T R G\right) \\ & = A - KC - \frac{1}{\gamma^2} P_F G^T R G \end{aligned} \tag{166}$$

This matrix is indeed block diagonalized by the similarity transformation Γ. However, the H_∞ optimal Kalman filter defined in (144) has the feedback matrix given by

$$\begin{aligned} & A - P_F C^T V^{-1} C = A - KC \\ & = \begin{bmatrix} A_1 - K_1 C_1 & A_2 - K_1 C_2 \\ \frac{1}{\epsilon}(A_3 - K_2 C_1) & \frac{1}{\epsilon}(A_4 - K_2 C_2) \end{bmatrix} \end{aligned} \tag{167}$$

This singularly perturbed matrix can be diagonalized by using another Chang transformation of the form

$$\mathbf{T}_{\mathrm{F}} = \begin{bmatrix} I - \epsilon HL & -\epsilon H \\ L & I \end{bmatrix}, \quad \mathbf{T}_{\mathrm{F}}^{-1} = \begin{bmatrix} I & \epsilon H \\ -L & I - \epsilon LH \end{bmatrix} \tag{168}$$

where L and H matrices satisfy the Chang decoupling equations

$$\begin{gathered} (A_4 - K_2C_2)L - (A_3 - K_2C_1) \\ -\epsilon[(A_1 - K_1C_1) - (A_2 - K_1C_2)L] = 0 \\ -H(A_4 - K_2C_2) + (A_2 - K_1C_2) \\ -\epsilon HL(A_2 - K_1C_2) \\ +\epsilon[(A_1 - K_1C_1) - (A_2 - K_1C_2)L]H = 0 \end{gathered} \tag{169}$$

The unique solutions of these equations exist under the assumption that the matrix $A_4 - K_2C_2$ is nonsingular. Note that based on theory of singular perturbations [4] the matrix $A_4 - P_{3F}C_2^TV^{-1}C_2 - \frac{1}{\gamma^2}P_{3F}G_2^TRG_2$ is nonsingular since it represents the fast feedback matrix. By the result from [74], the stability of the matrix $A_4 - P_{3F}C_2^TV^{-1}C_2 - \frac{1}{\gamma^2}P_{3F}G_2^TRG_2$ implies that the matrix $A_4 - P_{3F}C_2^TV^{-1}C_2$ is stable also. Using (141) we see that $A_4 - K_2C_2 + O(\epsilon)$ is a stable matrix. Thus, the matrix $A_4 - K_2C_2$ is stable for sufficiently small values of the small singular perturbation parameter ϵ. The unique solutions of equations (169) can be easily obtained either by using the Newton method or the fixed point iterations starting with the following initial conditions

$$\begin{aligned} L^{(0)} &= (A_4 - K_2C_2)^{-1}(A_3 - K_2C_1) \\ M^{(0)} &= (A_2 - K_1C_2)(A_4 - K_2C_2)^{-1} \end{aligned} \tag{170}$$

Hence, the optimal Kalman filter obtained by applying the following similarity transformation

$$\begin{bmatrix} \hat{\zeta}_s(t) \\ \hat{\zeta}_f(t) \end{bmatrix} = \mathbf{T}_{\mathrm{F}}^{-1} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \tag{171}$$

produces in the new coordinates the optimal pure-slow and optimal pure-fast, reduced-order, H_∞ Kalman filters, that is

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\zeta}}_s(t) \\ \dot{\hat{\zeta}}_f(t) \end{bmatrix} &= \mathbf{T}_{\mathrm{F}}^{-1} \begin{bmatrix} A_1 - K_1C_1 & A_2 - K_1C_2 \\ \frac{1}{\epsilon}(A_3 - K_2C_1) & \frac{1}{\epsilon}(A_4 - K_2C_2) \end{bmatrix} \\ &\quad \times \mathbf{T}_{\mathrm{F}} \begin{bmatrix} \hat{\zeta}_s(t) \\ \hat{\zeta}_f(t) \end{bmatrix} + \mathbf{T}_{\mathrm{F}}^{-1} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon}K_2 \end{bmatrix} y(t) \\ &= \begin{bmatrix} a_s & 0 \\ 0 & \frac{1}{\epsilon}a_f \end{bmatrix} \begin{bmatrix} \hat{\zeta}_s(t) \\ \hat{\zeta}_f(t) \end{bmatrix} + \begin{bmatrix} K_s \\ \frac{1}{\epsilon}K_f \end{bmatrix} y(t) \end{aligned} \tag{172}$$

where the pure-slow and pure-fast Kalman filter gains are agiven by

$$\begin{bmatrix} K_s \\ \frac{1}{\epsilon}K_f \end{bmatrix} = \mathbf{T}_{\mathrm{F}}^{-1} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon}K_2 \end{bmatrix} \tag{173}$$

Using the expression for the similarity transformation defined in (168) we can obtain analytical expressions for a_s, a_f, K_s, K_f as follows

$$\begin{aligned} a_s &= (A_1 - K_1C_1) - (A_2 - K_1C_2)L \\ a_f &= (A_4 - K_2C_2) + \epsilon L(A_2 - K_1C_2) \\ K_s &= K_1 - HK_2 - \epsilon HLK_1 \\ K_f &= K_2 + \epsilon LK_1 \end{aligned} \tag{174}$$

The reduced-order, independent, pure-slow and pure-fast, filtering equations (172) represent the main result of this section. Due to complete independence of the slow and fast filters, the slow and fast signals can be now processed with different sampling rates. In contrast, the original, full-order, filter (144) requires the fast sampling rate for processing of both the slow and fast signals.

III. COMMENTS

In Sections 1 and 2 we have presented solutions by the Hamiltonian approach to continuous-time optimal control and filtering steady state problems of linear singularly perturbed systems. The discrete-time linear-quadratic Gaussian control problem of singularly perturbed systems is solved via the Hamiltonian approach in [45], [75]. The open-loop discrete-time linear-quadratic optimal control problem is solved in [76]-[77]. The results for the H_∞ optimal control and filtering for singularly perturbed system in discrete-time by the Hamiltonian approach are not obtained yet. This might be an interesting and challenging research topic. In addition, the finite time feedback solution to the linear quadratic optimal control by using the presented methodology is an interesting subject for future research.

The open-loop high gain (cheap control) problem in continuous-time and the problem of complete and exact decomposition of the corresponding high gain (cheap control) algebraic Riccati equation is presented in [6]. The small measurement noise Kalman filtering problem via the Hamiltonian approach is solved in [78]. In a recent paper [30], the eigenvector method is introduced for simultaneous pure-fast and pure-slow block diagonalization of the Hamiltonian matrix and the solution of Chang's algebraic equations required for such a decomposition. The most fundamental results for the so-called multimodeling control problem [5] have been recently obtained in [79]. The solution to the cheap control problem of sampled data systems has been obtained in [80]. Several other research problem remain open in the context of the Hamiltonian approach to singularly perturbed linear optimal control and filtering problems.

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