12.1 The Essence of Feedback Systems

We can find the response of an open-loop system due to given input using any of the methods presented in this textbook. The response obtained is strictly determined by information about the system mathematical model, system input, and system initial conditions. In control theory and its applications, given system input and initial conditions, the goal is for the system output to have a desired response (shape). The system output can be shaped by closing the feedback loop around the system (see the next figure), and using particular static and/or dynamic elements in the feedback path. By closing the feedback loop around the system several interesting and important features can be achieved:

- (a) feedback can stabilize the system (the most important one);
- (b)feedback reduces system output sensitivity to system parameter variations (due to aging of system components, due to internal system disturbances);
- (c) feedback reduces the impact of external system disturbances on the system output;
- (d) feedback can improve system output transient response;
- (e) feedback can improve the system output steady state response;
- (f) feedback can reduce system sensitivity to inaccuracies in the system measurements (measurement noise).

Within this section we will discuss features (a), (b), and (c). Feedback features (d) and (e) will become evident in Sections 12.3, 12.4 and 12.6. Complete clarification of the feedback feature (f) is outside of the scope of this textbook.

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(a) Feedback Stabilization of Linear Time Invariant Systems

Consider first the system with unity feedback represented in Figure 4.7a.



Figure 4.7: Closed-loop system configurations:

(a) unity feedback; (b) nonunity feedback

Assume that the open-loop transfer function H(s) is represented by the ratio of two polynomials, that is, H(s) = N(s)/D(s). The zeros of D(s) represent the open-loop system poles, and we say that the open-loop transfer function is stable

(asymptotically stable and BIBO stable) if all the poles of H(s) are strictly in the left half complex plane (none of them is on the imaginary axis).

The closed-loop transfer function of the feedback system in Figure 4.7a is

$$M(s) = \frac{Y(s)}{F(s)} = \frac{H(s)}{1 + H(s)} = \frac{N(s)/D(s)}{1 + N(s)/D(s)} = \frac{N(s)}{D(s) + N(s)}$$

In this formula we have used: $E(s) = F(s) - Y(s) = F(s) - H(s)E(s) \Rightarrow$ Y(s) = H(s)F(s)/(1 + H(s)). The algebraic equation

$$1 + H(s) = 0 \quad \Rightarrow \quad D(s) + N(s) = 0$$

defines the *characteristic equation of the closed-loop system* with unity feedback. The zeros of the closed-loop characteristic equation represent the closed-loop system poles. The *open-loop characteristic equation* is simply D(s) = 0. Hence, the feedback changes the system characteristic equation and the system poles.

It can happen that the open-loop system is unstable, but the closed-loop system is stable as demonstrated in the next example.

Example 12.1: Consider an open-loop unstable linear system represented by

$$H(s) = \frac{1}{(s - 0.5)(s + 1)}$$

Its open-loop poles are located at -1 and 0.5. The closed-loop characteristic equation is

$$(s - 0.5)(s + 1) + 1 = s^{2} + 0.5s + 0.5 = 0$$

The solutions of this algebraic equation, given by $s_{1,2} = -0.25 \pm j0.25\sqrt{7}$, represent the closed-loop system poles. Obviously, both poles are in the open left half complex plane and the system is asymptotically stable.

Sometimes unity feedback is not sufficient to stabilize the open-loop unstable system, as demonstrated below.

Example 12.2: Consider the following open-loop unstable system

$$H(s) = \frac{1}{(s-1)(s+2)}$$

Its closed-loop characteristic equation is

$$(s-1)(s+2) + 1 = s^2 + s - 1 = 0$$

The solutions of this algebraic equation are $s_{1,2} = -1 \pm \sqrt{2}$, hence the closed-loop system is unstable since the pole $\sqrt{2} - 1$ is in the right half complex plane.

The simplest form of non unity feedback is when the element in the feedback path is static, that is, G(s) = g = const. In such a case, the closed-loop transfer function is

$$M(s) = \frac{H(s)}{1+gH(s)} = \frac{N(s)}{D(s)+gN(s)}$$

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The closed-loop system characteristic equation is now given by

$$1 + gH(s) = 0 \quad \Rightarrow \quad D(s) + gN(s) = 0$$

Example 12.3: Consider the open-loop transfer function from Example 12.2 in configuration with non unity static feedback elements G(s) = g. The closed-loop characteristic equation is given by

$$(s-1)(s+2) + g = s^{2} + s + (g-2) = 0$$

It can be easily checked that for all static feedback elements with g > 2 both closed-loop poles are in the open left half complex plane, which implies that the closed-loop system is asymptotically stable for any g > 2.

For more complex linear systems, systematic procedures are developed that find the range of values for the static element g that provides stability of closed-loop The slides contain the copyrighted material from *Linear Dynamic Systems and Signals*, Prentice Hall, 2003. Prepared by Professor Zoran Gajic **12–7** linear systems. The Roth-Hurwitz stability test, presented in Section 7.7.2, can be used for that purpose, as demonstrated in Example 7.28.

In some cases, it can happen that for all choices of the static element g the closed-loop system remains unstable. In such cases, we should try to stabilize the system using a dynamic feedback element G(s). The corresponding closed-loop characteristic equation is given by

$$1 + H(s)G(s) = 0 \quad \Rightarrow \quad D(s) + N(s)G(s) = 0$$

G(s) can be also chosen as a ratio of two polynomials. It should be pointed out that system stabilization with a dynamic feedback element is more complicated than stabilization with a static feedback element. Detailed consideration of the stabilization problem with dynamic feedback is beyond the scope of this textbook.

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(b) Feedback Reduction of System Output Sensitivity

A unity feedback system is presented in Figure 4.7a. The perturbed form of this system is given in Figure 12.1, where $\Delta H(s)$ denotes perturbations of the



Figure 12.1: A unity feedback system and its perturbation

system open-loop transfer function. The perturbations in the system open-loop transfer function are primarily due to aging of the system components and due to system internal disturbances (this also includes the case when some of the system parameters are not perfectly known—parameter uncertainties). For the closed-loop unity feedback system without perturbations, the system output is given by

$$Y(s) = M(s)F(s) = \frac{H(s)}{1 + H(s)}F(s)$$

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In the case of perturbations, we obtain from Figure 12.1

$$Y(s) + \Delta Y(s) = \frac{H(s) + \Delta H(s)}{1 + H(s) + \Delta H(s)}F(s)$$

which implies

$$\Delta Y(s) = \frac{H(s) + \Delta H(s)}{1 + H(s) + \Delta H(s)} F(s) - Y(s)$$
$$= \frac{H(s) + \Delta H(s)}{1 + H(s) + \Delta H(s)} F(s) - \frac{H(s)}{1 + H(s)} F(s)$$
$$= \frac{\Delta H(s)}{(1 + H(s) + \Delta H(s))(1 + H(s))} F(s)$$

Since in general $|\Delta H(s)| << |H(s)|$, we can approximate the last expression as

$$\Delta Y(s) \approx \frac{\Delta H(s)}{(1+H(s))(1+H(s))} F(s)$$

= $\frac{\Delta H(s)H(s)}{H(s)(1+H(s))(1+H(s))} F(s) = \frac{\Delta H(s)}{H(s)} \frac{1}{(1+H(s))} Y(s)$

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In summary, we have derived the following relationship

$$rac{\Delta Y(s)}{Y(s)} = rac{1}{(1+H(s))} rac{\Delta H(s)}{H(s)} riangleq S(s) rac{\Delta H(s)}{H(s)}$$

which indicates that the relative change in the system output is proportional to the relative change in the system transfer function. The proportionality factor S(s) is called the system output *sensitivity function*. Due to the fact that |S(s)| = |1/(1 + H(s))| < 1 we can conclude that the impact of system parameter variations on the system output is relatively reduced by the factor of |S(s)| < 1.

For systems with non unity feedback, $G(s) \neq 1$, we can follow the same derivations, and obtain the following form for the system output sensitivity function

$$S(s) = \frac{1}{1 + G(s)H(s)}$$

(c) Feedback Disturbance Rejection

Let the feedback system be under the influence of an external disturbance as demonstrated in the block diagram presented in Figure 12.2.



Figure 12.2: A feedback system with an external disturbance

Using the basic transfer function rules, we obtain

$$Y(s) = D(s) + H(s)(F(s) - Y(s))$$

which implies

$$Y(s) = \frac{H(s)}{1 + H(s)}F(s) + \frac{1}{1 + H(s)}D(s) = \frac{H(s)}{1 + H(s)}F(s) + S(s)D(s)$$

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It follows from the last formula that the sensitivity function causes the disturbance to be attenuated due to the fact that |S(s)| < 1. In general, S(s) is a function of frequency, hence we need that S(s) is very small at least in the frequency range of the system disturbance. We can conclude that the feedback configuration has a nice property of rejecting the system disturbance by reducing its impact on the system output. Similarly, like in the previous cases, we can perform the same derivations for the case of a non unity feedback control system, that is with $G(s) \neq 1$, which implies

$$Y(s) = \frac{H(s)}{1 + G(s)H(s)}F(s) + S(s)D(s)$$