

Problem 8.34

The system is represented in the linearized form by

$$\Delta \dot{x}(t) = a_0 \Delta x(t) + b_0 \Delta f(t)$$

where

$$a_0 = \frac{\partial}{\partial x} (x f e^{-f})|_{x_n=1, f_n=0} = f_n e^{-f_n} = 0, \quad b_0 = \frac{\partial}{\partial f} (x f e^{-f})|_{x_n=1, f_n=0} = x_n e^{-f_n} - f_n x_n e^{-f_n} = 1$$

The linearized system and its initial condition are given by

$$\Delta \dot{x}(t) = 0 \Delta x(t) + \Delta f(t) = \Delta f(t), \quad \Delta x(0) = x(0) - x_n(0) = 0.9 - 1 = -0.1$$

Problem 8.35

For the second-order nonlinear system

$$\ddot{x}(t) = -2\dot{x}(t) \cos(f(t)) - (1 + f(t))x(t) + f^2(t) + 1 = \mathcal{F}(x(t), \dot{x}(t), f(t))$$

the linearized equation is given by

$$\Delta \ddot{x}(t) + a_1 \Delta \dot{x}(t) + a_0 \Delta x(t) = b_1 \Delta \dot{f}(t) + b_0 \Delta f(t)$$

where

$$a_1 = -\frac{\partial \mathcal{F}(x(t), \dot{x}(t), f(t))}{\partial \dot{x}(t)}|_n = 2 \cos(f(t))|_{f(t)=f_n(t)=0} = 2$$

$$a_0 = -\frac{\partial \mathcal{F}(x(t), \dot{x}(t), f(t))}{\partial x(t)}|_n = (1 + f(t))|_{f(t)=f_n(t)=0} = 1$$

$$b_1 = \frac{\partial \mathcal{F}(x(t), \dot{x}(t), f(t))}{\partial \dot{f}(t)} = 0$$

$$b_0 = \frac{\partial \mathcal{F}(x(t), \dot{x}(t), f(t))}{\partial f(t)}|_{x_n(t)=1, f_n(t)=0} = 2\dot{x}(t) \sin(f(t)) - x(t)|_{x_n(t)=1, f_n(t)=0} = -1$$

The initial conditions for the linearized system are

$$\Delta x(0) = x(0) - x_n(0) = 1.1 - 1 = 0.1, \quad \Delta \dot{x}(0) = \dot{x}(0) - \dot{x}_n(0) = 0.1 - 0 = 0.1$$

Hence, we have the following linearized second-order system

$$\Delta \ddot{x}(t) + 2\Delta \dot{x}(t) + \Delta x(t) = -\Delta f(t), \quad \Delta x(0) = 0.1, \quad \Delta x_n(0) = 0.1$$

The corresponding state space form is given by

$$\begin{bmatrix} \Delta \dot{x}_1(t) \\ \Delta \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta f(t)$$

$$y(t) = [1 \quad 0] \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} = \Delta x_1(t)$$

The linearized system response due to $\Delta f(t) = e^{-2t}$ can be obtained via the use of the Laplace transform

$$(s^2 \Delta X(s) - s \Delta x(0) - \Delta \dot{x}(0)) + 2(s \Delta X(s) - \Delta x(0)) + \Delta X(s) = -\Delta F(s)$$

which leads to

$$\Delta X(s) = -\frac{1}{(s+1)^2(s+2)} + \frac{0.1s+0.3}{(s+1)^2} = \frac{(s+2)(0.1s+0.3)-1}{(s+1)^2(s+2)} = \frac{1.1}{s+1} - \frac{0.8}{(s+1)^2} - \frac{1}{s+2}$$

The inverse Laplace transform implies

$$\Delta x(t) = \mathcal{L}^{-1}\{\Delta X(s)\} = (1.1e^{-t} - 0.8te^{-t} - e^{-2t})u(t)$$

Problem 8.36

The nominal values for the state variable $x_n(t)$ are obtained for the nominal system input $f_n(t) = 1$ for the nominal system initial conditions, that is, by solving the following differential equation

$$\ddot{x}_n(t) + 2\dot{x}_n(t) = 2, \quad x_n(0) = 0, \quad \dot{x}_n(0) = 1.1$$

Applying the Laplace transform we have

$$(s^2 X_n(s) - sx_n(0) - \dot{x}_n(0)) + 2(sX_n(s) - x_n(0)) = \frac{2}{s}$$

The complex domain solution is given by

$$X_n(s) = \frac{1.1s+2}{s^2(s+2)} = \frac{0.05}{s} + \frac{1}{s^2} - \frac{0.05}{s+2}$$

so that in the time domain we have

$$x_n(t) = 0.05 + t - 0.05e^{-2t}, \quad t > 0 \quad \Rightarrow \quad \dot{x}_n(t) = 1 + 0.1e^{-2t}$$

For the given second-order nonlinear system

$$\ddot{x}(t) = -2\dot{x}(t)f(t) - (1-f(t))x(t) + f^2(t) + 1 = \mathcal{F}(x(t), \dot{x}(t), f(t))$$

the linearized equation is given by

$$\Delta \ddot{x}(t) + a_1 \Delta \dot{x}(t) + a_0 \Delta x(t) = b_1 \Delta \dot{f}(t) + b_0 \Delta f(t)$$

where

$$a_1 = -\frac{\partial \mathcal{F}(x(t), \dot{x}(t), f(t))}{\partial \dot{x}(t)}|_n = 2f(t)|_{f(t)=f_n(t)=1} = 2$$

$$a_0 = -\frac{\partial \mathcal{F}(x(t), \dot{x}(t), f(t))}{\partial x(t)}|_n = (1-f(t))|_{f(t)=f_n(t)=1} = 0$$

$$b_1 = \frac{\partial \mathcal{F}(x(t), \dot{x}(t), f(t))}{\partial \dot{f}(t)} = 0$$

$$b_0 = \frac{\partial \mathcal{F}(x(t), \dot{x}(t), f(t))}{\partial f(t)}|_{x_n(t)=1, f_n(t)=0} = -2\dot{x}(t) + x(t) + 2f(t)|_n = 0.05 + t - 0.25e^{-2t} = b_0(t)$$

The linearized second-order system is given by

$$\Delta \ddot{x}(t) + 2\Delta \dot{x}(t) = (0.05 + t - 0.25e^{-2t})\Delta f(t)$$

with the initial conditions

$$\Delta x(0) = x(0) - x_n(0) = 0, \quad \Delta \dot{x}(0) = \dot{x}(0) - \dot{x}_n(0) = 1 - 1.1 = -0.1$$

Choosing the state space variables as $x_1(t) = \Delta x(t)$ and $x_2(t) = \Delta \dot{x}(t)$ with $f(t) = \Delta f(t)$ and $y(t) = x_1(t)$, we obtain the following state space form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b_0(t) \end{bmatrix} f(t)$$

$$y(t) = [1 \ 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Problem 8.37

Represent first the pendulum equation as

$$\ddot{\theta}(t) = -\frac{mgI}{I} \sin(\theta(t)) + \frac{I}{I} f(t) = \mathcal{F}(\theta(t), f(t))$$

The linearized pendulum equation is given by

$$\Delta \ddot{\theta}(t) + a_1 \Delta \dot{\theta}(t) + a_0 \Delta \theta(t) = b_1 \Delta \dot{f}(t) + b_0 \Delta f(t)$$

where

$$a_1 = -\frac{\partial \mathcal{F}(\theta(t), f(t))}{\partial \dot{\theta}(t)} = 0, \quad a_0 = -\frac{\partial \mathcal{F}(\theta(t), f(t))}{\partial \theta(t)}|_{\theta_n(t)=0, f_n(t)=0} = -\frac{mgI}{I} \cos(\theta(t))|_{\theta_n(t)=0, f_n(t)=0} = -\frac{mgI}{I}$$

$$b_1 = \frac{\partial \mathcal{F}(\theta(t), f(t))}{\partial \dot{f}(t)} = 0, \quad b_0 = \frac{\partial \mathcal{F}(\theta(t), f(t))}{\partial f(t)}|_{\theta(t)=0, f(t)=0} = \frac{l}{I}$$

leading to

$$\Delta \ddot{\theta}(t) - \frac{mgI}{I} \Delta \dot{\theta}(t) = \frac{l}{I} \Delta f(t)$$

The initial conditions for the linearized system are given by

$$\Delta \theta(t_0) = \theta(t_0) - \theta_n(t_0) = \theta_0 - 0 = \theta_0, \quad \Delta \dot{\theta}(t_0) = \dot{\theta}(t_0) - \dot{\theta}_n(t_0) = \omega_0 - 0 = \omega_0$$

Problem 8.38

The linearized system is given by

$$\Delta \dot{x}(t) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Delta \mathbf{x}(t)$$

with

$$a_{11} = \frac{\partial(x_1(t)x_2(t) - \sin(x_1(t)))}{\partial x_1(t)}|_{(x_{1n}, x_{2n}, x_{3n})=(0, 1, 1)} = x_2(t) - \cos(x_1(t))|_{(x_{1n}, x_{2n}, x_{3n})=(0, 1, 1)} = 0$$

$$a_{12} = \frac{\partial(x_1(t)x_2(t) - \sin(x_1(t)))}{\partial x_2(t)}|_{(x_{1n}, x_{2n}, x_{3n})=(0, 1, 1)} = x_1(t)|_{(x_{1n}, x_{2n}, x_{3n})=(0, 1, 1)} = 0$$

$$a_{13} = \frac{\partial(x_1(t)x_2(t) - \sin(x_1(t)))}{\partial x_3(t)}|_{(x_{1n}, x_{2n}, x_{3n})=(0, 1, 1)} = 0$$

$$a_{21} = \frac{\partial(1 - 3x_2(t)e^{-x_1(t)})}{\partial x_1(t)}|_{(x_{1n}, x_{2n}, x_{3n})=(0,1,1)} = 3x_2(t)e^{-x_1(t)}|_{(x_{1n}, x_{2n}, x_{3n})=(0,1,1)} = 3$$

$$a_{22} = \frac{\partial(1 - 3x_2(t)e^{-x_1(t)})}{\partial x_2(t)}|_{(x_{1n}, x_{2n}, x_{3n})=(0,1,1)} = -3e^{-x_1(t)}|_{(x_{1n}, x_{2n}, x_{3n})=(0,1,1)} = -3$$

$$a_{23} = \frac{\partial(1 - 3x_2(t)e^{-x_1(t)})}{\partial x_3(t)}|_{(x_{1n}, x_{2n}, x_{3n})=(0,1,1)} = 0$$

$$a_{31} = \frac{\partial(x_1(t)x_2(t)x_3(t))}{\partial x_1(t)}|_{(x_{1n}, x_{2n}, x_{3n})=(0,1,1)} = x_2(t)x_3(t)|_{(x_{1n}, x_{2n}, x_{3n})=(0,1,1)} = 1$$

$$a_{32} = \frac{\partial(x_1(t)x_2(t)x_3(t))}{\partial x_2(t)}|_{(x_{1n}, x_{2n}, x_{3n})=(0,1,1)} = x_1(t)x_3(t)|_{(x_{1n}, x_{2n}, x_{3n})=(0,1,1)} = 0$$

$$a_{33} = \frac{\partial(x_1(t)x_2(t)x_3(t))}{\partial x_3(t)}|_{(x_{1n}, x_{2n}, x_{3n})=(0,1,1)} = x_1(t)x_2(t)|_{(x_{1n}, x_{2n}, x_{3n})=(0,1,1)} = 0$$

Hence, the linearized system is given by

$$\Delta \dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 3 & -3 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Delta x(t)$$

Problem 8.39

For the given nonlinear system

$$\begin{aligned} \dot{x}_1(t) &= f(t) \ln(x_1(t)) + x_2(t)e^{-f(t)} \\ \dot{x}_2(t) &= x_1(t) \sin(f(t)) - \sin(x_2(t)) \\ y(t) &= \sin(x_1(t)) \end{aligned}$$

the linearized system matrices are given by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \mathbf{C} = [c_1 \quad c_2], \quad \mathbf{D} = 0$$

where

$$\begin{aligned} a_{11} &= \frac{\partial(f(t) \ln(x_1(t)) + x_2(t)e^{-f(t)})}{\partial x_1(t)} = \frac{f(t)}{x_1(t)}, \quad a_{12} = \frac{\partial(f(t) \ln(x_1(t)) + x_2(t)e^{-f(t)})}{\partial x_2(t)} = e^{-f(t)} \\ a_{21} &= \frac{\partial(x_1(t) \sin(f(t)) - \sin(x_2(t)))}{\partial x_1(t)} = \sin(f(t)), \quad a_{22} = \frac{\partial(x_1(t) \sin(f(t)) - \sin(x_2(t)))}{\partial x_2(t)} = -\cos(x_2(t)) \\ b_1 &= \frac{\partial(f(t) \ln(x_1(t)) + x_2(t)e^{-f(t)})}{\partial f(t)} = \ln(x_1(t)) - x_2(t)e^{-f(t)} \\ b_2 &= \frac{\partial(x_1(t) \sin(f(t)) - \sin(x_2(t)))}{\partial f(t)} = x_1(t) \cos(f(t)) \\ c_1 &= \frac{\partial(\sin(x_1(t)))}{\partial x_1(t)} = \cos(x_1(t)), \quad c_2 = \frac{\partial(\sin(x_1(t)))}{\partial x_2(t)} = 0 \end{aligned}$$

The state space linearized model, with the coefficients evaluated at the nominal points x_{1n} , x_{2n} and f_n is given by

$$\begin{bmatrix} \Delta \dot{x}_1(t) \\ \Delta \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{f_n}{x_{1n}} & e^{-f_n} \\ \sin(f_n) & -\cos(x_{2n}) \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + \begin{bmatrix} \ln(x_{1n}) - x_{2n}e^{-f_n} \\ x_{1n}\cos(f_n) \end{bmatrix} \Delta f(t)$$

Problem 8.40

For the Volterra predator-prey mathematical model

$$\dot{x}_1(t) = -x_1(t) + x_1(t)x_2(t)$$

$$\dot{x}_2(t) = x_2(t) - x_1(t)x_2(t)$$

the linearized system matrices are given by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \mathbf{C} = [c_1 \quad c_2], \quad \mathbf{D} = 0$$

where

$$a_{11} = \frac{\partial(-x_1(t) + x_1(t)x_2(t))}{\partial x_1(t)} = x_2(t) - 1, \quad a_{12} = \frac{\partial(-x_1(t) + x_1(t)x_2(t))}{\partial x_2(t)} = x_1(t)$$

$$a_{21} = \frac{\partial(x_2(t) - x_1(t)x_2(t))}{\partial x_1(t)} = x_2(t), \quad a_{22} = \frac{\partial(x_2(t) - x_1(t)x_2(t))}{\partial x_2(t)} = 1 - x_1(t)$$

The state space linearized model, with the coefficients evaluated at the nominal points $x_{1n} = 0$ and $x_{2n} = 0$ is given by

$$\begin{bmatrix} \Delta \dot{x}_1(t) \\ \Delta \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}$$

Problem 8.41

Introducing the following change of the variables $x_1(t) = \theta_l(t)$, $x_2(t) = \dot{\theta}_l(t)$, $x_3(t) = \theta_m(t)$, $x_4(t) = \dot{\theta}_m(t)$, we obtain a system of four first-order differential equations

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{k}{J_l}x_1(t) - \frac{B_l}{J_l}x_2(t) + \frac{k}{J_l}kx_3(t)$$

$$\dot{x}_3(t) = x_4(t)$$

$$\dot{x}_4(t) = \frac{k}{J_m}x_1(t) - \frac{k}{J_m}x_3(t) - \frac{B_m}{J_m}x_4(t) + f(t)$$

which can be put in the state space matrix form as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_l} & -\frac{B_l}{J_l} & \frac{k}{J_l} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_m} & 0 & -\frac{k}{J_m} & -\frac{B_m}{J_m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_m} \end{bmatrix} f(t)$$