

Linear-Quadratic Optimal Controller

10.3 Optimal Linear Control Systems

In Chapters 8 and 9 of this book we have designed dynamic controllers such that the closed-loop systems display the desired transient response and steady state characteristics. The design techniques presented in those chapters have sometimes been limited to trial and error methods while searching for controllers that meet the best given specifications. Furthermore, we have seen that in some cases it has been impossible to satisfy all the desired specifications, due to contradictory requirements, and to find the corresponding controller.

Controller design can also be done through rigorous mathematical optimization techniques. One of these, which originated in the sixties (Kalman, 1960)—called modern optimal control theory in this book—is a time domain technique. During the sixties and seventies, the main contributor to modern optimal control theory was Michael Athans, a professor at the Massachusetts Institute of Technology (Athans and Falb, 1966). Another optimal control theory, known as H_∞ , is a trend of the eighties and nineties. H_∞ optimal control theory started with the work of Zames (1981). It combines both the time and frequency domain optimization techniques to give a unified answer, which is optimal from both the time domain and frequency domain points of view (Francis, 1987). Similarly to H_∞ optimal control theory, the so-called H_2 optimal control theory optimizes systems in both time and frequency domains (Doyle *et al.*, 1989, 1992; Saberi *et al.*, 1995) and is the trend of the nineties. Since H_∞ optimal control theory is mathematically quite involved, in this section we will present results only for the modern optimal linear control theory due to Kalman. It is worth mentioning that

very recently a new philosophy has been introduced for system optimization based on linear matrix inequalities (Boyd *et al.*, 1994).

In the context of the modern optimal linear control theory, we present results for the deterministic optimal linear regulator problem, the optimal Kalman filter, and the optimal stochastic linear regulator. Only the main results are given without derivations. This is done for both continuous- and discrete-time domains with emphasis on the infinite time optimization (steady state) and continuous-time problems. In some places, we also present the corresponding finite time optimization results. In addition, several examples are provided to show how to use MATLAB to solve the corresponding optimal linear control theory problems.

10.3.1 Optimal Deterministic Regulator Problem

In modern optimal control theory of linear deterministic dynamic systems, represented in continuous-time by

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

we use linear state feedback, that is

$$\mathbf{u}(t) = -\mathbf{F}(t)\mathbf{x}(t)$$

and optimize the value for the feedback gain, $\mathbf{F}(t)$, such that the following performance criterion is minimized

$$J = \min_{\mathbf{u}(t)} \left\{ \frac{1}{2} \int_{t_0}^{t_f} \left[\mathbf{x}^T(t) \mathbf{R}_1 \mathbf{x}(t) + \mathbf{u}^T(t) \mathbf{R}_2 \mathbf{u}(t) \right] dt \right\}$$

$$\mathbf{R}_1 \geq 0, \quad \mathbf{R}_2 > 0$$

This choice for the performance criterion is quite logical. It requires minimization of the “square” of input, which means, in general, minimization of the input energy required to control a given system, and minimization of the “square” of the state variables. Since the state variables—in the case when a linear system is obtained through linearization of a nonlinear system—represent deviations from the nominal system trajectories, control engineers are interested in minimizing the “square” of this difference, i.e. the “square” of $\mathbf{x}(t)$. In the case when the linear mathematical model (10.27) represents the “pure” linear system, the minimization of (10.29) can be interpreted as the goal of bringing the system as close as possible to the origin ($\mathbf{x}(t) = \mathbf{0}$) while optimizing the energy. This regulation to zero can easily be modified (by shifting the origin) to regulate state variables to any constant values.

It is shown in Kalman (1960) that the linear feedback law (10.28) produces the global minimum of the performance criterion (10.29). The solution to this optimization problem, obtained by using one of two mathematical techniques for dynamic optimization—dynamic programming (Bellman, 1957) and calculus of variations—is given in terms of the solution of the famous Riccati equation (Bittanti *et al.*, 1991; Lancaster and Rodman, 1995). It can be shown (Kalman, 1960; Kirk, 1970; Sage and White, 1977) that the required optimal solution for the feedback gain is given by

$$\mathbf{F}_{opt}(t) = -\mathbf{R}_2^{-1}\mathbf{B}^T(t)\mathbf{P}(t)$$

where $\mathbf{P}(t)$ is the positive semidefinite solution of the matrix differential Riccati equation

$$\begin{aligned} -\dot{\mathbf{P}}(t) &= \mathbf{A}^T(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{R}_1 - \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}_2^{-1}\mathbf{B}^T(t)\mathbf{P}(t) \\ \mathbf{P}(t_f) &= 0 \end{aligned}$$

In the case of time invariant systems and for an infinite time optimization period, i.e. for $t_f \rightarrow \infty$, the differential Riccati equation becomes an algebraic one

$$\mathbf{0} = \mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} + \mathbf{R}_1 - \mathbf{P} \mathbf{B} \mathbf{R}_2^{-1} \mathbf{B}^T \mathbf{P}$$

If the original system is both controllable and observable (or only stabilizable and detectable) the unique positive definite (semidefinite) solution of (10.32) exists, such that the closed-loop system

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} - \mathbf{B} \mathbf{R}_2^{-1} \mathbf{B}^T \mathbf{P} \right) \mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

is asymptotically stable. In addition, the optimal (minimal) value of the performance criterion is given by (Kirk, 1970; Kwakernaak and Sivan, 1972; Sage and White, 1977)

$$J_{opt} = J_{min} = \frac{1}{2} \mathbf{x}_0^T \mathbf{P} \mathbf{x}_0$$

Example 10.2: Consider the linear deterministic regulator for the F-8 aircraft whose matrices \mathbf{A} and \mathbf{B} are given in Section 5.7. The matrices in the performance criterion together with the system initial condition are taken from Teneketzis and Sandell (1977)

$$\mathbf{R}_1 = \text{diag}\{0.01 \ 0 \ 3260 \ 3260\}, \quad \mathbf{R}_2 = 3260$$

$$\mathbf{x}_0 = [100 \ 0 \ 0.2 \ 0]^T$$

The linear deterministic regulator problem is also known as the linear-quadratic optimal control problem since the system is linear and the performance criterion is quadratic. The MATLAB function `lqr` and the corresponding instruction

$$[\mathbf{F}, \mathbf{P}, \text{ev}] = \text{lqr}(\mathbf{A}, \mathbf{B}, \mathbf{R}_1, \mathbf{R}_2);$$

produce values for optimal gain \mathbf{F} , solution of the algebraic Riccati

equation \mathbf{P} , and closed-loop eigenvalues. These quantities are obtained as

$$\mathbf{F} = [-0.004 \quad 0.5557 \quad -0.2521 \quad 0.0590]$$

$$\mathbf{P} = 10^4 \begin{bmatrix} 0.0000 & -0.0016 & -0.0003 & -0.0003 \\ -0.0016 & 1.6934 & 0.1499 & 0.2199 \\ -0.0003 & 0.1499 & 0.8211 & -0.0713 \\ -0.0003 & 0.2199 & -0.0713 & 0.1361 \end{bmatrix}$$

$$\lambda_{1,2} = -0.9631 \pm j3.0061, \quad \lambda_{3,4} = -0.0373 \pm j0.0837$$

The optimal value for the performance criterion can be found from (10.34), which produces $J_{opt} = 743.9707$.

APPENDIX C. Some Results from Linear Algebra

Linear algebra plays a very important role in linear system control theory and applications (Laub, 1985; Skelton and Iwasaki, 1995). Here we review some standard and important linear algebra results.

Definite Matrices

Definition C.1: A square matrix \mathbf{M} is *positive definite* if all of its eigenvalues have positive real parts, $Re\{\lambda_i(\mathbf{M})\} > 0$. It is *positive semidefinite* if $Re\{\lambda_i(\mathbf{M})\} \geq 0, \forall i$. In addition, *negative definite* matrices are defined by $Re\{\lambda_i(\mathbf{M})\} < 0, \forall i$ and *negative semidefinite* by $Re\{\lambda_i(\mathbf{M})\} \leq 0, \forall i$.

Null Space

Definition C.2: The null space of a matrix \mathbf{M} of dimensions $m \times n$ is the space spanned by vectors \mathbf{v} that satisfy $\mathbf{A}\mathbf{v} = \mathbf{0}$.

Systems of Linear Algebraic Equations

Theorem C.1 *Consider a consistent (solvable) system of linear algebraic equations in n unknowns*

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (\text{c.1})$$

with $\dim\{\mathbf{M}\} = m \times n$. Equation (c.1) has a solution if and only if (consistency condition)

$$\text{rank}\{[\mathbf{M} \ \mathbf{b}]\} = \text{rank}\{\mathbf{M}\} \quad (\text{c.2})$$

In addition, if $\text{rank}\{\mathbf{M}\} = m$, then (c.1) always has a solution. For $n = m$ and $\text{rank}\{\mathbf{M}\} = m$ the solution obtained is unique.

Determinant of a Matrix Product

The following results hold for the determinant of a matrix product

$$\det\{\mathbf{M}_1\mathbf{M}_2\} = \det\{\mathbf{M}_1\}\det\{\mathbf{M}_2\} \quad (\text{c.3})$$

For the proof of the above statement the reader is referred to Stewart (1973). This result can be generalized to the product of a finite number of matrices.

Determinant of Matrix Inversion

By using the rule for the determinant of a product we are able to establish the following formula

$$\det\{\mathbf{M}^{-1}\} = \frac{1}{\det\{\mathbf{M}\}} \quad (\text{c.4})$$

This can be proved as follows

$$\begin{aligned} \det\{\mathbf{M}\mathbf{M}^{-1}\} &= \det\{\mathbf{I}\} = 1 = \det\{\mathbf{M}\}\det\{\mathbf{M}^{-1}\} \\ \Rightarrow \det\{\mathbf{M}^{-1}\} &= \frac{1}{\det\{\mathbf{M}\}} \end{aligned}$$

Inversion of a Matrix Product

Consider the problem of finding the matrix inversion to $\mathbf{M}_1\mathbf{M}_2$. The inversion is a matrix whose product with $\mathbf{M}_1\mathbf{M}_2$ produces an identity matrix, that is

$$\mathbf{M}_1\mathbf{M}_2[\text{Inverse}] = \mathbf{I} \quad (\text{c.5})$$

It can be checked that the inverse of the form $\mathbf{M}_2^{-1}\mathbf{M}_1^{-1}$ satisfies (c.5). This inversion of a product formula can be easily generalized to the product of a finite number of matrices.

Spectral Theorem

Theorem C.2 *If \mathbf{M} is a symmetric matrix, then its eigenvalues are real and \mathbf{M} is diagonalizable, i.e. there exists a similarity transformation \mathbf{P} such that $\mathbf{P}^{-1}\mathbf{M}\mathbf{P}$ is diagonal. Furthermore, the transformation is unitary, i.e. $\mathbf{P}^{-1} = \mathbf{P}^T$.*

Proof of this theorem can be found in many standard books on linear algebra and matrices (see for example Lancaster and Tismenetsky, 1985).

Integral of a Matrix Exponent

The following matrix integral formula is useful in some applications

$$\int_0^T e^{\mathbf{M}t} dt = \left(e^{\mathbf{M}T} - I \right) \mathbf{M}^{-1} \quad (\text{c.6})$$

provided that the matrix \mathbf{M} is nonsingular. In addition, if all eigenvalues of matrix \mathbf{M} are asymptotically stable, then

$$\int_0^{\infty} e^{\mathbf{M}t} dt = -\mathbf{M}^{-1}$$

Vector Derivatives

The following formulas for vector derivatives are used in Chapter 10.

$$\frac{\partial}{\partial \mathbf{y}}(\mathbf{M}\mathbf{y}) = \mathbf{M}$$

$$\frac{\partial}{\partial \mathbf{y}}(\mathbf{y}^T \mathbf{M}^T) = \mathbf{M}^T$$

$$\frac{\partial}{\partial \mathbf{y}}(\mathbf{y}^T \mathbf{M} \mathbf{y}) = \mathbf{M} \mathbf{y} + \mathbf{M}^T \mathbf{y}$$