Chapter Nine

Frequency Domain Controller Design

9.1 Introduction

Frequency domain techniques together with the root locus method have been very popular classical methods for both analysis and design of control systems. As they are still used extensively in industry for solving controller design problems for many industrial processes and systems, they have to be included in academic curricula and modern textbooks on control systems. This chapter is organized as follows.

In Section 9.2 we study the open- and closed-loop frequency transfer functions and identify the frequency response parameters such as system frequency bandwidth, peak resonance, and resonant frequency.

In Section 9.3 we show how to read the phase and gain stability margins, and the values of the steady state error parameters K_p, K_v, K_a from the corresponding frequency diagrams, known as Bode diagrams. Bode diagrams represent the magnitude and phase plots of the open-loop transfer function with respect to the angular frequency ω. They can be obtained either analytically or experimentally. In this chapter we present only the analytical study of Bode diagrams, though it should be mentioned that Bode diagrams can be obtained from experimental measurements performed on a real physical system driven by a sinusoidal input with a broad range of frequencies.

The controller design technique based on Bode diagrams is considered in Section 9.4. It is shown how to use the phase-lag, phase-lead, and phase-lag-lead controllers such that the compensated systems have the desired phase and gain stability margins and steady state errors. It is also possible in some cases to improve the transient response since the damping ratio is proportional to the phase margin, and the response rise time is inversely proportional to the system bandwidth.

Section 9.5 contains a case study for a ship positioning control system, and Section 9.7 represents a laboratory experiment. In Section 9.6 we comment on discrete-time controller design.

Chapter Objectives

The main objective of this chapter is to show how to use Bode diagrams as a tool for controller design such that compensated systems have, first of all, the desired phase and gain stability margins and steady state errors. Controllers based on Bode diagrams can also be used to improve some of the transient response parameters, but their design is far more complicated and far less accurate than the design of the corresponding controllers based on the root locus method.

9.2 Frequency Response Characteristics

The open- and closed-loop system transfer functions are defined in Chapter 2. For a feedback control system these transfer functions are respectively given by

$$\text{Open–loop: } G(s)H(s), \quad \text{Closed–loop: } \frac{G(s)}{1+G(s)H(s)} = M(s) \tag{9.1}$$

The frequency transfer functions are defined for sinusoidal inputs having all possible frequencies $\omega \in [0, +\infty)$. They are obtained from (9.1) by simply setting $s = j\omega$, that is

$$\text{Open–loop: } G(j\omega)H(j\omega), \quad \text{Closed–loop: } \frac{G(j\omega)}{1+G(j\omega)H(j\omega)} = M(j\omega) \tag{9.2}$$

Using the frequency transfer functions in the system analysis gives complete information about the system's steady state behavior, but not about the system's transient response. That is why controller design techniques based on frequency transfer functions improve primarily the frequency domain specifications such as phase and gain relative stability margins and steady state errors. However, some frequency domain specifications, to be defined soon, can be related to certain time domain specifications. For example, the wider the system bandwidth, the faster system response, which implies a shorter response rise time.

Typical diagrams for the magnitude and phase of the open-loop frequency transfer function are presented in Figure 9.1. From this figure one is able to read directly the phase and gain stability margins and the corresponding phase and gain crossover frequencies. In Section 9.3, where we present the Bode diagrams, which also represent the magnitude and phase plots of the open-loop frequency transfer function with respect to frequency, with magnitude being calculated in decibels (dB), we will show how to read the values for K_p, K_v, K_a.

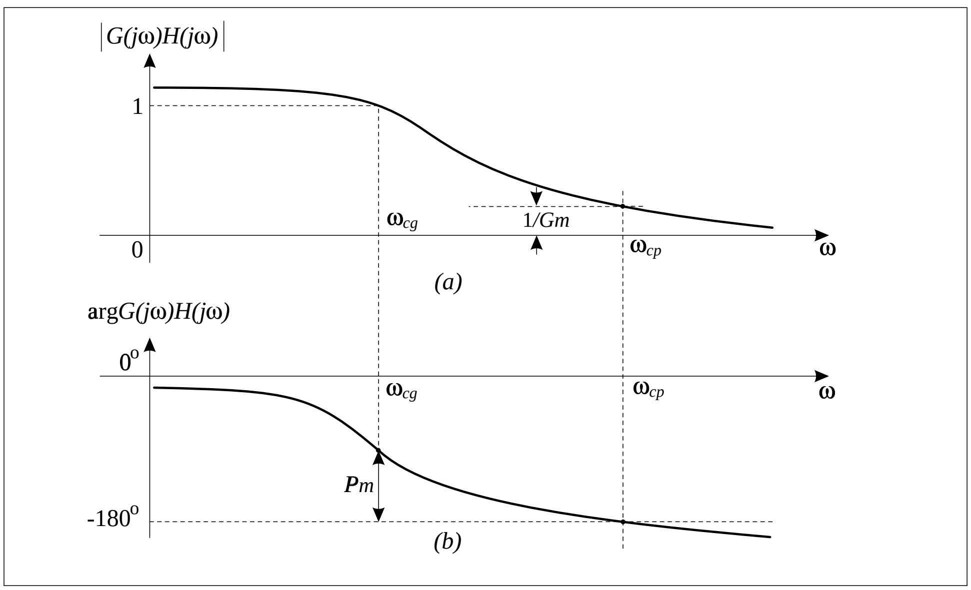

Figure 9.1: Magnitude (a) and phase (b) of the open-loop transfer function

In addition to the phase and gain margins, obtained from the *open-loop* frequency transfer function, other frequency response parameters can be obtained from the frequency plot of the magnitude of the *closed-loop* frequency transfer function. This plot is given in Figure 9.2. The main *closed-loop* frequency response parameters are: system bandwidth, peak resonance, and resonant frequency. They are formally defined below.

System Bandwidth: This represents the frequency range in which the magnitude of the closed-loop frequency transfer function drops no more than $3\,\mathrm{dB}$ (decibels) from its zero-frequency value. The system bandwidth can be obtained from the next equality, which indicates the attenuation of $3\,\mathrm{dB}$, as

$$|M(j\omega_{BW})| = \frac{1}{\sqrt{2}}|M(0)| \Rightarrow \omega_{BW} \tag{9.3}$$

It happens to be computationally very involved to solve equation (9.3) for higher-order systems, and hence the system bandwidth is mostly determined experimentally. For second-order systems the frequency bandwidth can be found analytically (see Problem 9.2).

Peak Resonance: This is obtained by finding the maximum of the function $|M(j\omega)|$ with respect to frequency ω. It is interesting to point out that the systems having large maximum overshoot have also large peak resonance. This is analytically justified for a second-order system in Problem 9.1.

Resonant Frequency: This is the frequency at which the peak resonance occurs. It can be obtained from

$$\frac{d}{d\omega}|M(j\omega)| = 0 \quad \Rightarrow \quad \omega_r$$

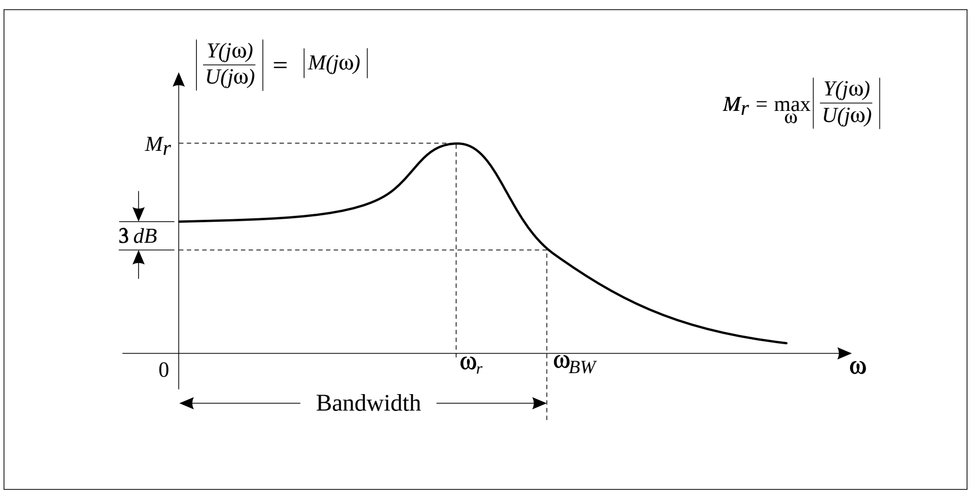

Figure 9.2: Magnitude of the closed-loop transfer function

9.3 Bode Diagrams

Bode diagrams are the main tool for frequency domain controller design, a topic that will be presented in detail in Section 9.4. In this section we show how to plot these diagrams and how to read from them certain control system characteristics, such as the phase and gain stability margins and the constants required to determine steady state errors.

Bode diagrams represent the frequency plots of the magnitude and phase of the open-loop frequency transfer function $G(j\omega)H(j\omega)$. *The magnitude is plotted in* dB *(decibels) on the* $\log\omega$ *scale.* In general, the open-loop frequency transfer function contains elementary frequency transfer functions representing a constant term (static gain) and dynamic elements like system real poles and zeros and complex conjugate poles and zeros. We first study independently the magnitude and frequency plots of each of these elementary frequency transfer functions. Since the open-loop frequency transfer function $G(j\omega)H(j\omega)$ is given in terms of products and ratios of elementary transfer functions, it is easy to see that *the phase of* $G(j\omega)H(j\omega)$ *is obtained by summing and subtracting phases of the elementary transfer functions*. Also, by expressing the magnitude of the open-loop transfer function in decibels, *the magnitude* $|G(j\omega)H(j\omega)|_{dB}$ *is obtained by adding the magnitudes of the elementary frequency transfer functions.* For example

$$
\begin{aligned}
|G(j\omega)H(j\omega)|_{dB} &= 20\log_{10}\left|\frac{K(j\omega+z_1)(j\omega+z_2)}{(j\omega)(j\omega+p_2)(j\omega+p_3)}\right| \\
&= 20\log_{10}|K| + 20\log_{10}|j\omega+z_1| + 20\log_{10}|j\omega+z_2| \\
&\quad + 20\log_{10}\left|\frac{1}{j\omega}\right| + 20\log_{10}\left|\frac{1}{j\omega+p_2}\right| + 20\log_{10}\left|\frac{1}{j\omega+p_3}\right|
\end{aligned}
$$

and

$$
\begin{aligned}
\arg\{G(j\omega)H(j\omega)\} &= \arg\{K\} + \arg\{j\omega+z_1\} + \arg\{j\omega+z_2\} \\
&\quad - \arg\{j\omega\} - \arg\{j\omega+p_2\} - \arg\{j\omega+p_3\}
\end{aligned}
$$

In the following we show how to draw Bode diagrams for elementary frequency transfer functions.

Constant Term: Since

$$K_{dB} = 20\log_{10} K = \begin{cases} \text{positive number} & K > 1 \\ \text{negative number} & K < 1 \end{cases}$$
$$\arg K = \begin{cases} 0^\circ, & K > 0 \\ -180^\circ, & K < 0 \end{cases} \tag{9.4}$$

the magnitude and phase of this element are easily drawn and are presented in Figure 9.3.

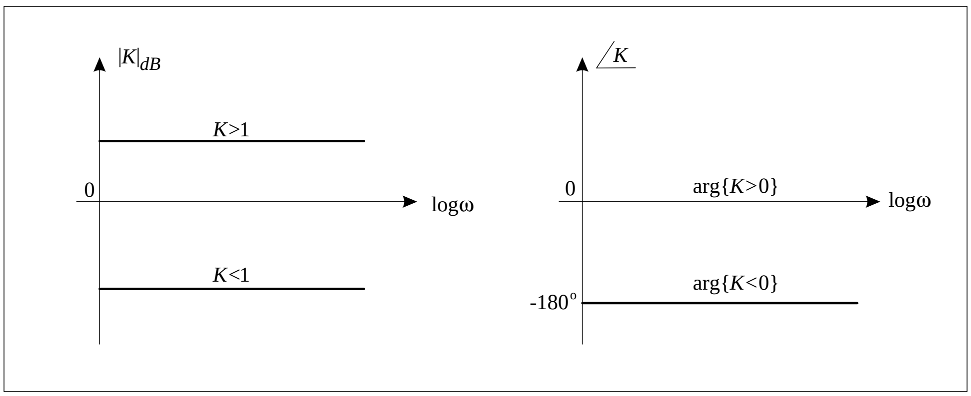

Figure 9.3: Magnitude and phase diagrams for a constant

Pure Integrator: The transfer function of a pure integrator, given by

$$G(j\omega) = \frac{1}{j\omega} \tag{9.5}$$

has the following magnitude and phase

$$|G(j\omega)|_{dB} = 20\log_{10}\frac{1}{\omega} = -20\log_{10}\omega, \qquad \arg G(j\omega) = -90^\circ \tag{9.6}$$

It can be observed that the phase for a pure integrator is constant, whereas the magnitude is represented by a straight line intersecting the frequency axis at $\omega = 1$ and having the slope of $-20\,\mathrm{dB/decade}$. Both diagrams are represented in Figure 9.4. Thus, *a pure integrator introduces a phase shift of* -90° *and a gain attenuation of* $-20\,\mathrm{dB/decade}$.

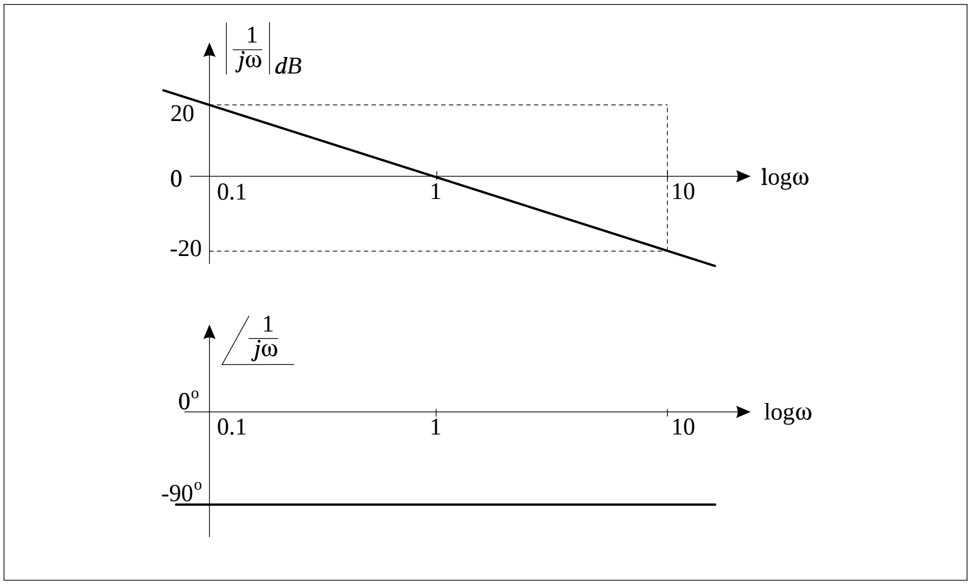

Figure 9.4: Magnitude and phase diagrams for a pure integrator

Pure Differentiator: The transfer function of a pure differentiator is given by

$$G(j\omega) = j\omega \tag{9.7}$$

Its magnitude and phase are easily obtained as

$$|G(j\omega)|_{dB} = 20\log_{10}\omega, \qquad \arg G(j\omega) = 90^\circ \tag{9.8}$$

The corresponding frequency diagrams are presented in Figure 9.5. *It can be concluded that a pure differentiator introduces a positive phase shift of* 90° *and an amplification of* $20\,\mathrm{dB/decade}$.

Real Pole: The transfer function of a real pole, given by

$$G(j\omega) = \frac{p}{p + j\omega} = \frac{1}{1 + j\frac{\omega}{p}} \tag{9.9}$$

has the following magnitude and phase

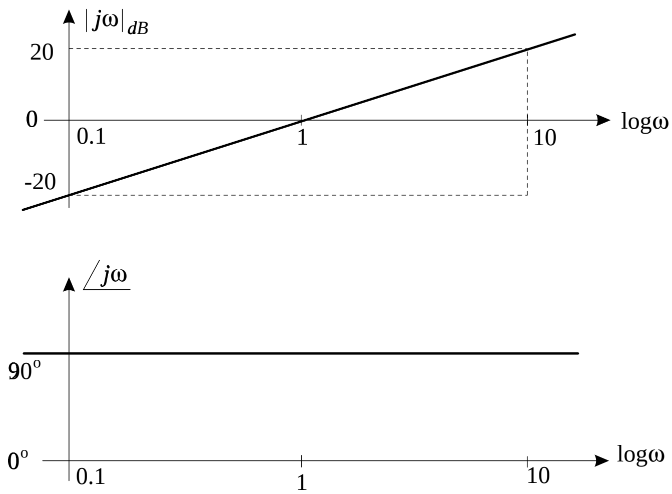

Figure 9.5: Magnitude and phase diagrams for a pure differentiator

$$|G(j\omega)|_{dB} = -20\log_{10}\left[1+\left(\frac{\omega}{p}\right)^2\right]^{1/2}, \qquad \arg G(j\omega) = -\tan^{-1}\left(\frac{\omega}{p}\right) \tag{9.10}$$

The phase diagram for a real pole can be plotted directly from (9.10). It can be seen that for large values of ω, $\omega \gg p$, the phase contribution is $-90°$. For ω small, $\omega \ll p$, the phase is close to zero, and for $\omega = p$ the phase contribution is $-45°$. This information is sufficient to sketch $\arg G(j\omega)$ as given in Figure 9.6.

For the magnitude, we see from (9.10) that for small ω the magnitude is very close to zero. For large values of ω we can neglect 1 compared to ω/p so that we have a similar result as for a pure integrator, i.e. we obtain an attenuation of $20\,\mathrm{dB/decade}$. For small and large frequencies we have straight-line approximations. These straight lines intersect at $\omega = p$, which is also known as a *corner frequency*. The actual magnitude curve is below the straightline approximations. It has the biggest deviation from the asymptotes at the corner frequency (see Figure 9.6).

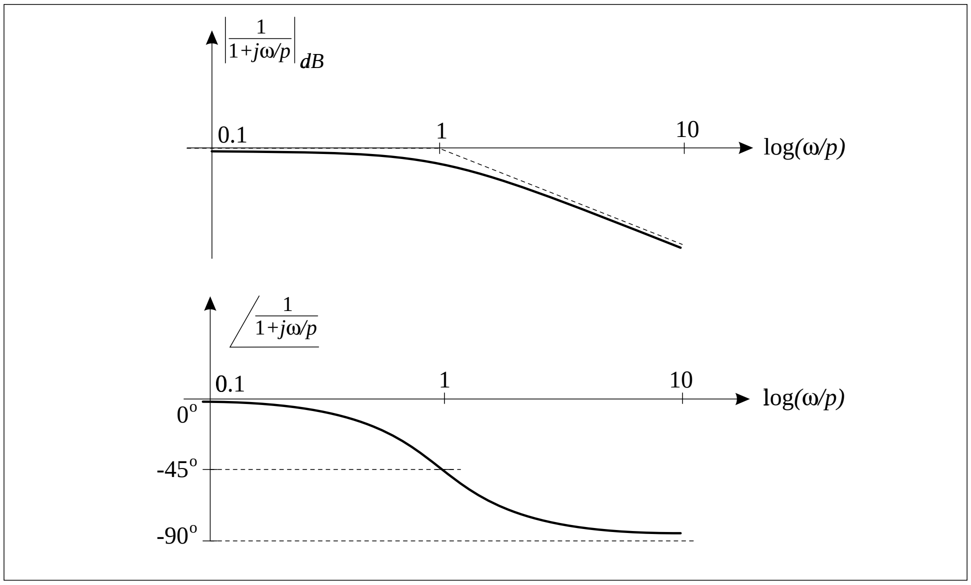

Figure 9.6: Magnitude and phase diagrams for a real pole

Real Zero: The transfer function of an element representing a real zero is given by

$$G(j\omega) = \frac{1}{z}(z + j\omega) = 1 + j\left(\frac{\omega}{z}\right) \tag{9.11}$$

Its magnitude and phase are

$$|G(j\omega)|_{dB} = 20\log_{10}\left[1 + \left(\frac{\omega}{z}\right)^2\right]^{1/2}, \quad \arg G(j\omega) = \tan^{-1}\left(\frac{\omega}{z}\right) \tag{9.12}$$

Using analysis similar to that performed for a real pole, we can conclude that for small frequencies an asymptote for the magnitude is equal to zero and for large frequencies the magnitude asymptote has a slope of $20\,\mathrm{dB/decade}$ and intersects the real axis at $\omega = z$ (the corner frequency). The phase diagram for small frequencies also has an asymptote equal to zero and for large frequencies an asymptote of $90°$. The magnitude and phase Bode diagrams for a real-zero element are represented in Figure 9.7.

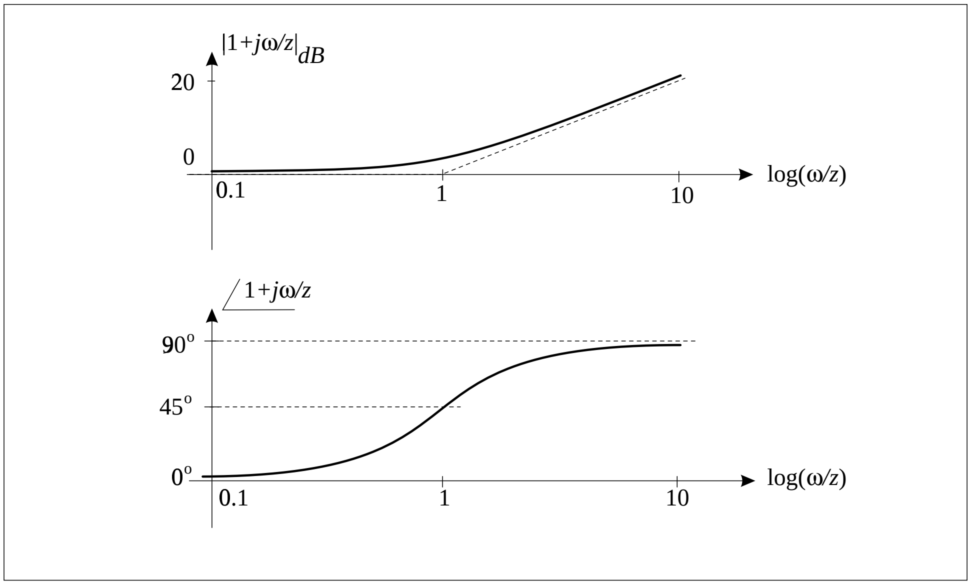

Figure 9.7: Magnitude and phase diagrams for a real zero

Complex Conjugate Poles: The transfer function of an element representing a pair of complex conjugate poles is in fact a transfer function of a second-order system, which has the form

$$G(j\omega) = \frac{w_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\zeta\frac{\omega}{\omega_n}} \tag{9.13}$$

The magnitude and phase of this second-order system are given by

$$\begin{aligned} |G(j\omega)|_{dB} &= -20\log_{10}\left[\left(\frac{2\zeta\omega}{\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{\omega_n^2}\right)^2\right]^{1/2} \\ \arg G(j\omega) &= -\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right) \end{aligned} \tag{9.14}$$

For large values of ω the corresponding approximations of (9.14) are

$$|G(j\omega)|_{dB} \approx -20\log_{10}\left(\frac{\omega^2}{\omega_n^2}\right) = -40\log_{10}\left(\frac{\omega}{\omega_n}\right)$$

$$\arg\{G(j\omega)\} \approx -\tan^{-1}\left(\frac{2\zeta\omega_n}{-\omega}\right) \to -\tan^{-1}\left(0^-\right) = -180°$$

At low frequencies the approximations can be obtained directly from (9.13), that is

$$G(j\omega) \approx \frac{\omega_n^2}{\omega_n^2} = 1 \;\Rightarrow\; |G(j\omega)|_{dB} = 0, \quad \arg\{G(j\omega)\} = 0°$$

Thus, the corresponding asymptotes for small and large frequencies are, respectively, zero and $-40\,\mathrm{dB/decade}$ (with the corner frequency at $\omega = \omega_n$) for the magnitude, and zero and $-180°$ for the phase. At the corner frequency ω_n the phase is equal to $-90°$. The corresponding Bode diagrams are represented in Figure 9.8. Note that the actual plot in the neighborhood of the corner frequency depends on the values of the damping ratio ζ. Several curves are shown for $0.1 \leq \zeta \leq 1$. It can be seen from Figure 9.8 that the smaller ζ, the higher peak of the magnitude plot.

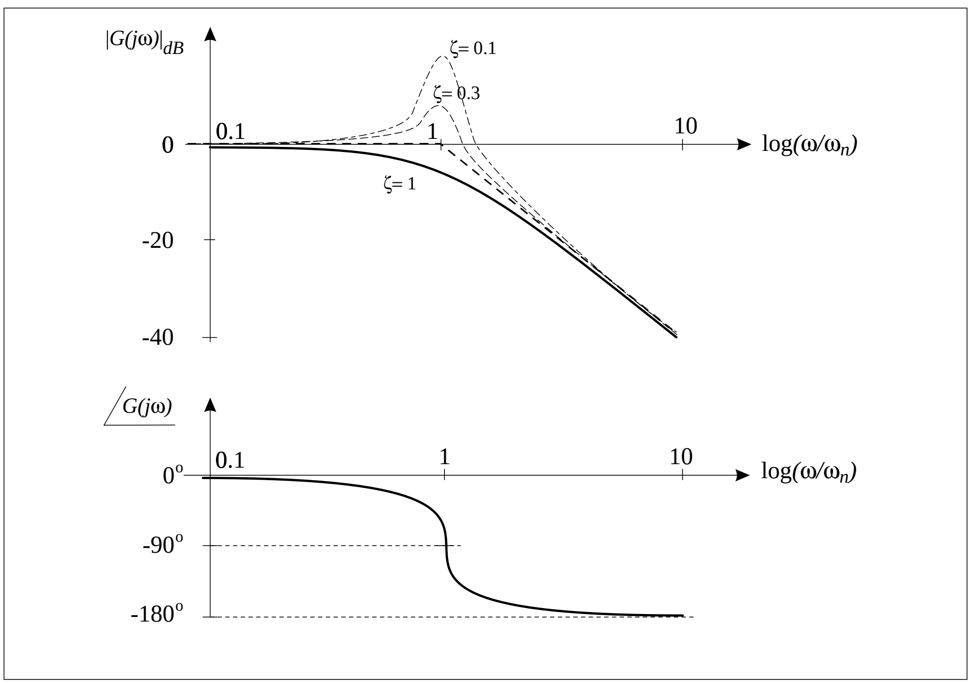

Figure 9.8: Magnitude and phase diagrams for complex conjugate poles

Complex Conjugate Zeros: An element that has complex conjugate zeros can be represented in the form

$$G(j\omega) = 1 + 2\zeta j\left(\frac{\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2 = 1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right) \tag{9.15}$$

so that the corresponding Bode diagrams will be the mirror images of the Bode diagrams obtained for the complex conjugate poles represented by (9.13). In the case of complex conjugate zeros, the asymptotes for small frequencies are equal to zero for both the magnitude and phase plots; for high frequencies the magnitude asymptote has a slope of $40\,\mathrm{dB/decade}$ and starts at the corner frequency of $\omega = \omega_n$, and the phase plot asymptote is $180°$.

9.3.1 Phase and Gain Stability Margins from Bode Diagrams

It has been already indicated in Figure 9.1 how to read the phase and gain stability margins from the frequency magnitude and phase plots of the open-loop feedback transfer function. In the case of Bode diagrams the magnitude plot is expressed in dB (decibels) so that the gain crossover frequency is obtained at the point of intersection of the Bode magnitude plot and the frequency axis. Bearing in mind the definition of the phase and gain stability margins given in (4.54) and (4.55), and the corresponding phase and gain crossover frequencies defined in (4.56) and (4.57), it is easy to conclude that these margins can be found from Bode diagrams as indicated in Figure 9.9.

Example 9.1: In this example we use MATLAB to plot Bode diagrams for the following open-loop frequency transfer function

$$G(j\omega)H(j\omega) = \frac{(j\omega + 1)}{j\omega(j\omega + 2)\left[(j\omega)^2 + 2(j\omega) + 2\right]}$$

Bode diagrams are obtained by using the MATLAB function `bode(num,den)`. The phase and gain stability margins and the phase and gain crossover frequencies can be obtained by using `[Gm,Pm,wcp,wcg]=margin(num,den)`. Note that the open-loop frequency transfer function has to be specified in terms of polynomials `num` (numerator) and `den` (denominator). The MATLAB function `conv` helps to multiply polynomials as explained below in the program written to plot Bode diagrams and find the phase and gain margins for Example 9.1.

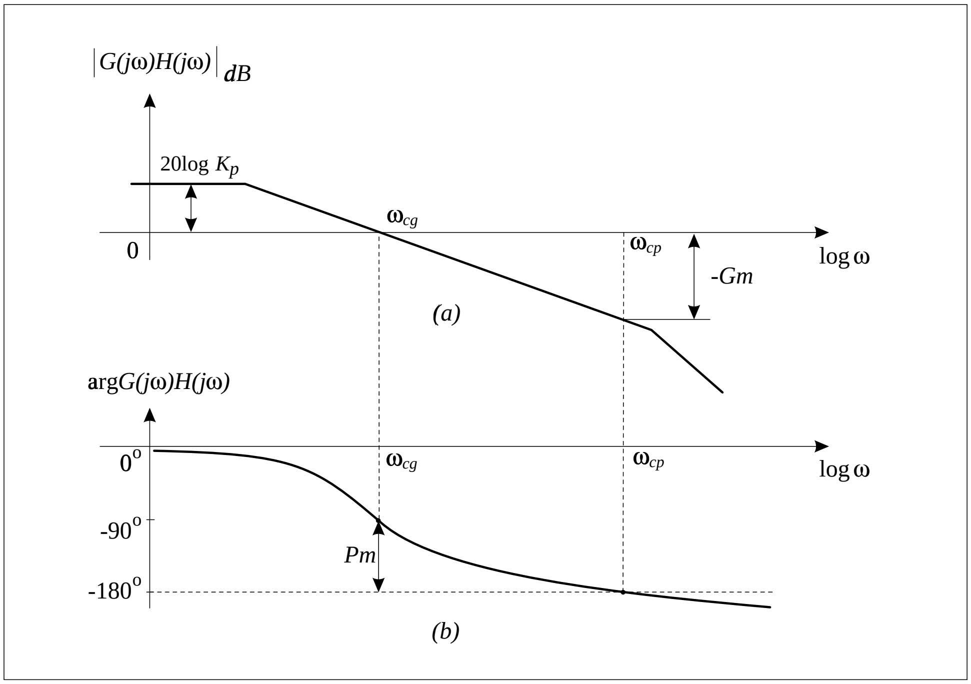

Figure 9.9: Gain and phase margins and Bode diagrams

```
num=[1 1];
d1=[1 0];
d2=[1 2];
d3=[1 2 2];
den1=conv(d1,d2);
den=conv(den1,d3);
bode(num,den);
[Gm,Pm,wcp,wcg]=margin(num,den);
```

The corresponding Bode diagrams are presented in Figure 9.10. The phase and gain stability margins and the corresponding crossover frequencies are obtained as

$$Gm = 8.9443\,\mathrm{dB},\ Pm = 82.2462^\circ,\ \omega_{cp} = 1.7989\,\mathrm{rad/s},\ \omega_{cg} = 0.2558\,\mathrm{rad/s}$$

◇

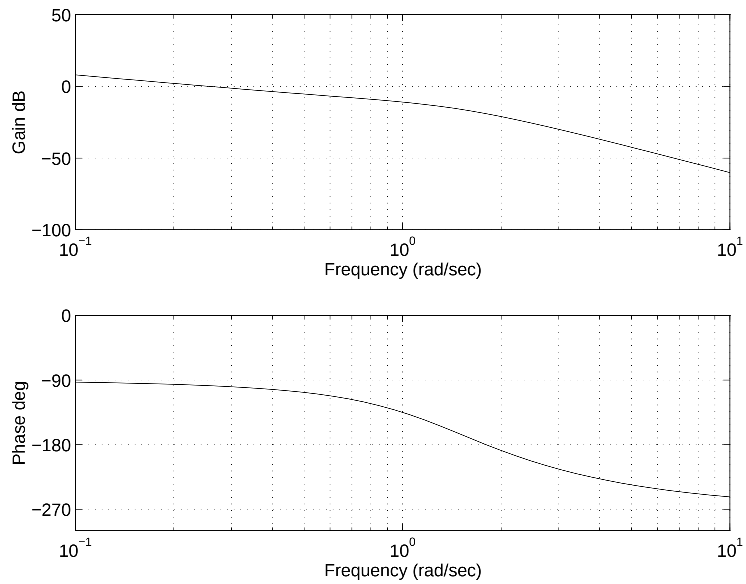

Figure 9.10: Bode diagrams for Example 9.1

Example 9.2: Consider the following open-loop frequency transfer function

$$G(j\omega)H(j\omega) = \frac{10(j\omega + 4)}{(j\omega + 1)(j\omega + 2)(j\omega + 3)}$$

The corresponding Bode diagrams obtained by following the same MATLAB instructions as in Example 9.1 are given in Figure 9.11. The phase and gain margins and the phase and gain crossover frequencies are obtained as

$$Gm = \infty, \quad Pm = 43.1488^\circ, \quad \omega_{cp} = \infty, \quad \omega_{cg} = 3.0576 \, \mathrm{rad/s}$$

◇

9.3.2 Steady State Errors and Bode Diagrams

Steady state errors can be indirectly determined from Bode diagrams by reading the values for constants K_p, K_v, K_a from them. Knowing these constants, the corresponding errors are easily found by using formulas (6.30), (6.32), and (6.34). The steady state errors and corresponding constants K_p, K_v, K_a are first of all determined by the system type, which represents the multiplicity of the pole at the origin of the open-loop feedback transfer function, in general, represented by

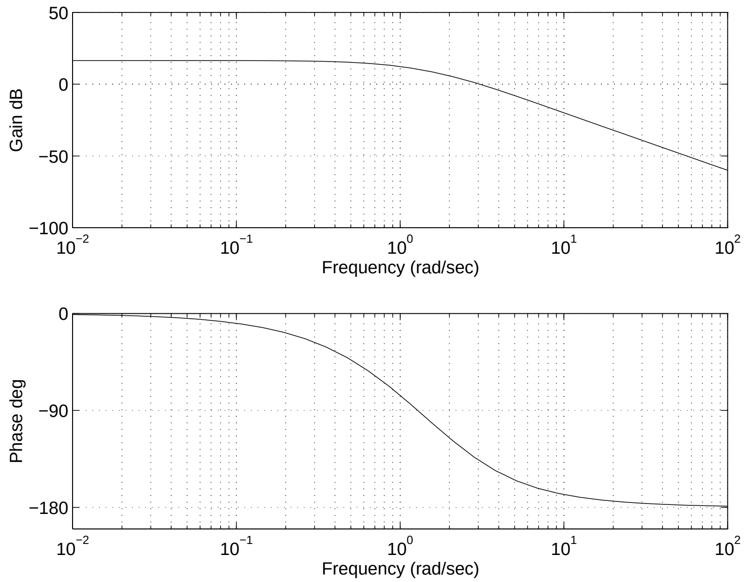

Figure 9.11: Bode diagrams for Example 9.2

$$G(j\omega)H(j\omega) = \frac{K(j\omega + z_1)(j\omega + z_2)\ \cdots}{(j\omega)^r(j\omega + p_1)(j\omega + p_2)\ \cdots} \tag{9.16}$$

This can be rewritten as

$$\begin{aligned} G(j\omega)H(j\omega) &= \frac{K z_1 z_2 \cdots \left(1 + \frac{j\omega}{z_1}\right)\left(1 + \frac{j\omega}{z_2}\right)\cdots}{p_1 p_2 \cdots (j\omega)^r \left(1 + \frac{j\omega}{p_1}\right)\left(1 + \frac{j\omega}{p_2}\right)\ \cdots} \\ &= \frac{K_B\left(1 + \frac{j\omega}{z_1}\right)\left(1 + \frac{j\omega}{z_2}\right)\ \cdots}{(j\omega)^r\left(1 + \frac{j\omega}{p_1}\right)\left(1 + \frac{j\omega}{p_2}\right)\ \cdots} \end{aligned} \tag{9.17}$$

where

$$K_B = \frac{K z_1 z_2\ \cdots}{p_1 p_2\ \cdots} \tag{9.18}$$

is known as *Bode's gain*, and r is the type of feedback control system.

For control systems of type $r = 0$, the position constant according to formula (6.31) is obtained from (9.17) as

$$K_p = \frac{K_B\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\ \cdots}{(j\omega)^0\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)\ \cdots}\ \Big|_{j\omega=0} = K_B \tag{9.19}$$

It follows from (9.17)–(9.19) that the corresponding magnitude Bode diagram of type zero control systems for small values of ω is flat (has a slope of $0\,\mathrm{dB}$) and the value of $20\log K_B = 20\log K_p$. This is graphically represented in Figure 9.12.

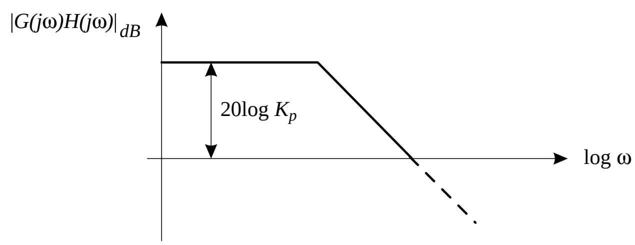

Figure 9.12: Magnitude Bode diagram of type zero control systems at small frequencies

For control systems of type $r = 1$, the open-loop frequency transfer function is approximated at low frequencies by

$$\frac{K_B\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\cdots}{(j\omega)^1\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)\cdots} \approx \frac{K_B}{(j\omega)^1} \tag{9.20}$$

It follows that the corresponding magnitude Bode diagram of type one control systems for small values of ω has a slope of $-20\,\mathrm{dB/decade}$ and the values of

$$20\log\left|\frac{K_B}{jw}\right| = 20\log|K_B| - 20\log|\omega| \tag{9.21}$$

From (9.20) and (6.33) it is easy to conclude that for type one control systems the velocity constant is $K_v = K_B$. Using this fact and the frequency plot of

(9.21), we conclude that K_v is equal to the frequency ω^* at which the line (9.21) intersects the frequency axis, that is

$$0 = 20\log|K_B| - 20\log|\omega^*| \quad \Rightarrow \quad K_B = \omega^* = K_v \tag{9.22}$$

This is graphically represented in Figure 9.13.

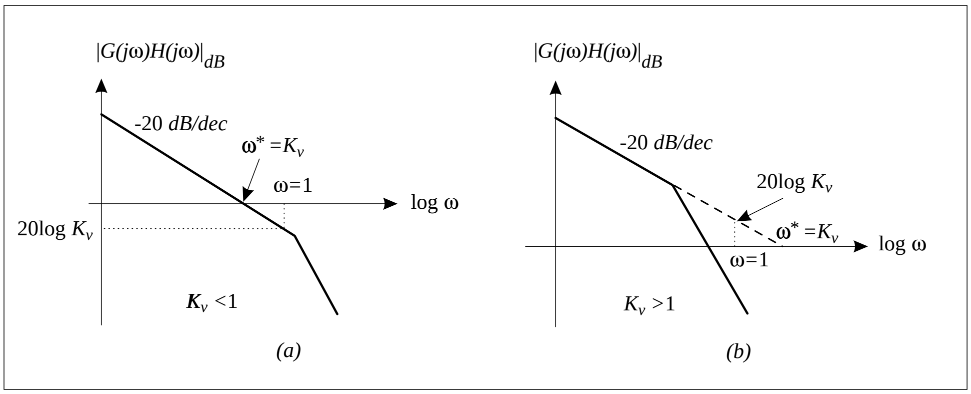

Figure 9.13: Magnitude Bode diagram of type one control systems at small frequencies

Note that if $K_v = \omega^* > 1$, the corresponding frequency ω^* is obtained at the point where the extended initial curve, which has a slope of $-20\,\mathrm{dB/decade}$, intersects the frequency axis (see Figure 9.13b).

Similarly, for type two control systems, $r = 2$, we have at low frequencies

$$\frac{K_B\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\ \cdots}{(j\omega)^2\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)\ \cdots} \approx \frac{K_B}{(j\omega)^2} \tag{9.23}$$

which indicates an initial slope of $-40\ \mathrm{dB/decade}$ and a frequency approximation of

$$20\log\left|\frac{K_B}{(jw)^2}\right| = 20\log|K_B| - 20\log\left|\omega^2\right| = 20\log|K_B| - 40\log\omega \tag{9.24}$$

From (9.23) and (6.35) it is easy to conclude that for type two control systems the acceleration constant is $K_a = K_B$. From the frequency plot of the straight line (9.24), it follows that $K_B = (\omega^{**})^2$, where ω^{**} represents the intersection of the initial magnitude Bode plot with the frequency axis as represented in Figure 9.14.

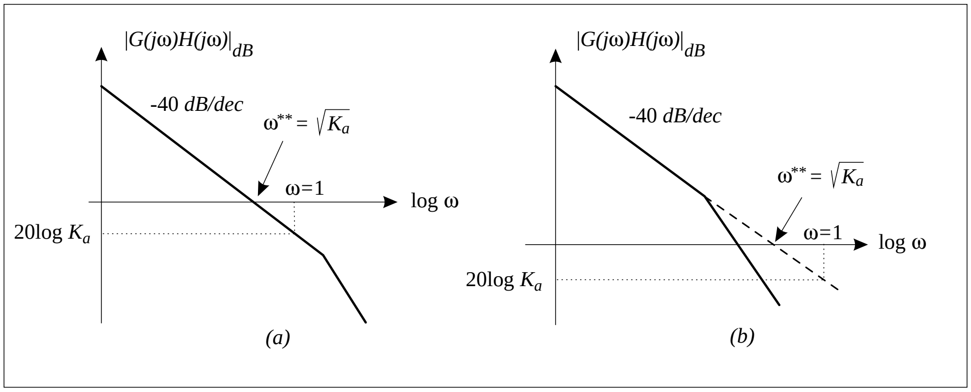

Figure 9.14: Magnitude Bode diagram of type two control systems at small frequencies

It can be seen from Figures 9.12–9.14 that by increasing the values for the magnitude Bode diagrams at low frequencies (i.e. by increasing K_B), the constants K_p, K_v, and K_a are increased. According to the formulas for steady state errors, given in (6.30), (6.32), and (6.34) as

$$e_{ss_{step}} = \frac{1}{1+K_p}, \quad e_{ss_{ramp}} = \frac{1}{K_v}, \quad e_{ss_{parabolic}} = \frac{2}{K_a}$$

we conclude that in this case the steady state errors are decreased. Thus, *the bigger K_B, the smaller the steady state errors.*

Example 9.3: Consider Bode diagrams obtained in Examples 9.1 and 9.2. The Bode diagram in Figure 9.10 has an initial slope of $-20\,\mathrm{dB/decade}$ which intersects the frequency axis at roughly $\omega^* = 0.2\,\mathrm{rad/s}$. Thus, we have for the Bode diagram in Figure 9.10

$$K_p = \infty, \quad K_v \approx 0.2, \quad K_a = 0$$

Using the exact formula for K_v, given by (6.33), we get

$$K_v = \lim_{s \to 0} \left\{ s \frac{(s+1)}{s(s+2)(s^2+2s+2)} \right\} = 0.25$$

In Figure 9.11 the initial slope is $0\,\mathrm{dB}$, and hence we have from this diagram

$$20\log K_p \approx 15 \quad \Rightarrow \quad K_p \approx 5.62, \quad K_v = 0, \quad K_a = 0$$

Using the exact formula for K_p as given by (6.31) produces

$$K_p = \lim_{s\to 0}\left\{\frac{10(s+4)}{(s+1)(s+2)(s+3)}\right\} = 6.67$$

Note that the accurate results about steady state error constants are obtained easily by using the corresponding formulas; hence the Bode diagrams are used only for quick and rough estimates of these constants.

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9.4 Compensator Design Using Bode Diagrams

In this section we show how to use Bode diagrams in order to design controllers such that the closed-loop system has the desired specifications. Three main types of controllers—phase-lead, phase-lag, and phase-lag-lead controllers—have been introduced in the time domain in Chapter 8. Here we give their interpretation in the frequency domain.

We present the design procedure for the general controllers mentioned above. Similar and simpler procedures can be developed for PD, PI, and PID controllers. After mastering the design with phase-lead, phase-lag, and phase-lag-lead controllers, students will be able to propose their own algorithms for PD, PI, and PID controllers.

Controller design techniques in the frequency domain will be governed by the following facts:

(a) Steady state errors are improved by increasing Bode's gain K_B.
(b) System stability is improved by increasing phase and gain margins.
(c) Overshoot is reduced by increasing the phase stability margin.
(d) Rise time is reduced by increasing the system's bandwidth.

However, very often it is not possible to satisfy all of these requirements at the same time, and control engineers have to compromise between several contradicting requirements.

The first two items, (a) and (b), have been already clarified. In order to justify item (c), we consider the open-loop transfer function of a second-order system given by

$$G(j\omega)H(j\omega) = \frac{\omega_n^2}{(j\omega)(jw + 2\zeta\omega_n)} \tag{9.25}$$

whose gain crossover frequency can be easily found from

$$|G(j\omega_{cg})H(j\omega_{cg})| = \frac{\omega_n^2}{\omega\sqrt{\omega^2 + 4\zeta^2\omega_n^2}} = 1 \tag{9.26}$$

leading to

$$\omega_{cg} = \omega_n\sqrt{\sqrt{1+4\zeta^2} - 2\zeta^2} \tag{9.27}$$

The phase of (9.25) at the gain crossover frequency is

$$\arg\{G(j\omega_{cg})H(j\omega_{cg})\} = -90^\circ - \tan^{-1}\frac{\omega_{cg}}{2\zeta\omega_n} \tag{9.28}$$

so that the corresponding phase margin becomes

$$Pm = \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^2} - 2\zeta^2}} = Pm(\zeta) \tag{9.29}$$

Plotting the function $Pm(\zeta)$, it can be shown that it is a monotonically increasing function with respect to ζ; we therefore conclude that *the higher phase margin, the larger the damping ratio, which implies the smaller the overshoot.*

Item (d) cannot be analytically justified since we do not have an analytical expression for the response rise time. However, it is very well known from undergraduate courses on linear systems and signals that rapidly changing signals have a wide bandwidth. Thus, *systems that are able to accommodate fast signals must have a wide bandwidth*.

In the remainder of this section, we first present standard controllers (phase-lag, phase-lead, and phase-lag-lead) in the frequency domain, and then show how to use these in order to achieve the desired system specifications. Each design technique will be given in an algorithmic form, and each will be demonstrated by an example.

9.4.1 Phase-Lag Controller

The transfer function of a phase-lag controller is given by

$$G_{lag}(j\omega) = \left(\frac{p_1}{z_1}\right)\frac{z_1 + j\omega}{p_1 + j\omega} = \frac{1 + j\frac{\omega}{z_1}}{1 + j\frac{\omega}{p_1}}, \quad z_1 > p_1 \tag{9.30}$$

The corresponding magnitude and phase frequency diagrams for a phase-lag controller are presented in Figure 9.15.

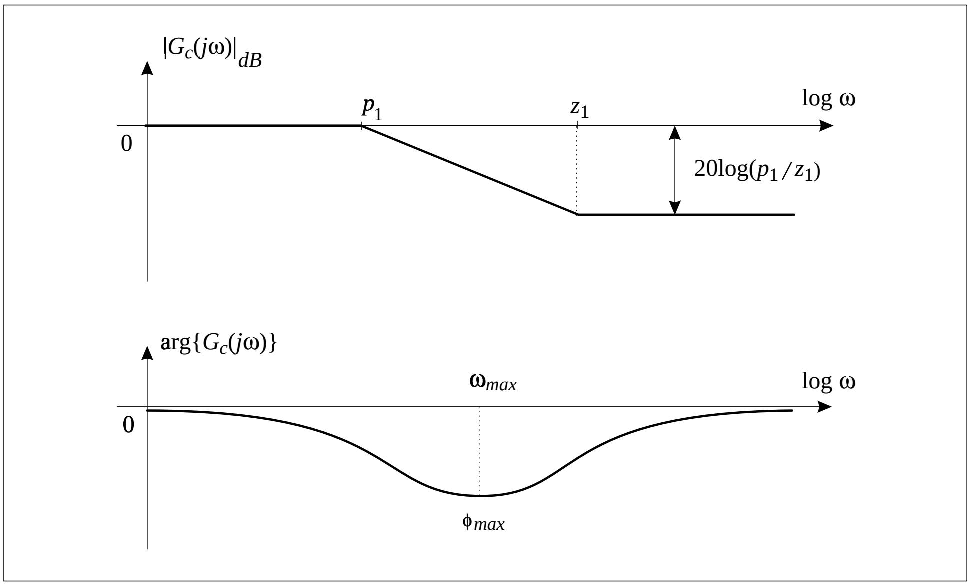

Figure 9.15: Magnitude approximation and exact phase of a phase-lag controller

Note that for the magnitude diagram it is sufficient to use only the straightline approximations for a complete understanding of the role of this controller. In general, straightline approximations can be used for almost all Bode magnitude diagrams in controller design problems. However, phase Bode diagrams are very sensitive to changes in frequency in the neighborhood of the corner frequencies, and so should be drawn as accurately as possible.

Due to attenuation of the phase-lag controller at high frequencies, the frequency bandwidth of the compensated system (controller and system in series) is reduced. Thus, *the phase-lag controllers are used in order to decrease the* *system bandwidth (to slow down the system response).* In addition, they can be used *to improve the stability margins (phase and gain) while keeping the steady state errors constant*.

Expressions for ω_{max} and ϕ_{max} of a phase-lag controller will be derived in the next subsection in the context of the study of a phase-lead controller. As a matter of fact, both types of controllers have the same expressions for these two important design quantities.

9.4.2 Phase-Lead Controller

The transfer function of a phase-lead controller is

$$G_{lead}(j\omega) = \left(\frac{p_2}{z_2}\right)\frac{z_2 + j\omega}{p_2 + j\omega} = \frac{1 + j\frac{\omega}{z_2}}{1 + j\frac{\omega}{p_2}}, \qquad p_2 > z_2 \tag{9.31}$$

and the corresponding magnitude and phase Bode diagrams are shown in Figure 9.16.

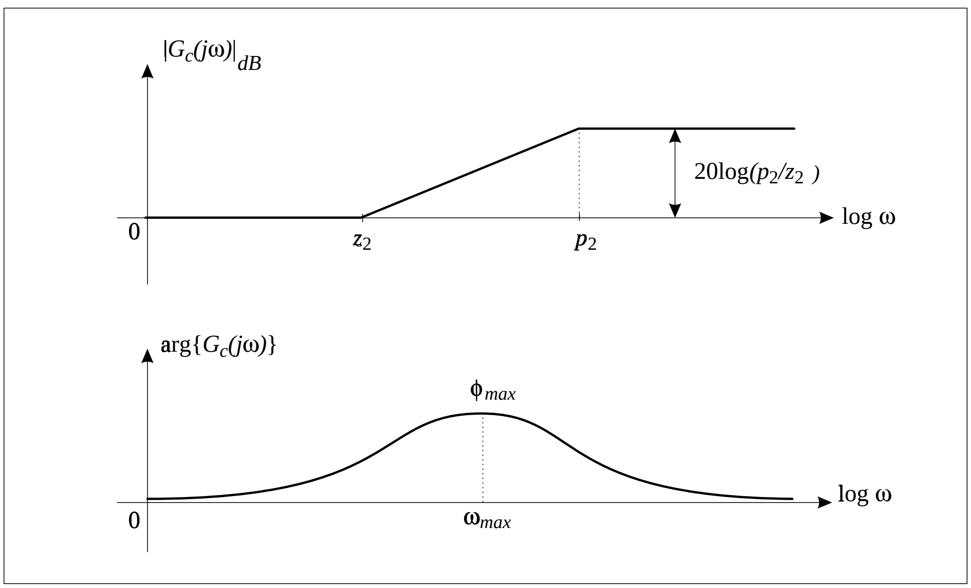

Figure 9.16: Magnitude approximation and exact phase of a phase-lead controller

Due to phase-lead controller (compensator) amplification at higher frequencies, it increases the bandwidth of the compensated system. *The phase-lead controllers* *are used to improve the gain and phase stability margins and to increase the system bandwidth (decrease the system response rise time).*

It follows from (9.31) that the phase of a phase-lead controller is given by

$$\arg\{G_{lead}(j\omega)\} = \tan^{-1}\left(\frac{\omega}{z_2}\right) - \tan^{-1}\left(\frac{\omega}{p_2}\right) \tag{9.32}$$

so that

$$\frac{d}{d\omega}\arg\{G_{lead}(j\omega)\} = 0 \;\Rightarrow\; \omega_{max} = \sqrt{z_2 p_2} \tag{9.33}$$

Assume that

$$p_2 = a z_2, \quad a > 1 \quad \Rightarrow \quad \omega_{max} = \frac{p_2}{\sqrt{a}} \tag{9.34}$$

Substituting ω_{max} in (9.32) implies

$$\tan\phi_{max} = \frac{a-1}{2\sqrt{a}} \tag{9.35}$$

It is left as an exercise for students to give detailed derivations of formula (9.35)—see Problem 9.3.

It is easy to find, from (9.35), that the value for parameter a in terms of ϕ_{max} is given by

$$a = \frac{1+\sin\phi_{max}}{1-\sin\phi_{max}} \tag{9.36}$$

Note that the same formulas for ω_{max}, (9.33), and the parameter a, (9.36), hold for a *phase-lag controller* with p_1, z_1 replacing p_2, z_2 and with $p_1 = a z_1$, $a < 1$.

9.4.3 Phase-Lag-Lead Controller

The phase-lag-lead controller has the features of both phase-lag and phase-lead controllers and can be used to improve both the transient response and steady state errors. However, its design is more complicated than the design of either phase-lag or phase-lead controllers. The frequency transfer function of the phase-lag-lead controller is given by

$$
\begin{aligned}
G_c(j\omega) &= \frac{(j\omega + z_1)(j\omega + z_2)}{(j\omega + p_1)(j\omega + p_2)} = \frac{z_1 z_2}{p_1 p_2} \frac{\left(1 + j\frac{\omega}{z_1}\right)\left(1 + j\frac{\omega}{z_2}\right)}{\left(1 + j\frac{\omega}{p_1}\right)\left(1 + j\frac{\omega}{p_2}\right)} \\
&= \frac{\left(1 + j\frac{\omega}{z_1}\right)\left(1 + j\frac{\omega}{z_2}\right)}{\left(1 + j\frac{\omega}{p_1}\right)\left(1 + j\frac{\omega}{p_2}\right)}, \qquad z_1 z_2 = p_1 p_2, \qquad p_2 > z_2 > z_1 > p_1
\end{aligned}
\tag{9.37}
$$

The Bode diagrams of this controller are shown in Figure 9.17.

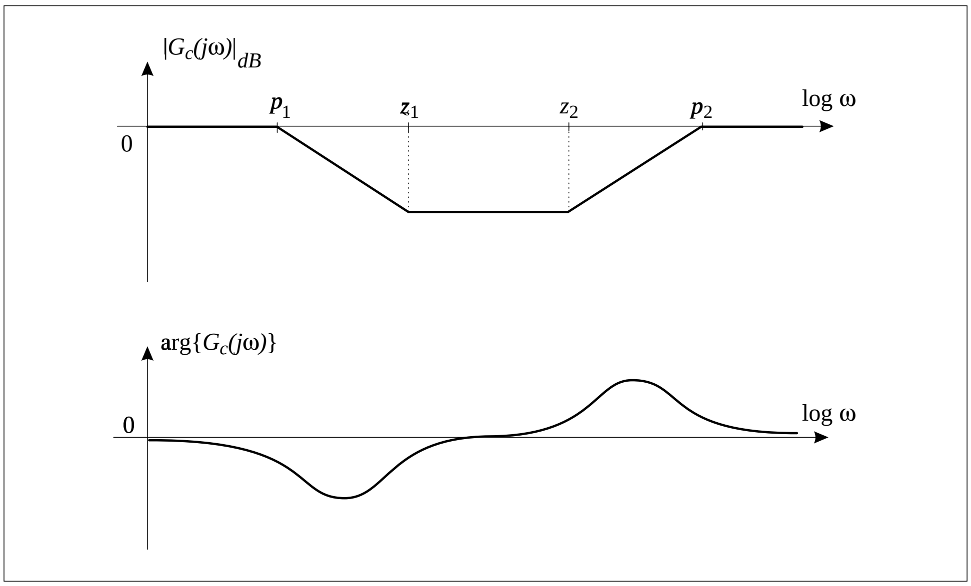

Figure 9.17: Bode diagrams of a phase-lag-lead controller

9.4.4 Compensator Design with Phase-Lead Controller

The following algorithm can be used to design a controller (compensator) with a phase-lead network.

Algorithm 9.1:

1. Determine the value of the Bode gain K_B given by (9.18) as

$$K_B = \frac{K z_1 z_2 \cdots}{p_1 p_2 \cdots}$$

such that the steady state error requirement is satisfied.

2. Find the phase and gain margins of the original system with K_B determined in step 1.

3. Find the phase difference, $\Delta\phi$, between the actual and desired phase margins and take ϕ_{max} to be $5°$–$10°$ greater than this difference. Only in rare cases should this be greater than $10°$. This is due to the fact that we have to give an estimate of a new gain crossover frequency, which can not be determined very accurately (see step 5).

4. Calculate the value for parameter a from formula (9.36), i.e. by using

$$a = \frac{1 + \sin\phi_{max}}{1 - \sin\phi_{max}} > 1$$

5. Estimate a value for a compensator's pole such that ω_{max} is roughly located at the new gain crossover frequency, $\omega_{max} \approx \omega_{cgnew}$. As a rule of thumb, add the gain of $\Delta G = 20\log(a)[\mathrm{dB}]$ at high frequencies to the uncompensated system and estimate the intersection of the magnitude diagram with the frequency axis, say ω_1. The new gain crossover frequency is somewhere in between the old ω_{cg} and ω_1. Some authors (Kuo, 1991) suggest fixing the new gain crossover frequency at the point where the magnitude Bode diagram has the value of $-0.5\Delta G[\mathrm{dB}]$. Using the value for parameter a obtained in step 4 find the value for the compensator pole from (9.34) as $-p_c = -\omega_{max}\sqrt{a}$ and the value for compensator's zero as $-z_c = -p_c/a$. Note that one can also guess a value for p_c and then evaluate z_c and ω_{max}. The phase-lead compensator now can be represented by

$$G_c(s) = \frac{as + p_c}{s + p_c}$$

6. Draw the Bode diagram of the given system with controller and check the values for the gain and phase margins. If they are satisfactory, the controller design is done, otherwise repeat steps 1–5.

The next example illustrates controller design using Algorithm 9.1.

Example 9.4: Consider the following open-loop frequency transfer function

$$G(j\omega)H(j\omega) = \frac{K(j\omega+6)}{(j\omega+1)(j\omega+2)(j\omega+3)}$$

Step 1. Let the design requirements be set such that the steady state error due to a unit step is less than 2% and the phase margin is at least 45°. Since

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+K_B}, \quad K_B = \frac{K \times 6}{1 \times 2 \times 3} = K$$

we conclude that $K \geq 50$ will satisfy the steady state error requirement of being less than 2%. We know from the root locus technique that high static gains can damage system stability, and so for the rest of this design problem we take $K = 50$.

Step 2. We draw Bode diagrams of the uncompensated system with the Bode gain obtained in step 1 and determine the phase and gain margins and the crossover frequencies. This can be done by using the following sequence of MATLAB functions.

```
[den]=input('enter denominator');
% for this example [den]=[1 6 11 6];
[num]=input('enter numerator');
% for this example [num]=[50 300];
[Gm,Pm,wcp,wcg]=margin(num,den);
bode(num,den)
```

The corresponding Bode diagrams are presented in Figure 9.18a. The phase and gain margins are obtained as $Gm = \infty$, $Pm = 5.59^\circ$ and the crossover frequencies are $\omega_{cg} = 7.5423\,\mathrm{rad/s}$, $\omega_{cp} = \infty$.

Step 3. Since the desired phase is well above the actual one, the phase-lead controller must make up for $45^\circ - 5.59^\circ = 39.41^\circ$. We add 10°, for the reason explained in step 3 of Algorithm 9.1, so that $\phi_{max} = 49.41^\circ$. The above operations can be achieved by using MATLAB as follows

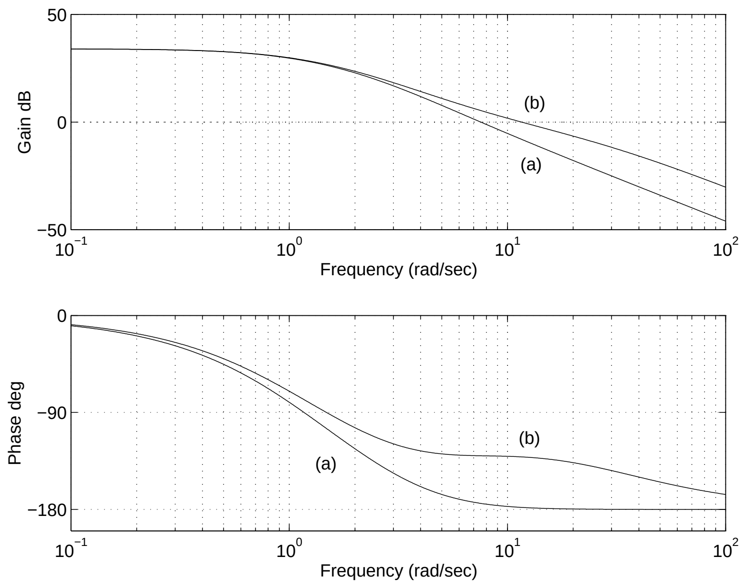

Figure 9.18: Bode diagrams for the original system (a) and compensated system (b) of Example 9.4

```
% estimate Phimax with Pmd = desired phase margin;
Pmd=input('enter desired value for phase margin');
Phimax=Pmd-Pm+10;
% converts Phimax into radians;
Phirad=(Phimax/180)*pi;
```

Step 4. Here we evaluate the parameter a according to the formula (9.36) and get $a = 7.3144$. This can be done in MATLAB by

```
a=(1+sin(Phirad))/(1-sin(Phirad));
```

Step 5. In order to obtain an estimate for the new gain crossover frequency we first find the controller amplification at high frequencies, which is equal to $20\log(a) = 17.2836\,\mathrm{dB} = \Delta G_{dB}$. The magnitude Bode diagram increased by ΔG_{dB} at high frequencies intersects the frequency axis at $\omega_{max} \approx 10.5\,\mathrm{rad/s}$. We guess (estimate) the value for p_c as $p_c = 25$, which is roughly equal to $\omega_{max}\sqrt{a}$. By using $p_c = 25$ and forming the corresponding compensator, we get for the compensated system $Pmc = 48.2891°$ at $\omega_{cgnew} = 13.8519\,\mathrm{rad/s}$, which is satisfactory. This step can be performed by MATLAB as follows.

```
% Find amplification at high frequencies, DG;
DG=20*log10(a);
% estimate value for pole –pc from Step 5;
pc=input('enter estimated value for pole pc');
% form compensator's numerator;
nc=[a pc];
% form compensator's denominator;
dc=[1 pc];
% find the compensated system transfer function;
numc=conv(num,nc);
denc=conv(den,dc);
[Gmc,Pmc,wcp,wcg]=margin(numc,denc);
bode(numc,denc)
```

The phase-lead compensator obtained is given by

$$G_c(s) = \frac{7.3144s + 25}{s + 25} = \frac{as + p_2}{s + p_2}$$

Step 6. The Bode diagrams of the compensated control system are presented in Figure 9.18b. Both requirements are satisfied, and therefore the controller design procedure is successfully completed.

It is interesting to compare the transient response characteristics of the compensated and uncompensated systems. This cannot be easily done analytically since the orders of both systems are greater than two, but it can be simply performed by using MATLAB. Note that `num`, `den`, `numc`, `denc` represent, respectively, the numerators and denominators of the open-loop transfer functions of the original and compensated systems. In order to find the corresponding closed-loop transfer functions, we use the MATLAB function `cloop`, that is

```
[cnum,cden]=cloop(num,den,-1);
% –1 indicates a negative unit feedback
[cnumc,cdenc]=cloop(numc,denc,-1);
```

The closed-loop step responses are obtained by

```
[y,x]=step(cnum,cden);
[yc,xc]=step(cnumc,cdenc);
```

and are represented in Figure 9.19. It can be seen from this figure that both the maximum percent overshoot and the settling time are drastically reduced. In addition, the rise time of the compensated system is shortened since the phase-lead controller increases the frequency bandwidth of the system.

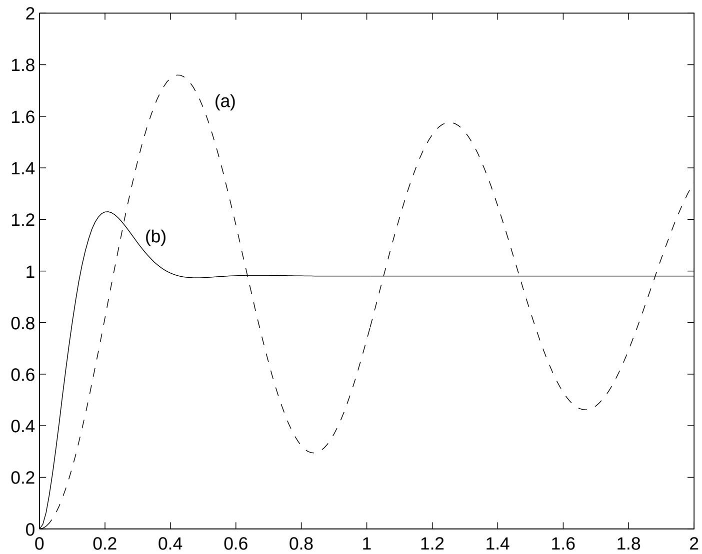

Figure 9.19: Step responses for the original (a) and compensated (b) systems

The highly oscillatory behavior of the step response of the original system can be fully understood from its root locus, which is given in Figure 9.20. It can be seen that the real part of a pair of complex conjugate poles is very small for almost all values of the static gain, which causes high-frequency oscillations and very slow convergence to the response steady state value.

The closed-loop eigenvalues of the original and compensated systems, obtained by MATLAB functions `roots(cden)` and `roots(cdenc)`, are given by

$$\lambda_1 = -5.3288, \;\; \lambda_{2,3} = -0.3356 \pm j7.5704$$
$$\lambda_{1,2c} = -9.6736 \pm j13.3989, \;\; \lambda_{3c} = -8.2628, \;\; \lambda_{4c} = -3.3900$$

which indicates a big difference in the real parts of complex conjugate poles for the original and compensated systems.

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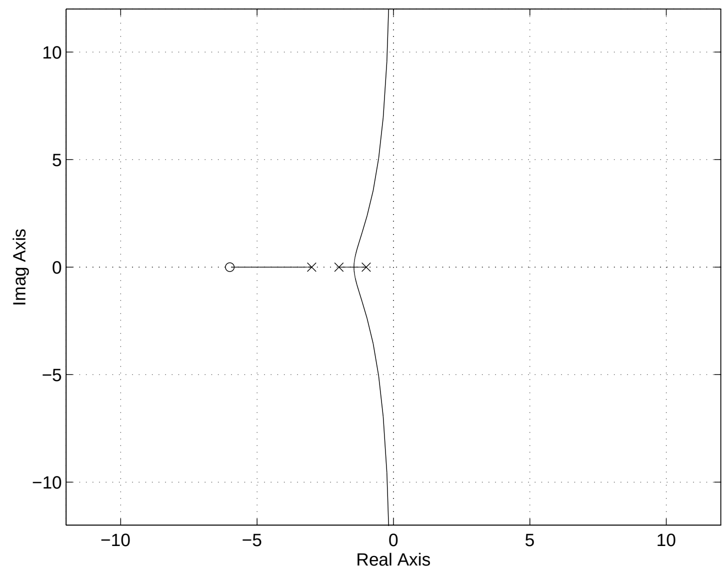

Figure 9.20: Root locus of the original system

9.4.5 Compensator Design with Phase-Lag Controller

Compensator design using phase-lag controllers is based on the compensator's attenuation at high frequencies, which causes a shift of the gain crossover frequency to the lower frequency region where the phase margin is high. The phase-lag compensator can be designed by the following algorithm.

Algorithm 9.2:

1. Determine the value of the Bode gain K_B that satisfies the steady state error requirement.

2. Find on the phase Bode plot the frequency which has the phase margin equal to the desired phase margin increased by $5°$ to $10°$. This frequency represents the new gain crossover frequency, ω_{cgnew}.

3. Read the required attenuation at the new gain crossover frequency, i.e. $|\Delta G(j\omega_{cgnew})|_{dB}$, and find the parameter a from

$$-20\log\left(\frac{p_1}{z_1}\right) = -20\log(a) = |\Delta G(j\omega_{cgnew})|_{dB}$$

which implies

$$a = 10^{-\frac{1}{20}|\Delta G(j\omega_{cgnew})|_{dB}} = \frac{1}{|\Delta G(j\omega_{cgnew})|}$$

Note that

$$|\Delta G(j\omega_{cgnew})| = \frac{|K||j\omega_{cgnew} + z_1||j\omega_{cgnew} + z_2|\cdots}{|j\omega_{cgnew} + p_1||j\omega_{cgnew} + p_2|\cdots}$$

4. Place the controller zero one decade to the left of the new gain crossover frequency, that is

$$z_c = \frac{\omega_{cgnew}}{10}$$

Find the pole location from $p_c = az_c = a\omega_{cgnew}/10$. The required compensator has the form

$$G_c(s) = \frac{as + p_c}{s + p_c}$$

5. Redraw the Bode diagram of the given system with the controller and check the values for the gain and phase margins. If they are satisfactory, the controller design is done, otherwise repeat steps 1–5.

Example 9.5: Consider a control system represented by

$$G(s) = \frac{K}{s(s+2)(s+30)}$$

Design a phase-lag compensator such that the following specifications are met: $e_{ss_{ramp}} \leq 0.05$, $Pm \geq 45°$. The minimum value for the static gain that produces the required steady state error is equal to $K = 1200$. The original system with this static gain has phase and gain margins given by $Pm = 6.6449°$, $Gm = 4.0824\,\mathrm{dB}$ and crossover frequencies of $\omega_{cg} = 6.1031\,\mathrm{rad/s}$, $\omega_{cp} = 7.746\,\mathrm{rad/s}$.

The new gain crossover frequency can be estimated as $\omega_{cgnew} = 1.4\,\mathrm{rad/s}$ since for that frequency the phase margin of the original system is approximately $50°$. At $\omega_{cgnew} = 1.4\,\mathrm{rad/s}$ the required gain attenuation is obtained by MATLAB as

```
wcgnew=1.4;
d1=1200;
g1=abs(j*1.4);
g2=abs(j*1.4+2);
g3=abs(j*1.4+30);
dG=d1/(g1*g2*g3);
```

which produces $|\Delta G(j1.4)| = 11.6906$ and $a = 1/|\Delta G(j1.4)| = 0.0855$. The compensator's pole and zero are obtained as $-z_c = -\omega_{cgnew}/10 = -0.14$ and $-p_c = -a\omega_{cgnew}/10 = -0.0120$ (see step 4 of Algorithm 9.2). The transfer function of the phase-lag compensator is

$$G_c(s) = \frac{0.0855s + 0.0120}{s + 0.0120}$$

The Bode diagrams of the original and compensated systems are given in Figure 9.21.

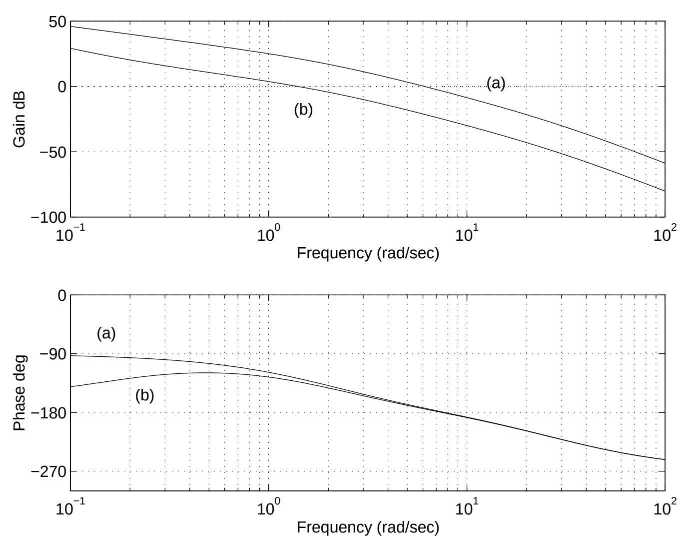

Figure 9.21: Bode diagrams for the original system (a) and compensated system (b) of Example 9.5

The new phase and gain margins and the actual crossover frequencies are $Pmc = 47.03°$, $Gmc = 24.82\,\mathrm{dB}$, $\omega_{cgnew} = 1.405\,\mathrm{rad/s}$, $\omega_{cpnew} = 7.477\,\mathrm{rad/s}$ and so the design requirements are satisfied. The step responses of the original and compensated systems are presented in Figure 9.22.

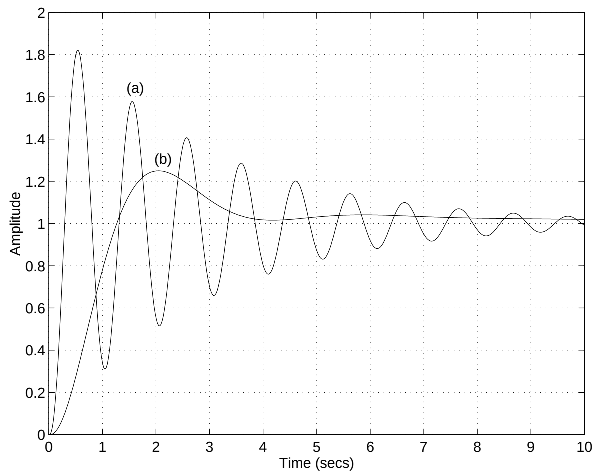

Figure 9.22: Step responses for the original system (a) and compensated system (b) of Example 9.5

It can be seen from this figure that the overshoot is reduced from roughly 0.83 to 0.3. In addition, it can be observed that the settling time is also reduced. Note that the phase-lag controller reduces the system bandwidth ($\omega_{cgnew} < \omega_{cg}$) so that the rise time of the compensated system is increased.

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9.4.6 Compensator Design with Phase-Lag-Lead Controller

Compensator design using a phase-lag-lead controller can be performed according to the algorithm given below, in which we first form a phase-lead compensator and then a phase-lag compensator. Finally, we connect them together in series. Note that several different algorithms for the phase-lag-lead controller design can be found in the control literature.

Algorithm 9.3:

1. Set a value for the static gain K_B such that the steady state error requirement is satisfied.
2. Draw Bode diagrams with K_B obtained in step 1 and find the corresponding phase and gain margins.

3. Find the difference between the actual and desired phase margins, $\Delta\phi = Pmd - Pm$, and take ϕ_{max} to be a little bit greater than $\Delta\phi$. Calculate the parameter a_2 of a phase-lead controller by using formula (9.36), that is

$$a_2 = \frac{1 + \sin \phi_{max}}{1 - \sin \phi_{max}}$$

4. Locate the new gain crossover frequency at the point where

$$20 \log |G(j\omega_{cgnew})| = -10 \log a_2 \qquad (9.38)$$

5. Compute the values for the phase-lead compensator's pole and zero from

$$p_{c2} = \omega_{cgnew}\sqrt{a_2}, \qquad z_{c2} = p_{c2}/a_2 \qquad (9.39)$$

6. Select the phase-lag compensator's zero and pole according to

$$z_{c1} = 0.1 z_{c2}, \qquad p_{c1} = z_{c1}/a_2 \qquad (9.40)$$

7. Form the transfer function of the phase-lag-lead compensator as

$$G_c(s) = G_{lag}(s) \times G_{lead}(s) = \frac{s + z_{c1}}{s + p_{c1}} \times \frac{s + z_{c2}}{s + p_{c2}}$$

8. Plot Bode diagrams of the compensated system and check whether the design specifications are met. If not, repeat some of the steps of the proposed algorithm—in most cases go back to steps 3 or 4.

The phase-lead part of this compensator helps to increase the phase margin (increases the damping ratio, which reduces the maximum percent overshoot and settling time) and broaden the system's bandwidth (reduces the rise time). The phase-lag part, on the other hand, helps to improve the steady state errors.

Example 9.6: Consider a control system that has the open-loop transfer function

$$G(s) = \frac{K(s + 10)}{(s^2 + 2s + 2)(s + 20)}$$

For this system we design a phase-lag-lead controller by following Algorithm 9.3 such that the compensated system has a steady state error of less than 4% and a phase margin greater than $50°$. In the first step, we choose a value for the static gain K that produces the desired steady state error. It is easy to check that $K = 100 \Rightarrow e_{ss} = 3.85\%$, and therefore in the following we stick with this value for the static gain. Bode diagrams of the original system with $K = 100$ are presented in Figure 9.23.

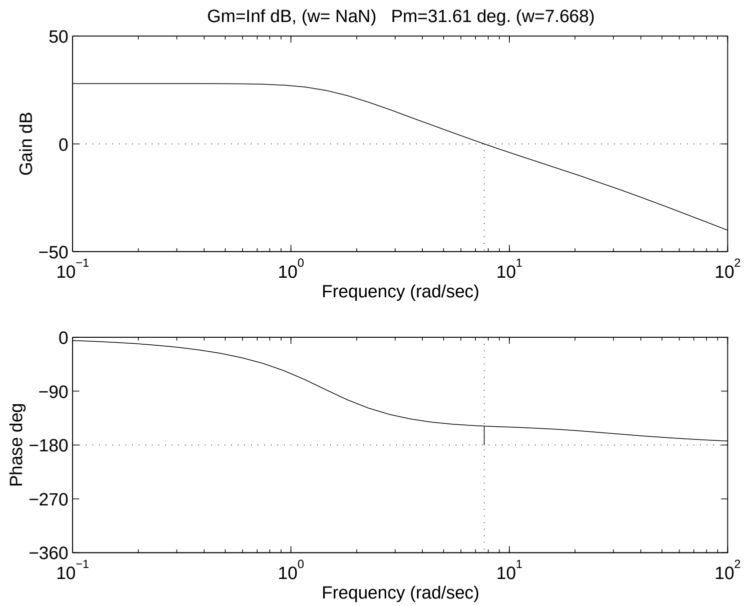

Figure 9.23: Bode diagrams of the original system

It can be seen from these diagrams—and with help of MATLAB determined accurately—that the phase and gain margins and the corresponding crossover frequencies are given by $Pm = 31.61°, Gm = \infty$, and $\omega_{cg} = 7.668\,\mathrm{rad/s}$, $\omega_{cp} = \infty$. According to step 3 of Algorithm 9.3, a controller has to introduce a phase lead of $18.39°$. We take $\phi_{max} = 25°$ and find the required parameter $a_2 = 2.4639$. Taking $\omega_{cgnew} = 20\,\mathrm{rad/s}$ in step 4 and completing the design steps 5–8 we find that $Pm = 39.94°$, which is not satisfactory. We go back to step 3 and take $\phi_{max} = 30° = 0.5236\,\mathrm{rad}$, which implies $a_2 = 3$.

Step 4 of Algorithm 9.3 can be executed efficiently by MATLAB by performing the following search. Since $-10\log 3 = -10.9861$ dB we search the magnitude diagram for the frequency where the attenuation is approximately equal to -11 dB. We start search at $\omega = 20\,\mathrm{rad/s}$ since at that point, according to Figure 9.23, the attenuation is obviously smaller than -11 dB. The following

MATLAB program is used to find the new gain crossover frequency, i.e. to solve approximately equation (9.38)

```
w=20;
while 20*log10(100*abs(j*w+10)/
abs(((j*w)^2+2*j*w+2)*(j*w+20)))<-11;
w=w-1;
end
```

This program produces $\omega_{cgnew} = 10\,\mathrm{rad/s}$. In steps 5 and 6 the phase-lag-lead controller zeros and poles are obtained as $-p_{c2} = -17.3205,\ -z_{c2} = -5.7735$ for the phase-lead part and $-p_{c1} = -0.1925,\ -z_{c1} = -0.5774$ for the phase-lag part; hence the phase-lag-lead controller has the form

$$G_c(s) = \frac{s+0.5774}{s+0.1925} \times \frac{s+17.3205}{s+5.7735}$$

The Bode diagrams of the compensated system are given in Figure 9.24.

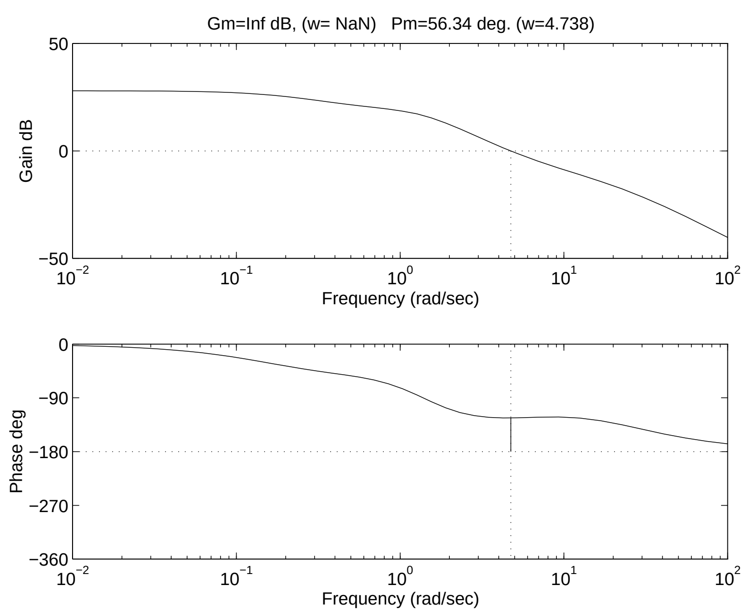

Figure 9.24 Bode diagrams of the compensated system

It can be seen that the phase margin obtained of $56.34°$ meets the design requirement and that the actual gain crossover frequency, $4.738\,\mathrm{rad/s}$, is considerably smaller than the one predicted. This contributes to the generally accepted inaccuracy of frequency methods for controller design based on Bode diagrams.

The step responses of the original and compensated systems are compared in Figure 9.25. The transient response of the compensated system is improved since the maximum percent overshoot is considerably reduced. However, the system rise time is increased due to the fact that the system bandwidth is shortened ($\omega_{cgnew} = 4.738\,\mathrm{rad/s} < \omega_{cg} = 7.668\,\mathrm{rad/s}$).

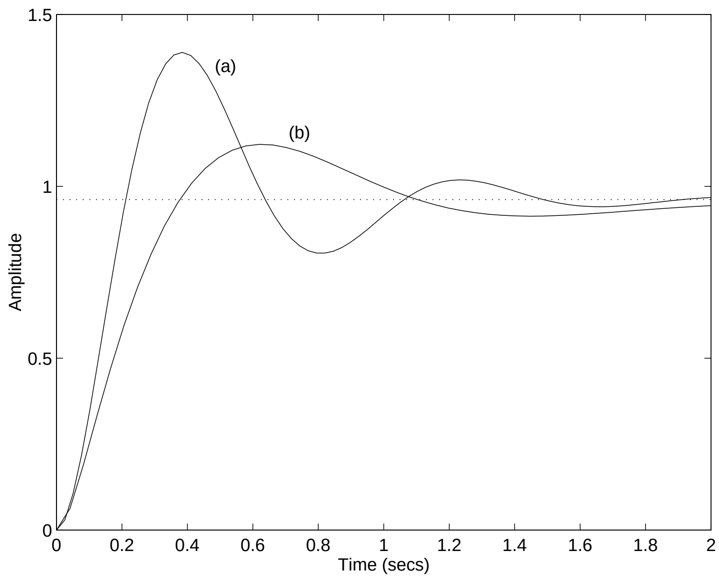

Figure 9.25 Step responses of the original (a) and compensated (b) systems

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9.5 MATLAB Case Study

Consider the problem of finding a controller for the ship positioning control system given in Problem 7.5. The goal is to increase stability phase margin above $30°$. The problem matrices are given by

$$\mathbf{A} = \begin{bmatrix} -0.0546 & 0 & 0.5435 \\ 1 & 0 & 0 \\ 0 & 0 & -1.55 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1.55 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{D} = 0$$

The transfer function of the ship positioning system is obtained by the MATLAB instruction `[num,den]=ss2tf(A,B,C,D)` and is given by

$$G(s) = \frac{0.8424}{s(s+1.55)(s+0.0546)}$$

The phase and gain stability margins of this system are $Pm = -19.94°$ and $Gm = -15.86\,\mathrm{dB}$, with the crossover frequencies $\omega_{cp} = 0.2909\,\mathrm{rad/s}$ and $\omega_{cg} = 0.7025\,\mathrm{rad/s}$ (see the Bode diagrams in Figure 9.26). From known values for the phase and gain margins, we can conclude that this system has very poor stability properties.

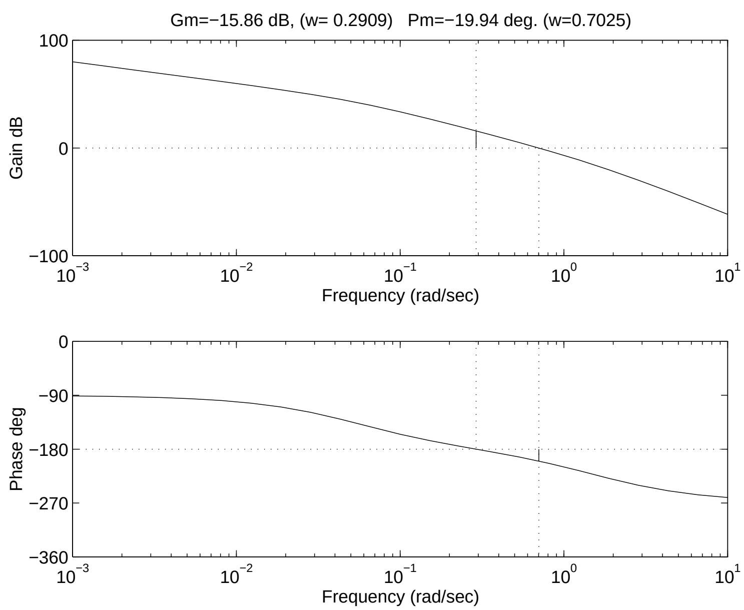

Figure 9.26: Bode diagrams of a ship positioning control system

Since the phase margin is well below the desired one, we need a controller which will make up for almost a $50°$ increase in phase. In general, it is hard to stabilize systems that have large negative phase and gain stability margins. In the following we will design phase-lead, phase-lag, and phase-lag-lead controllers to solve this problem and compare the results obtained.

Phase-Lead Controller: By using Algorithm 9.1 with $\phi_{max} = 50° + 10° = 60°$ we get a phase margin of only $23.536°$, which is not satisfactory. It is necessary to make up for $\phi_{max} = 50° + 27° = 87°$. In the latter case the compensator has the transfer function

$$G_c(s) = 76.31\frac{s + 0.2038}{s + 15.55}$$

Figure 9.27 shows Bode diagrams of both the original (a) and compensated (b) systems.

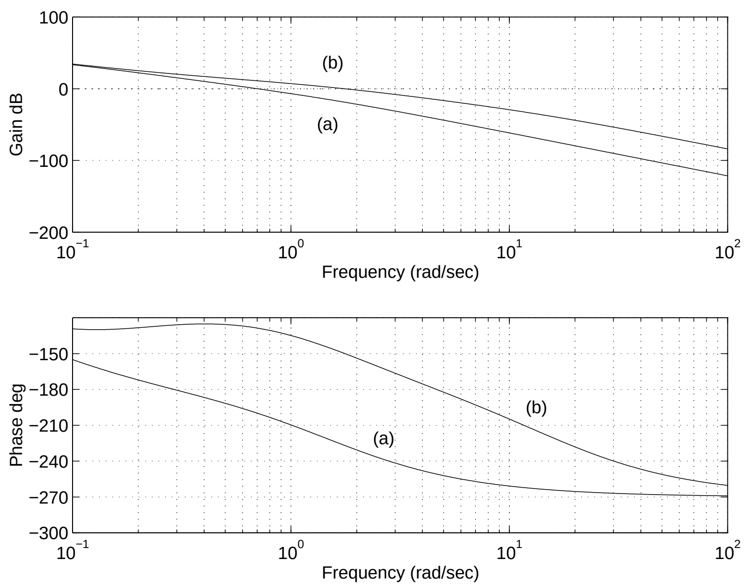

Figure 9.27: Bode diagrams for a ship positioning system: (a) original system, (b) phase-lead compensated system

The gain and phase stability margins of the compensated system are found from the above Bode diagrams as $Gmc = 15.1603\,\mathrm{dB}$, $Pmc = 30.0538°$, and the crossover frequencies are $\omega_{cpc} = 1.7618\,\mathrm{rad/s}$, $\omega_{cgc} = 4.6419\,\mathrm{rad/s}$. The step response of the compensated system exhibits an overshoot of 45.47% (see Figure 9.28).

Phase-Lag-Lead Controller: By using Algorithm 9.3 we find the compensator transfer function as

$$G_c(s) = 7.524\frac{s + 0.1599}{s + 1.203} \times 0.1329\frac{s + 0.016}{s + 0.002125}$$

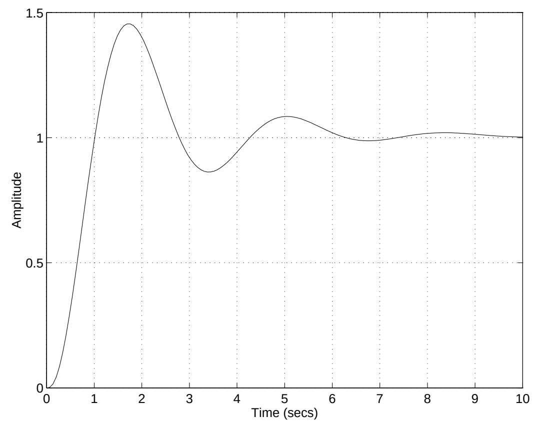

Figure 9.28: Step response of the compensated system with a phase-lead controller

The Bode diagrams of the original and compensated systems are shown in Figure 9.29.

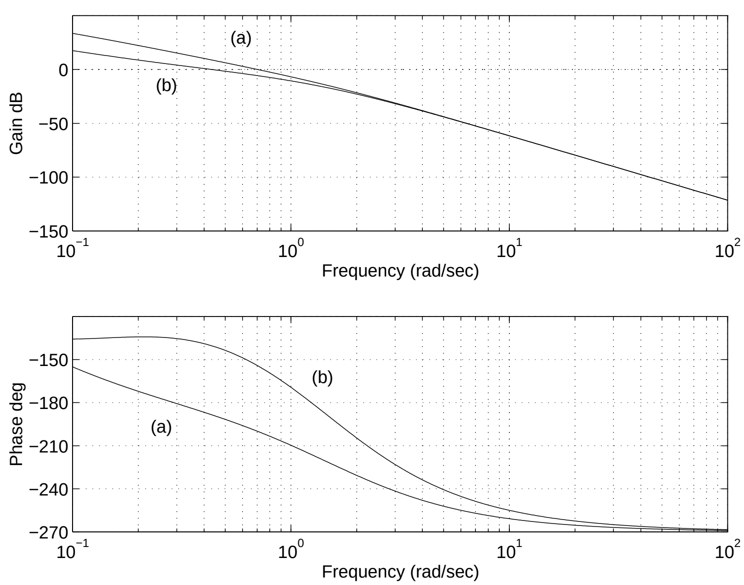

Figure 9.29: Bode diagrams for a ship positioning control system: (a) original system, (b) phase-lag-lead compensated system

The phase and gain margins of the compensated system are given by $Pmc = 39.6694°$, $Gmc = 14\,\mathrm{dB}$ and the crossover frequencies are $\omega_{cgc} = 0.4332\,\mathrm{rad/s}$, $\omega_{cpc} = 1.2401\,\mathrm{rad/s}$.

From the step response of the compensated system (see Figure 9.30), we can observe that this compensated system has a smaller overshoot and a larger rise time than the system compensated only by the phase-lead controller.

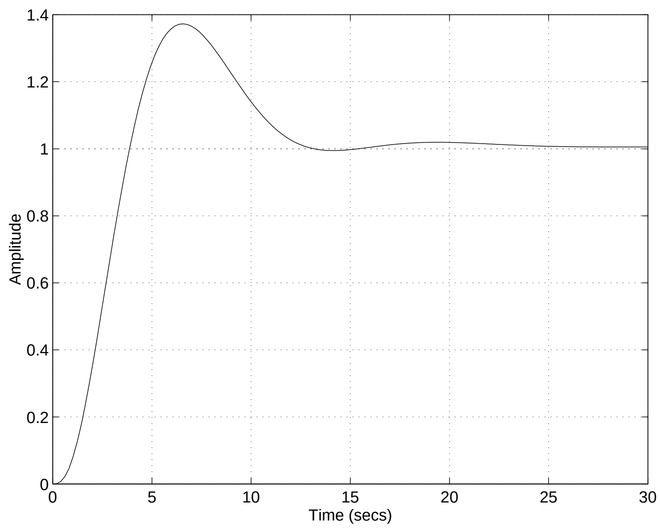

Figure 9.30: Step response of the compensated system with a phase-lag-lead controller

Phase-Lag Controller: If we choose a new gain crossover frequency at $\omega_{cgnew} = 0.03\,\mathrm{rad/s}$, the phase margin at that point will clearly be above $50°$. Proceeding with a phase-lag compensator design, according to Algorithm 9.2, we get $|\Delta G(j0.03)| = 290.7390$ and $a = 0.034$, which implies $z_c = 0.003$ and $p_c = 1.0319 \times 10^{-5}$. Using the corresponding phase-lag compensator produces very good stability margins for the compensated system, i.e. $Gm = 32.91$ dB and $Pm = 54.33°$. The maximum percent overshoot obtained is much better than with the previously used compensators and is equal to $MPOS = 18\%$. However, the closed-loop step response reveals that the obtained system is too sluggish since the response peak time is $t_p = 95.2381\,\mathrm{s}$ (note that in the previous two cases the peak time is only a few seconds).

One may try to get better agreement by designing a phase-lag compensator, which will reduce the phase margin of the compensated system to just above $30°$. In order to do this we write a MATLAB program, which searches the phase Bode diagram and finds the frequency corresponding to the prespecified value of the phase. That frequency is used as a new gain crossover frequency. Let $Pm = 35° = 0.6109\,\mathrm{rad}$. The MATLAB program is

```
w=0.1;
while pi+
angle(1/((j*w)*(j*w+1.55)*(j*w+0.0546)))<0.6109;
w=w-0.01;
end
dG=0.8424*abs(1/((j*w)*(j*w+1.55)*(j*w+0.0546)));
```

This program produces $\omega_{cgnew} = 0.07\,\mathrm{rad/s}$ and $|\Delta G(j0.07)| = 87.3677$. From step 4 of Algorithm 9.2 we obtain the phase-lag controller of the form

$$G_c(s) = \frac{s + 0.009}{s + 0.000081} \times 0.0114$$

The Bode diagrams of the compensated system are given in Figure 9.31.

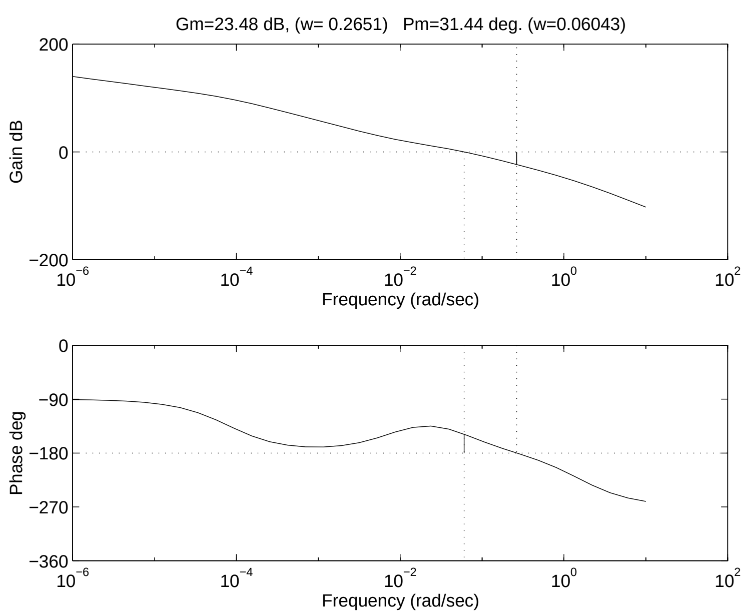

Figure 9.31: Bode diagram of the phase-lag compensated system

It can be seen that the phase and gain margins are satisfactory and given by $Pm = 31.44°$ and $Gm = 23.48\ \mathrm{dB}$. The actual gain crossover frequencies are $w_{cgnew} = 0.06043\,\mathrm{rad/s}$ and $\omega_{cpnew} = 0.2651\,\mathrm{rad/s}$.

The closed-loop step response of the phase-lag compensated system, given in Figure 9.32, shows that the peak time is reduced to $t_p = 50.15\,\mathrm{s}$—which is still fairly big—and that the maximum percent overshoot is increased to $MPOS = 45.82\%$, which is comparable to the phase-lead and phase-lag-lead compensation.

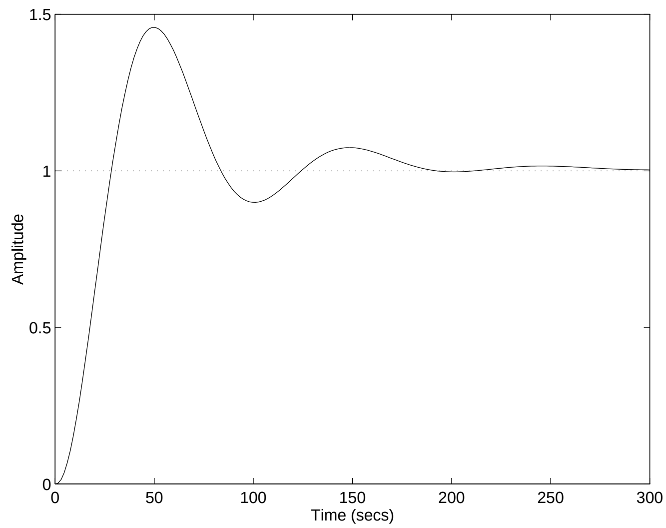

Figure 9.32: Step response of the phase-lag compensated system

Comparing all three controllers and their performances, we can conclude that, for this particular problem, the phase-lag compensation produces the worst result, and therefore either the phase-lead or phase-lag-lead controller should be used.

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9.6 Comments on Discrete-Time Controller Design

Bode diagrams were originally introduced for studying continuous-time systems (Bode, 1940). However, discrete-time systems can be studied using the same diagrams. The bilinear transformation, already used in this book in the stability study of discrete-time systems, that has the form

$$s = \frac{z-1}{z+1}, \qquad z = \frac{1+s}{1-s} \tag{9.41}$$

maps the imaginary axis from the s-plane into the unit circle in the z-plane and vice versa. Since on the unit circle $z = e^{j\omega_z T}$, it is easy to establish the relationship between angular frequencies in the s and z domains. It is left as an exercise for students to show that

$$w_s = \tan\frac{\omega_z T}{2}, \qquad \omega_z = \frac{2}{T}\tan^{-1}\omega_s \tag{9.42}$$

The above transformation allows one to map the discrete-time open-loop transfer function into the continuous-time open-loop transfer function, that is

$$G(z)_{|z=\frac{1+s}{1-s}} = G(s) \tag{9.43}$$

and to perform controller design in the continuous-time domain. The results obtained have to be mapped back into the discrete-time domain by using (9.42), that is

$$G_c(s)_{|s=\frac{z-1}{z+1}} = G_c(z) \tag{9.44}$$

Note that several bilinear transformations, which are just scaled versions of (9.41), can be found in the control literature. For more details the reader is referred, for example, to Franklin *et al.* (1990), DiStefano *et al.* (1990), and Phillips and Nagle (1995). MATLAB discrete-time controller design problems can be found in Shahian and Hassul (1993).

9.7 MATLAB Laboratory Experiment

Part 1. Consider the closed-loop system represented in Figure 9.33. This system has a transport lag-element, e^{-Ts}, which represents a time delay of T time units. The transport lag-element can be approximated for small values of time delay T by

$$\text{(i)} \quad e^{-Ts} \approx \frac{1}{1+Ts}\,, \qquad \text{(ii)} \;\; e^{-Ts} \approx 1 - Ts$$

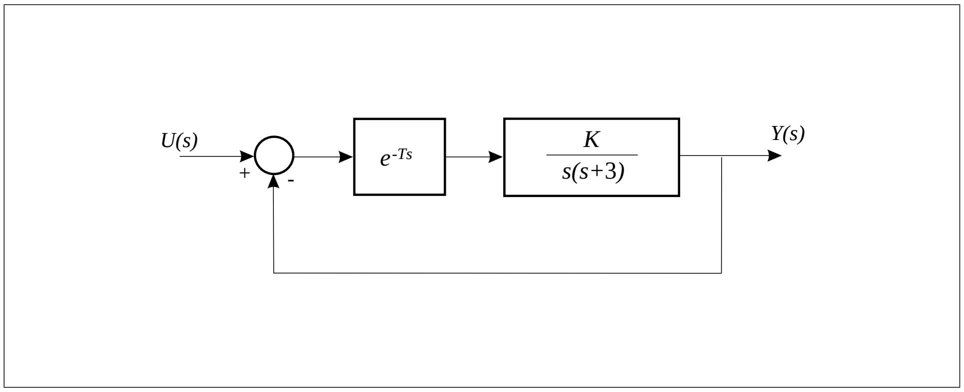

Figure 9.33: Block diagram of a control system

Using the following values for the time delay $T = 0,\ 0.01,\ 0.1$, design phase-lead, phase-lag, and phase-lag-lead compensators to meet the following closed-loop design requirements: steady state error $e_{ss_{ramp}} \leq 0.02$ and $Pm \geq 50°$. Consider both approximations (i) and (ii) and compare the results obtained.

Part 2. Draw the exact phase Bode diagrams of the systems compensated in Part 1 including the exact contribution from the time delay element. Note that the time delay element does not affect the magnitude Bode diagram, but modifies the phase Bode diagrams by a factor of $\measuredangle e^{-j\omega T}$. Compare the approximated and exact phase Bode diagrams. Draw conclusions about the impact of time delay elements on phase and gain stability margins.

Part 3. Consider the controller design problem for a system represented by its open-loop transfer function

$$G(s) = \frac{K(s+6)}{(s+10)(s^2+2s+2)}$$

This system has been studied in Example 8.9 and in the MATLAB laboratory experiment for the root locus controller design in Section 8.8.

(a) Design a phase-lag-lead controller using Bode diagrams such that the compensated system has the same specifications as those in Section 8.8, i.e. the steady state error is less than 1% and the phase margin is such that $t_s < 2\,\mathrm{s}$ and $MPOS < 10\%$. Note that the maximum percent overshoot and settling time are inversely proportional to the phase margin. Experiment with several values for the phase margin and take the one that satisfies both transient response requirements.

(b) Compare the results obtained with those from Section 8.8 and comment on the differences between root locus and Bode diagram phase-lag-lead controller design. Which one is easier to design? Which one is more accurate?

9.8 References

Bode, H., "Relations between attenuation and phase in feedback amplifier design," *Bell System Technical Journal*, vol. 19, 421–454, 1940.

DiStefano, J., A. Stubberud, and I. Williams, *Feedback and Control Systems*, McGraw-Hill, New York, 1990.

Franklin, G., J. Powel, and M. Workman, *Digital Control of Dynamic Systems*, Addison-Wesley, Reading, Massachusetts, 1990.

Kuo, B., *Automatic Control Systems*, Prentice Hall, Englewood Cliffs, New Jersey, 1991.

Phillips, C. and H. Nagle, *Digital Control System Analysis and Design*, Prentice Hall, Englewood Cliffs, New Jersey, 1995.

Shahian, B. and M. Hassul, *Control System Design with MATLAB*, Prentice Hall, Englewood Cliffs, New Jersey, 1993.

9.9 Problems

9.1 Show that for a second-order closed-loop system

$$M(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

the resonant frequency is given by

$$\omega_r = \omega_n\sqrt{1-\zeta^2}$$

and the peak resonance is

$$M_p = \frac{1}{2\zeta\sqrt{1-2\zeta^2}}$$

9.2 Show that the frequency bandwidth for a second-order closed-loop system given in Problem 9.1 is

$$\omega_{BW} = \omega_n \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

Since the 5%-settling time is given by formula (6.20) as

$$t_s \approx \frac{3}{\zeta\omega_n}$$

conclude that the settling time is inversely proportional to the system bandwidth, in other words, *the wider the system bandwidth, the shorter the settling time*.

9.3 Derive formula (9.35) for the maximum phase of a phase-lead controller.

9.4 Using MATLAB, draw Bode diagrams for a magnetic tape control system considered in Problem 5.12. Matrices $\mathbf{A}$ and $\mathbf{B}$ are given in Problem 5.12. The output matrices are

$$\mathbf{C} = [1 \quad 0 \quad 1 \quad 0], \qquad \mathbf{D} = 0$$

Find the phase and gain stability margins for this system.

9.5 Based on Algorithms 9.1–9.3, propose algorithms for controller design with

(a) a PD controller;
(b) a PI controller;
(c) a PID controller.

9.6 Solve the controller design problem defined in Example 9.5 by using both phase-lead and phase-lag-lead controllers.

9.7 Design a phase-lag-lead network for the system

$$G(s) = \frac{K}{s(s+3)}$$

such that $Pm \geq 45°$ and $e_{ss_{ramp}} \leq 0.03$.

9.8 The block diagram of a servo control system is shown in Figure 9.34.

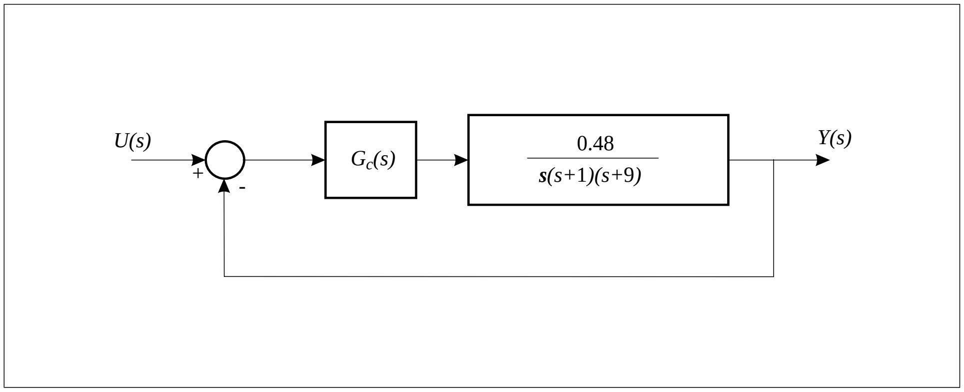

Figure 9.34: Block diagram for Problem 9.8

Design phase-lag, phase-lead, and phase-lag-lead controllers such that the phase margin is greater than $60°$.

9.9 A unit feedback system has the transfer function

$$G(s) = \frac{40K}{s(s+4)(s+10)}$$

(a) Construct Bode diagrams for $K = 1$.
(b) Using the MATLAB function `margin`, determine the phase and gain margins of the system.
(c) Determine the value for the static gain K such that the system has a gain margin of $Gm = 10\ \mathrm{dB}$. Find the corresponding steady state errors.
(d) From Bode diagrams determine the value for the static gain K such that the phase margin is $45°$. Determine the damping ratio and the natural frequency for the obtained value of K.

9.10 A zero type plant has the transfer function

$$G(s) = \frac{10}{(s+5)(s+20)}$$

(a) Determine the damping ratio and the natural frequency of the corresponding closed-loop system.

(b) Find the steady state errors and the system overshoot.
(c) Determine the phase and gain margins.
(d) Design a controller such that the compensated system has a phase margin of at least $50°$ and a steady state error less than 0.02.

9.11 A unit feedback system with transport delay has the transfer function

$$G(s) = \frac{10Ke^{-0.5s}}{(s+4)(s+10)}$$

Approximate time delay as $e^{-Ts} \approx 1/(1+Ts)$.

(a) Plot on the same figure the Bode diagrams for the system with and without transport delay for $K = 1$. Comment on system stability.
(b) Plot the unit step response of the system for both cases and compare the results (steady state errors, transient response parameters).
(c) Determine the value of the static gain K that gives a steady state unit step error of 0.02 for both the approximated system and the original system without time delay.
(d) Design phase-lag, phase-lead, and phase-lag-lead controllers to achieve a steady state error of $e_{ss} \leq 0.05$ and a phase margin $Pm \geq 50°$ for the approximated system.
(e) Draw the exact phase Bode diagrams of the compensated systems, including the time delay, and check the values obtained for the phase margins.

9.12 Determine a passive cascade compensator for a unit feedback system

$$G(s) = \frac{4}{s(s^2+4s+8)}$$

such that the compensated system has a phase margin of $45°$ and a steady state ramp error of less than 2%.

9.13 Derive formulas (9.42).

9.14 Solve the controller design problem defined in Example 9.4 by using a phase-lag-lead controller. What is the advantage of the controller's phase-lag part?

9.15 Consider a control system that has the open-loop transfer function

$$G(s) = \frac{K(s+20)}{(s+1)(s+3)(s+10)}$$

(a) Use MATLAB to design any controller by using Bode diagrams such that the compensated system has the best possible transient response and steady state specifications.
(b) Solve the same problem using the root locus technique for controller design.
(c) Compare the obtained results and comment on the simplicity (or complexity) of the root locus and Bode diagram methods for controller design.

9.16 Repeat Problem 9.15 for the open-loop control system defined in Problem 9.9.

9.17 In order to relate the phase margin (gain margin) and the real parts of the system eigenvalues as quantities for relative stability measure, perform the following experiment. Consider the system

$$G(s) = \frac{K(s+6)}{(s+10)(s^2+2s+2)}$$

(a) Vary the value for static gain K from zero to 100 in increments of 10, and for each value of K find the phase and gain margins. Plot both phase and gain margins as functions of K.
(b) For each value of K find the closed-loop eigenvalues and plot the magnitude of the eigenvalue real parts with respect to K.
(c) Compare diagrams obtained in (a) and (b) and draw the corresponding conclusion.

9.18 Repeat Problem 9.17 for the open-loop control system defined in Problem 9.9.

9.19 Consider a unit feedback control system that has the open-loop transfer function

$$G(s) = \frac{(2+s)e^{-Ts}}{(1+s)(s+10)}$$

Note that the term e^{-Ts} represents a time delay.

(a) Assuming that the time delay is negligible, draw the corresponding Bode diagrams and determine the phase and gain stability margins.
(b) Since the time delay affects only the phase diagram, draw the corrected phase Bode diagram for $T = 0.1$. Determine the phase

and gain stability margins and compare them to the corresponding quantities found in (a).

(c) Repeat part (b) for $T = 0.2$ and $T = 0.5$. Comment on the impact of the time delay element on the phase and gain stability margins.