Chapter Eight

Time Domain Controller Design

8.1 Introduction

In this chapter we study the problem of controller design such that the desired system specifications are achieved. Controller design is performed in the time domain using the root locus technique. Controller design techniques in the frequency domain, based on Bode diagrams, will be presented in Chapter 9. In this book we emphasize controller design in the time domain for the following reasons: (a) with the help of MATLAB very accurate results can be obtained for both desired transient response parameters and steady state errors; (b) while designing a controller in time system stability will can be easily monitored since the root locus technique is producing information about the location of all of the system poles so that one is able to design control systems that have a specified relative degree (extent) of stability; (c) *controller design using the root locus method is simpler than the corresponding one based on Bode diagrams*; (d) root locus controller design techniques are equally applicable to both minimum and nonminimum phase systems, whereas the *corresponding techniques based on Bode diagrams are very difficult to use (if applicable at all) for nonminimum phase systems* (Kuo, 1995).

Before we actually introduce the root locus techniques for dynamic controller design (Section 8.5), in Section 8.2 we consider a class of static controllers obtained through the pole placement technique based on full state feedback. The main purpose of this controller is to stabilize the closed-loop system, and it can sometimes be used to improve the system transient response.

In some cases it is possible to achieve the desired system performance by changing only the static gain K. In general, *as K increases, the steady state errors decrease, but the maximum percent overshoot increases*. However, very often a static controller is not sufficient and one is faced with the problem of designing dynamic controllers.

In Section 8.3 we present common controllers used in linear system control design. Two main classes of these controllers are discussed: PI and phase-lag controllers that improve steady state errors, and PD and phase-lead controllers that improve the system transient response. Combinations of these controllers, which simultaneously improve both the system transient response and steady state errors, are also considered.

A simple class of dynamic feedback controllers can be obtained by feeding back the derivative of the output variables. These controllers, known as the rate feedback controllers, are presented in Section 8.4. It is shown that a rate feedback controller increases the damping ratio of a second-order system while keeping the natural frequency unchanged so that both the response maximum percent overshoot and the settling time are reduced.

Actual controller design in the time domain is studied in Section 8.5. Design algorithms (procedures) are outlined for several types of controllers introduced in Section 8.3. The impact of particular controllers on transient response parameters and steady state errors is examined.

Several controller design case studies involving real physical systems are presented in Section 8.6. A the end of this chapter, in Section 8.7, a MATLAB laboratory experiment is formulated, in which students are exposed to the problem of controller design for real physical control systems.

Chapter Objectives

This chapter presents systematic procedures for time domain controller design techniques based on the root locus method. Students will learn how to design different types of controllers such that the closed-loop control systems have the desired steady state errors and transient response parameters (maximum percent overshoot, settling time). In addition, the eigenvalue (pole) placement technique, which for controllable systems allows location of system eigenvalues in any desired position in the complex plane, is fully explained for the case of single-input single-output systems.

8.2 State Feedback and Pole Placement

Consider a linear dynamic system in the state space form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{8.1}$$

In some cases one is able to achieve the goal (e.g. stabilizing the system or improving its transient response) by using the full state feedback, which represents a linear combination of the state variables, that is

$$\mathbf{u} = -\mathbf{F}\mathbf{x} \tag{8.2}$$

so that the closed-loop system, given by

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{A} - \mathbf{B}\mathbf{F})\mathbf{x} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{8.3}$$

has the desired specifications.

The main role of state feedback control is to stabilize a given system so that all closed-loop eigenvalues are placed in the left half of the complex plane. The following theorem gives a condition under which is possible to place system poles in the desired locations.

Theorem 8.1 *Assuming that the pair* $(\mathbf{A}, \mathbf{B})$ *is controllable, there exists a feedback matrix* $\mathbf{F}$ *such that the closed-loop system eigenvalues can be placed in arbitrary locations.*

This important theorem will be proved (justified) for *single-input single-output* systems. For the general treatment of the pole placement problem for multi-input multi-output systems, which is much more complicated, the reader is referred to Chen (1984).

If the pair $(\mathbf{A}, \mathbf{b})$ is controllable, the original system can be transformed into the phase variable canonical form, i.e. it exists a nonsingular transformation

$$\mathbf{x} = \mathbf{Q}\mathbf{z} \tag{8.4}$$

such that

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{8.5}$$

where a_i's are coefficients of the characteristic polynomial of $\mathbf{A}$, that is

$$\det(\lambda \mathbf{I} - \mathbf{A}) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0 \tag{8.6}$$

For single-input single-output systems the state feedback is given by

$$u(\mathbf{z}) = -f_1 z_1 - f_2 z_2 - \cdots - f_n z_n = -\mathbf{f}_c \mathbf{z} \tag{8.7}$$

After closing the feedback loop with $u(\mathbf{z})$, as given by (8.7), we get from (8.5)

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -(a_0 + f_1) & -(a_1 + f_2) & -(a_2 + f_3) & \dots & -(a_{n-1} + f_n) \end{bmatrix} \mathbf{z} \tag{8.8}$$

If the desired closed-loop eigenvalues are specified by $\lambda_1^d, \lambda_2^d, \dots, \lambda_n^d$, then the desired characteristic polynomial will be given by

$$\begin{aligned} \Delta^d(\lambda) &= \left(\lambda - \lambda_1^d\right)\left(\lambda - \lambda_2^d\right)\cdots\left(\lambda - \lambda_n^d\right) \\ &= s^n + a_{n-1}^d s^{n-1} + a_{n-2}^d s^{n-2} + \cdots + a_1^d s + a_0^d \end{aligned} \tag{8.9}$$

Since the last row in (8.8) contains coefficients of the characteristic polynomial of the original system after the feedback is applied, it follows from (8.8) and (8.9) that the required feedback gains must satisfy

$$\begin{aligned} a_0 + f_1 = a_0^d \ &\Rightarrow \ f_1 = a_0^d - a_0 \\ a_1 + f_2 = a_1^d \ &\Rightarrow \ f_2 = a_1^d - a_1 \\ &\dots \\ a_{n-1} + f_n = a_{n-1}^d \ &\Rightarrow \ f_n = a_{n-1}^d - a_{n-1} \end{aligned} \tag{8.10}$$

The pole placement procedure using the state feedback for a system which is already in phase variable canonical form is demonstrated in the next example.

Example 8.1: Consider the following system given in phase variable canonical form

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -10 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

It is required to find coefficients f_1, f_2, f_3 such that the closed-loop system has the eigenvalues located at $\lambda_{1,2}^d = -1 \pm j1,\ \lambda_3^d = -5$. The desired characteristic polynomial is obtained from (8.9) as

$$\Delta^d(\lambda) = (\lambda + 5)(\lambda + 1 + j1)(\lambda + 1 - j1) = \lambda^3 + 7\lambda^2 + 12\lambda + 10$$

so that from (8.10) we have

$$\begin{aligned} f_1 &= a_0^d - a_0 = 10 - 2 = 8 \\ f_2 &= a_1^d - a_1 = 12 - 5 = 7 \\ f_3 &= a_2^d - a_2 = 7 - 10 = -3 \end{aligned}$$

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In general, in order to be able to apply this technique to all controllable single-input single-output systems we need to find a nonsingular transformation which transfers the original system into phase variable canonical form. This transformation can be obtained by using the linearly independent columns of the system controllability matrix

$$\mathcal{C} = \left[\mathbf{b} \vdots \mathbf{A}\mathbf{b} \vdots \mathbf{A}^2\mathbf{b} \vdots \cdots \vdots \mathbf{A}^{n-1}\mathbf{b}\right]$$

It can be shown (Chen, 1984) that the required transformation is given by

$$\mathbf{Q} = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \cdots \quad \mathbf{q}_n] \tag{8.11}$$

where

$$\begin{aligned} \mathbf{q}_n &= \mathbf{b} \\ \mathbf{q}_{n-1} &= \mathbf{A}\mathbf{q}_n + a_{n-1}\mathbf{q}_n = \mathbf{A}\mathbf{b} + a_{n-1}\mathbf{b} \\ \mathbf{q}_{n-2} &= \mathbf{A}\mathbf{q}_{n-1} + a_{n-2}\mathbf{q}_n = \mathbf{A}^2\mathbf{b} + a_{n-1}\mathbf{A}\mathbf{b} + a_{n-2}\mathbf{b} \\ &\cdots \\ \mathbf{q}_1 &= \mathbf{A}\mathbf{q}_2 + a_1\mathbf{q}_n = \mathbf{A}^{n-1}\mathbf{b} + a_{n-1}\mathbf{A}^{n-2}\mathbf{b} + \cdots + a_1\mathbf{b} \end{aligned} \tag{8.12}$$

where a_i's are coefficients of the characteristic polynomial of matrix $\mathbf{A}$. After the feedback gain has been found for phase variable canonical form, $\mathbf{f}_c$, in the original coordinates it is obtained as (similarity transformation)

$$\mathbf{f} = \mathbf{f}_c\mathbf{Q}^{-1} \tag{8.13}$$

Example 8.2: Consider the following linear system given by

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -3 & 4 \\ -1 & 1 & -9 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial of this system is

$$\det(\lambda \mathbf{I} - \mathbf{A}) = s^3 + 13s^2 + 33s + 13$$

Its phase variable canonical form can be obtained either from its transfer function (see Section 3.1.2) or by using the nonsingular (similarity) transformation (8.11). The system transfer function is given by

$$\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \frac{46s + 13}{s^3 + 13s^2 + 33s + 13}$$

Using results from Section 3.1.2 we are able to write new matrices in phase variable canonical form representation as

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -13 & -33 & -13 \end{bmatrix}, \quad \mathbf{b}_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{c}_c = \begin{bmatrix} 13 & 46 & 0 \end{bmatrix}$$

The same matrices could have been obtained by using the similarity transformation with

$$\mathbf{A}_c = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}, \quad \mathbf{b}_c = \mathbf{Q}^{-1}\mathbf{b}, \quad \mathbf{c}_c = \mathbf{c}\mathbf{Q} \tag{8.14}$$

with $\mathbf{Q}$ obtained from (8.11) and (8.12) as

$$\mathbf{Q} = \begin{bmatrix} 51 & 16 & 1 \\ 19 & 17 & 2 \\ -5 & -3 & -1 \end{bmatrix}$$

Assume that we intend to find the feedback gain for the original system such that its closed-loop eigenvalues are located at $-1, -2, -3$, then

$$\Delta^d(\lambda) = (\lambda + 1)(\lambda + 2)(\lambda + 3) = \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

From equation (8.10) we get expressions for the feedback gains for the system in phase variable canonical form as

$$f_1 = a_0^d - a_0 = 6 - 13 = -7$$
$$f_2 = a_1^d - a_1 = 11 - 33 = -22$$
$$f_3 = a_2^d - a_2 = 6 - 13 = -7$$

In the original coordinates the feedback gain is obtained from (8.13)

$$\mathbf{f} = \mathbf{f}_c\mathbf{Q}^{-1} = [0.8149 \quad -1.0540 \quad 5.7069]$$

Using this gain in order to close the state feedback around the system we get

$$\dot{\mathbf{x}} = \begin{bmatrix} -1.8149 & 3.0540 & -5.7069 \\ -0.6298 & -0.8920 & -7.4139 \\ -0.1851 & -0.0540 & -3.2931 \end{bmatrix} \mathbf{x}$$

It is easy to check by MATLAB that the eigenvalues of this systems are located at $-1, -2, -3$.

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Comment: Exactly the same procedure as the one given in this section can be used for placing the observer poles in the desired locations. The observers have been considered in Section 5.6. Choosing the observer gain $\mathbf{K}$ such that the closed-loop observer matrix $\mathbf{A} - \mathbf{KC}$ has the desired poles corresponds to the problem of choosing the feedback gain $\mathbf{F}$ such that the closed-loop system matrix $\mathbf{A}^T - \mathbf{F}^T\mathbf{B}^T$ has the same poles. Thus, for the observer pole placement problem, matrix $\mathbf{A}$ should be replaced by $\mathbf{A}^T$, $\mathbf{B}$ replaced by $\mathbf{C}^T$ and $\mathbf{F}$ replaced by $\mathbf{K}^T$. In addition, it is known from Chapter 5 that the observability of the pair $(\mathbf{A}, \mathbf{C})$ is equal to the controllability of the pair $(\mathbf{A}^T, \mathbf{C}^T)$, and hence the controllability condition stated in Theorem 8.1—the pair $(\mathbf{A}, \mathbf{B})$ is controllable, which for observer pole placement requires that the pair $(\mathbf{A}^T, \mathbf{C}^T)$ be controllable—is satisfied by assuming that the pair $(\mathbf{A}, \mathbf{C})$ is observable.

8.3 Common Dynamic Controllers

Several common dynamic controllers appear very often in practice. They are known as PD, PI, PID, phase-lag, phase-lead, and phase-lag-lead controllers. In this section we introduce their structures and indicate their main properties. In the follow-up sections procedures for designing these controllers by using the root locus technique such that the given systems have the desired specifications are presented. In the most cases these controllers are placed in the forward path at the front of the plant (system) as presented in Figure 8.1.

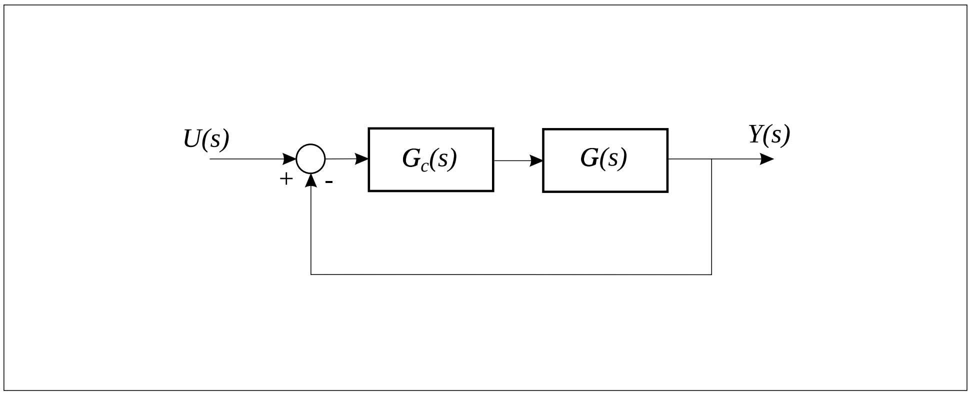

Figure 8.1: A common controller-plant configuration

8.3.1 PD Controller

PD stands for a proportional and derivative controller. The output signal of this controller is equal to the sum of two signals: the signal obtained by multiplying the input signal by a constant gain K_p and the signal obtained by differentiating and multiplying the input signal by K_d, i.e. its transfer function is given by

$$G_c(s) = K_p + K_d s \tag{8.15}$$

This controller is used to improve the system transient response.

8.3.2 PI Controller

Similarly to the PD controller, the PI controller produces as its output a weighted sum of the input signal and its integral. Its transfer function is

$$G_c(s) = K_p + K_i \frac{1}{s} = \frac{K_p s + K_i}{s} \tag{8.16}$$

In practical applications the PI controller zero is placed very close to its pole located at the origin so that the angular contribution of this "dipole" to the root locus is almost zero. *A PI controller is used to improve the system response steady state errors* since it increases the control system type by one (see Definition 6.1).

8.3.3 PID Controller

The PID controller is a combination of PD and PI controllers; hence its transfer function is given by

$$G_c(s) = K_p + K_d s + K_i \frac{1}{s} = \frac{K_i + K_p s + K_d s^2}{s} \tag{8.17}$$

The PID controller can be used to improve both the system transient response and steady state errors. This controller is very popular for industrial applications.

8.3.4 Phase-Lag Controller

The phase-lag controller belongs to the same class as the PI controller. The phase-lag controller can be regarded as a generalization of the PI controller. It introduces a negative phase into the feedback loop, which justifies its name. It has a zero and pole with the pole being closer to the imaginary axis, that is

$$\begin{gathered} G_c(s) = \left(\frac{p_1}{z_1}\right)\frac{s+z_1}{s+p_1}, \quad z_1 > p_1 > 0 \\ \arg G_c(s) = \arg(s+z_1) - \arg(s+p_1) = \theta_{z_1} - \theta_{p_1} < 0 \end{gathered} \tag{8.18}$$

where p_1/z_1 is known as the lag ratio. The corresponding angles θ_{z_1} and θ_{p_1} are given in Figure 8.2a. *The phase-lag controller is used to improve steady state errors.*

8.3.5 Phase-Lead Controller

The phase-lead controller is designed such that its phase contribution to the feedback loop is positive. It is represented by

$$\begin{gathered} G_c(s) = \frac{s+z_2}{s+p_2}, \quad p_2 > z_2 > 0 \\ G_c(s) = \arg(s+z_2) - \arg(s+p_2) = \theta_{z_2} - \theta_{p_2} > 0 \end{gathered} \tag{8.19}$$

where θ_{z_2} and θ_{p_2} are given in Figure 8.2b. This controller introduces a positive phase shift in the loop (phase lead). *It is used to improve the system response transient behavior.*

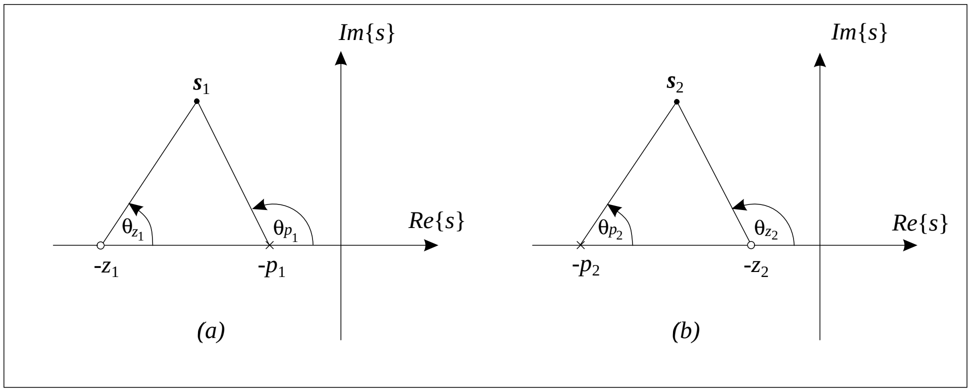

Figure 8.2: Poles and zeros of phase-lag (a) and phase-lead (b) controllers

8.3.6 Phase-Lag-Lead Controller

The phase-lag-lead controller is obtained as a combination of phase-lead and phase-lag controllers. Its transfer function is given by

$$G_c(s) = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}, \qquad p_2 > z_2 > z_1 > p_1 > 0, \quad z_1 z_2 = p_1 p_2 \tag{8.20}$$

It has features of both phase-lag and phase-lead controllers, i.e. *it can be used to improve simultaneously both the system transient response and steady state errors*. However, it is harder to design phase-lag-lead controllers than either phase-lag or phase-lead controllers.

Note that all controllers presented in this section can be realized by using active networks composed of operational amplifiers (see, for example, Dorf, 1992; Nise, 1992; Kuo, 1995).

8.4 Rate Feedback Control

The controllers considered in the previous section have simple forms and in most cases they are placed in the forward loop in the front of the system to be controlled. Another simple controller that is always used in the feedback loop is known as the rate feedback controller. The rate feedback controller is obtained by feeding back the derivative of the output of a second-order system (or a system which can be approximated by a second-order system, i.e. a system with dominant complex conjugate poles) according to the block diagram given in Figure 8.3.

The rate feedback control helps to increase the system damping. This follows from the fact that the closed-loop transfer function for this configuration is given by

$$\frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\left(\zeta + \frac{1}{2}K_t\omega_n\right)\omega_n s + \omega_n^2} = \frac{\omega_n^2}{s^2 + 2\zeta_c\omega_n s + \omega_n^2}, \quad K_t > 0 \quad (8.21)$$

Compared with the closed-loop transfer function of the second-order system without control (6.4), we see that the damping factor is now increased to

$$\zeta_c = \zeta + \frac{1}{2}K_t\omega_n \quad (8.22)$$

Since the natural frequency is unchanged, this controller decreases the response settling time (see (6.20)). The system response maximum percent overshoot is also decreased (see Problem 8.3).

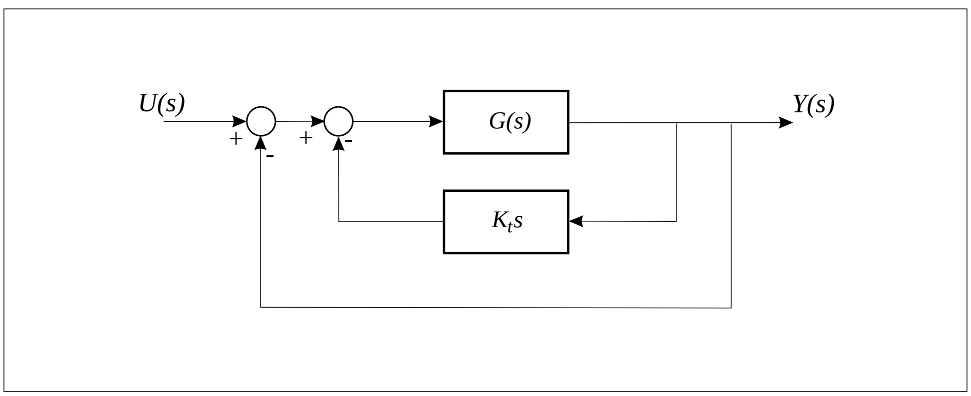

Figure 8.3: Block diagram for a rate feedback controller

Example 8.3: Design a rate feedback controller such that the damping ratio of the second-order system considered in Example 6.1 is increased to $\zeta_c = 0.75$ and determine the system response settling time and maximum percent overshoot.

From Example 6.1 we know

$$\omega_n = 2\,\mathrm{rad/s}, \quad \zeta = 0.5, \quad t_s \approx 3\,\mathrm{s}, \quad MPOS = 16.3\,\%$$

The gain for the rate controller is obtained from (8.22) as

$$K_t = \frac{1}{\omega_n}2(\zeta_c - \zeta) = \frac{1}{2}2(0.75 - 0.5) = 0.25$$

The new values for the response settling time and maximum percent overshoot are given by

$$t_{sc} \approx \frac{3}{\zeta_c \omega_n} = \frac{3}{0.75 \times 2} = 2\,\mathrm{s}, \quad MPOS = e^{-\frac{\zeta_c \pi}{\sqrt{1-\zeta_c^2}}}100(\%) = 2.83\,\%$$

It can be seen that both the system response settling time and maximum percent overshoot are reduced.

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8.5 Compensator Design by the Root Locus Method

Sometimes one is able to improve control system specifications by changing the static gain K only. It can be observed that *as K increases, the steady state errors decrease (assuming system's asymptotic stability), but the maximum percent overshoot increases*. However, using large values for K may damage system stability. Even more, in most cases the desired operating points for the system dominant poles, which satisfy the transient response requirements, do not lie on the original root locus. Thus, in order to solve the transient response and steady state errors improvement problem, one has to design dynamic controllers, considered in Section 8.3, and put them in series with the plant (system) to be controlled (see Figure 8.1).

In the following we present dynamic controller design techniques in three categories: improvement of steady state errors (PI and phase-lag controllers), improvement of system transient response (PD and phase-lead controllers), and improvement of both steady state errors and transient response (PID and phase-lag-lead controllers). Note that transient response specifications are obtained under the assumption that a given system has a pair of dominant complex conjugate closed-loop poles; hence this assumption has to be checked after a controller is added to the system. This can be easily done using the root locus technique.

8.5.1 Improvement of Steady State Errors

It has been seen in Chapter 6 that the steady state errors can be improved by increasing the type of feedback control system, in other words, by adding a pole at the origin to the open-loop system transfer function. The simplest way to achieve this goal is to add in series with the system a PI controller as defined in (8.16), i.e. to get

$$G_c(s)G(s) = \frac{K_p s + K_i}{s} G(s)$$

Since this controller also introduces a zero at $-K_i/K_p$, *the zero should be placed as close as possible to the pole*. In that case the pole at $p = 0$ and the zero at $z \approx p$ act as a dipole, and so their mutual contribution to the root locus is almost negligible. Since the root locus is practically unchanged, the system transient response remains the same and the effect due to the PI controller is to increase the type of the control system by one, which produces improved steady state errors. The effect of a dipole on the system response is studied in the next example.

Example 8.4: Consider the open-loop transfer functions

$$G_1(s) = \frac{(s+2)}{(s+1)(s+3)}$$

and

$$G_2(s) = \frac{(s+2)(s+5)}{(s+1)(s+3)(s+5.1)}$$

Note that the second transfer function has a dipole with a stable pole at -5.1. The corresponding step responses are given in Figure 8.4. It can be seen from this figure that the system with a stable dipole and the system without a stable dipole have almost identical responses. These responses have been obtained by the following sequence of MATLAB instructions.

```
num1=[1 2];
den1=[1 4 3];
num2=[1 7 10];
d1=[1 1];
d2=[1 3];
d3=[1 5.1];
d12=conv(d1,d2);
```

```
den2=conv(d12,d3);
[cnum1,cden1]=feedback(num1,den1,1,1,-1);
[cnum2,cden2]=feedback(num2,den2,1,1,-1);
t=0:0.1:2;
step(cnum1,cden1,t)
step(cnum2,cden2,t)
```

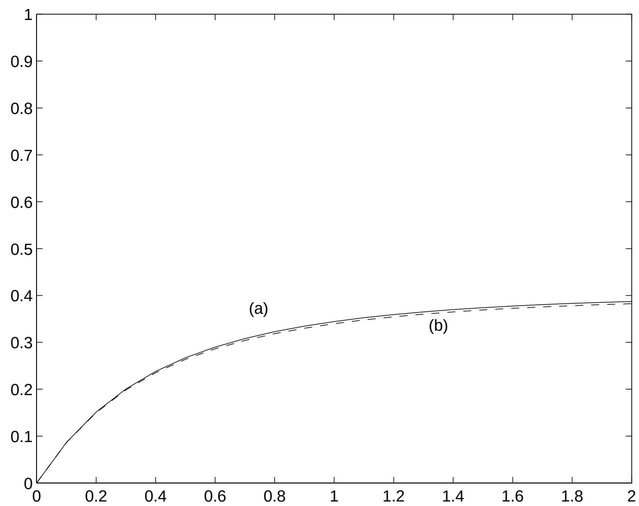

Figure 8.4: Step responses of a system without a stable dipole (a) and with a stable dipole (b)

It is important to point out that in the case of an *unstable dipole* the effect of a dipole is completely different. Consider, for example, the open-loop transfer function given by

$$G_3(s) = \frac{(s+2)(s-5)}{(s+1)(s+3)(s-5.1)}$$

Its step response is presented in Figure 8.5b and compared with the corresponding step response after a dipole is eliminated (Figure 8.5a). In fact, the system without a dipole is stable and the system with a dipole is unstable; hence their responses are drastically different. Thus, we can conclude that *it is not correct to cancel an* *unstable dipole* since it has a big impact on the system response.

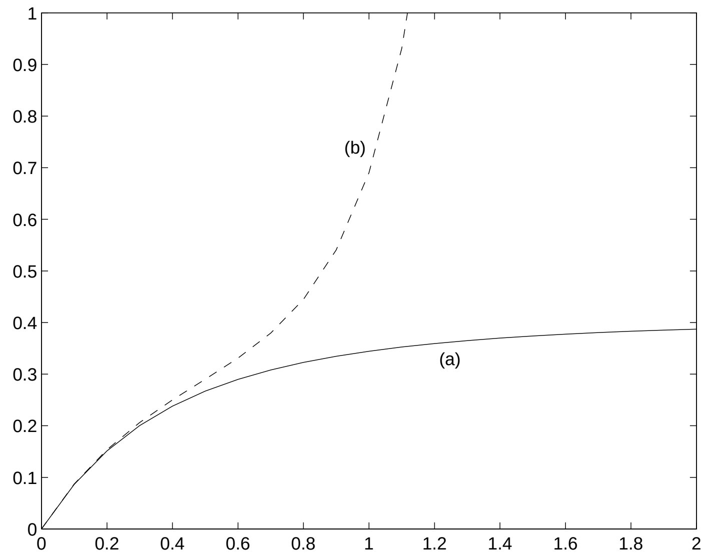

Figure 8.5: Step responses of a system without an unstable dipole (a) and with an unstable dipole (b)

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Both the PI and phase-lag controller use this "*stable dipole effect*". They do not change the system transient response, but they do have an important impact on the steady state errors.

PI Controller Design

As we have already indicated, the PI controller represents a stable dipole with a pole located at the origin and a stable zero placed near the pole. Its impact on the transient response is negligible since it introduces neither significant phase shift nor gain change (see root locus rules 9 and 10 in Table 7.1). Thus, the transient response parameters with the PI controller are almost the same as those for the original system, but the steady state errors are drastically improved due to the fact that the feedback control system type is increased by one.

The PI controller is represented, in general, by

$$G_c(s) = K_p \frac{s + \frac{K_i}{K_p}}{s}, \qquad K_i \ll K_p \tag{8.23}$$

where K_p represents its static gain and K_i/K_p is a stable zero near the origin. Very often it is implemented as

$$G_c(s) = \frac{s + z_c}{s} \tag{8.24}$$

This implementation is sufficient to justify its main purpose. The design algorithm for this controller is extremely simple.

Design Algorithm 8.1:

1. Set the PI controller's pole at the origin and locate its zero arbitrarily close to the pole, say $z_c = 0.1$ or $z_c = 0.01$.
2. If necessary, adjust for the static loop gain to compensate for the case when K_p is different from one. Hint: Use $K_p = 1$, and avoid gain adjustment problem.

Comment: Note that while drawing the root locus of a system with a PI controller (compensator), the stable open-loop zero of the compensator will attract the compensator's pole located at the origin as the static gain increases from 0 to $+\infty$ so that there is no danger that the closed-loop system may become unstable due to addition of a PI compensator (controller).

The following example demonstrates the use of a PI controller in order to reduce the steady state errors.

Example 8.5: Consider the following open-loop transfer function

$$G(s) = \frac{K(s+6)}{(s+10)(s^2+2s+2)}$$

Let the choice of the static gain $K = 10$ produce a pair of dominant poles on the root locus, which guarantees the desired transient specifications. The corresponding position constant and the steady state unit step error are given by

$$K_p = \frac{10 \times 6}{10 \times 2} = 3 \;\Rightarrow\; e_{ss} = \frac{1}{1+K_p} = 0.25$$

Using a PI controller in the form of (8.24) with the zero at -0.1 ($z_c = 0.1$), we obtain the improved values as $K_p = \infty$ and $e_{ss} = 0$. The step responses of the original system and the compensated system, now given by

$$G_c(s)G(s) = \frac{10(s+0.1)(s+6)}{s(s+10)(s^2+2s+2)}$$

are presented in Figure 8.6.

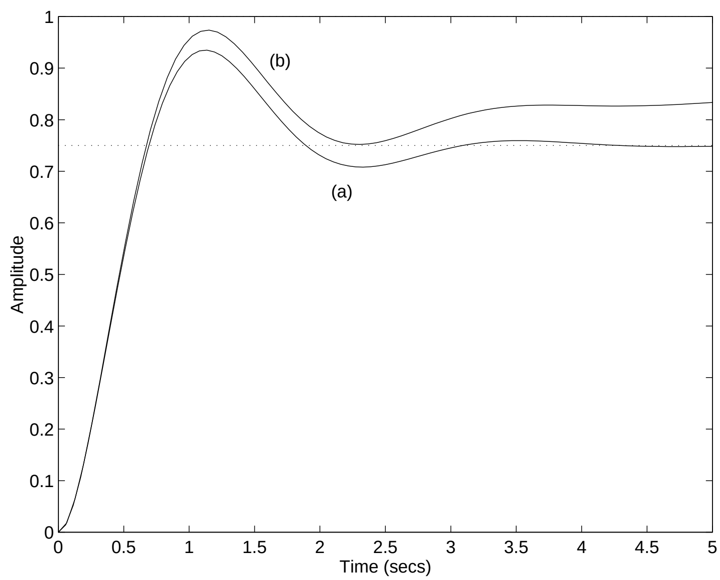

Figure 8.6: Step responses of the original (a) and compensated (b) systems for Example 8.5

The closed-loop poles of the original system are given by

$$\lambda_1 = -9.5216, \qquad \lambda_{2,3} = -1.2392 \pm j2.6204$$

For the compensated system they are

$$\lambda_{1c} = -9.5265, \quad \lambda_{2c,3c} = -1.1986 \pm j2.6109$$

Having obtained the closed-loop system poles, it is easy to check that the dominant system poles are preserved for the compensated system and that the damping ratio and natural frequency are only slightly changed. Using information about the dominant system poles and relationships obtained from Figure 6.2, we get

$$\zeta\omega_n = 1.2392, \;\; \omega_n^2 = (1.2392)^2 + (2.6204)^2 \;\Rightarrow\; \omega_n^2 = 2.9019, \; \zeta = 0.4270$$

and

$$\zeta_c\omega_{nc} = 1.1986, \;\; \omega_{nc}^2 = (1.1986)^2 + (2.6109)^2$$
$$\Rightarrow \; \omega_{nc}^2 = 2.8901, \quad \zeta_c = 0.4147$$

In Figure 8.7 we draw the step response of the compensated system over a long period of time in order to show that the steady state error of this system is theoretically and practically equal to zero.

Figures 8.6 and 8.7 are obtained by using the same MATLAB functions as those used in Example 8.4.

The root loci of the original and compensated systems are presented in Figures 8.8 and 8.9. It can be seen from these figures that the root loci are almost identical, with the exception of a tiny dipole branch near the origin.

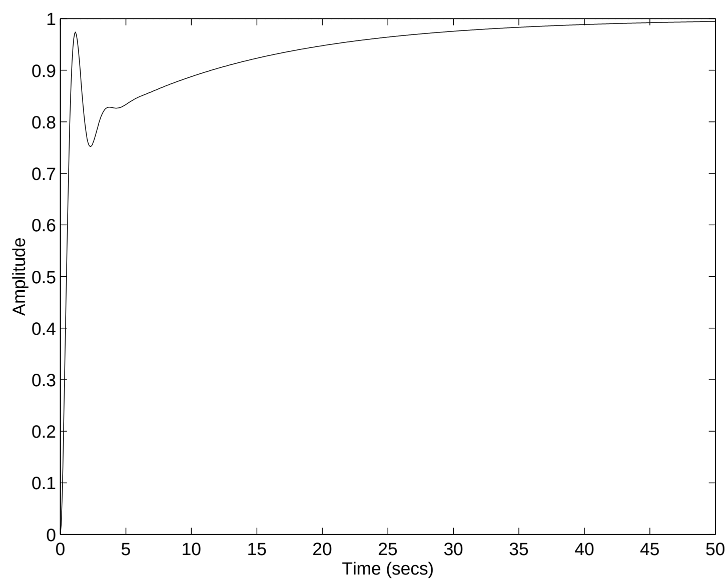

Figure 8.7: Step response of the compensated system for Example 8.5

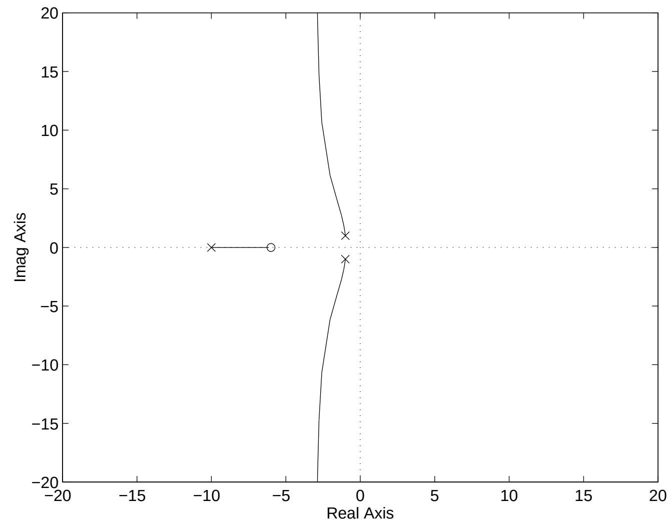

Figure 8.8: Root locus of the original system for Example 8.5

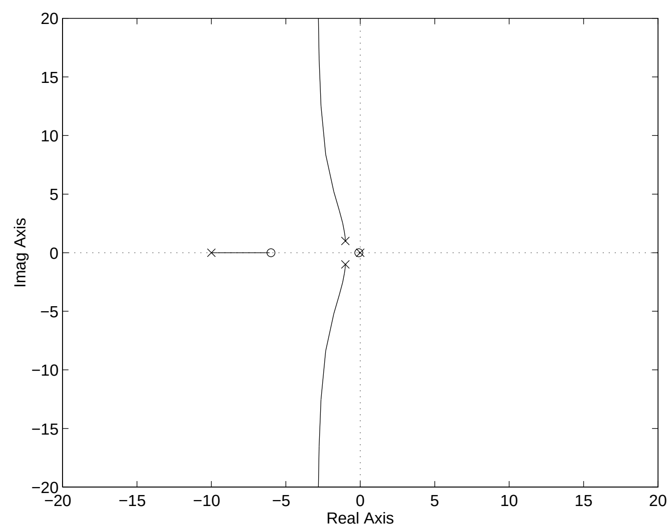

Figure 8.9: Root locus of the compensated system for Example 8.5

◇

Phase-Lag Controller Design

The phase-lag controller, in the context of root locus design methodology, is also implemented as a dipole that has no significant influence on the root locus, and thus on the transient response, but increases the steady state constants and reduces the corresponding steady state errors. Since it is implemented as a dipole, its zero and pole have to be placed very close to each other.

The lag controller's impact on the steady state errors can be obtained from the expressions for the corresponding steady state constants. Namely, from (6.31), (6.33), and (6.35) we know that

$$K_p = \lim_{s\to 0}\{H(s)G(s)\}, \quad K_v = \lim_{s\to 0}\{sH(s)G(s)\}, \quad K_a = \lim_{s\to 0}\left\{s^2H(s)G(s)\right\}$$

and from (6.30), (6.32), and (6.34) we have

$$e_{ss_{step}} = \frac{1}{1+K_p}, \quad e_{ss_{ramp}} = \frac{1}{K_v}, \quad e_{ss_{parabolic}} = \frac{2}{K_a}$$

For control systems of type zero, one, and two, respectively, the constants K_p, K_v, and K_a are all given by the same expression, that is

$$K_l = K\frac{z_1z_2\cdots}{p_1p_2\cdots}, \quad l = p, v, a \tag{8.25}$$

Consider, first, a phase-lag compensator of the form

$$G_c(s) = \frac{s+z_c}{s+p_c}, \quad z_c > p_c > 0 \tag{8.26}$$

If we put this controller in series with the system, the corresponding steady state constants of the compensated system will be given by

$$K_{lc} = K\frac{z_1z_2\cdots}{p_1p_2\cdots}\frac{z_c}{p_c} = K_l\frac{z_c}{p_c}, \quad lc, l = p, v, a \tag{8.27}$$

In order to increase these constants and reduce the steady state errors, the ratio of z_c/p_c should be as large as possible. Since at the same time z_c must be close to p_c (they form a dipole), a large value for the ratio z_c/p_c can be achieved if both of them are placed close to zero. For example, the choice of $z_c = 0.1$ and $p_c = 0.01$ increases the constants $K_l, l = p, v, a$, ten times and reduces the corresponding steady state errors ten times.

Now consider a phase-lag controller defined by (8.18), that is

$$G_c(s) = \left(\frac{p_c}{z_c}\right)\frac{s+z_c}{s+p_c}, \quad z_c > p_c > 0$$

This controller will change the value of the static gain K by a factor of p_c/z_c, which will produce a movement of the desired operating point along the root locus in the direction of smaller static gains. Thus, the plant static gain has to be adjusted to a higher value in order to preserve the same operating point. The consequence of using a phase-lag controller as defined in (8.18) is that *the same (desired) operating point is obtained with higher static gain*. We already know that by increasing the static gain, the steady state errors are reduced. In this case, the static gain adjustment has to be done by choosing a new static gain $\widetilde{K} = Kz_c/p_c$. Note that the effects of both phase-lag controllers (8.18) and (8.26) are exactly the same, since the gain adjustment in the case of controller (8.18) in fact cancels its lag ratio p_c/z_c.

The following simple algorithm is used for phase-lag controller design.

Design Algorithm 8.2:

1. Choose a point that has the desired transient specifications on the root locus branch with dominant system poles. Read from the root locus the value for the static gain K at the chosen point, and determine the corresponding steady state errors.
2. Set both the phase-lag controller's pole and zero near the origin with the ratio z_c/p_c obtained from (8.27) such that the desired steady state error requirement is satisfied.
3. In the case of controller (8.18), adjust for the static loop gain, i.e. take a new static gain as $\widetilde{K} = Kz_c/p_c$.

The next example demonstrates the controller design procedure with a phase-lag compensator according to the steps outlined in Design Algorithm 8.2.

Example 8.6: The steady state errors of the system considered in Example 8.5 can be improved by using a phase-lag controller of the form

$$G_c(s) = \frac{s+0.1}{s+0.01}$$

Since $z_c/p_c = 10$, the position constant is increased ten times, that is

$$K_{pc} = K_p\frac{z_c}{p_c} = 3 \times 10 = 30$$

so that the steady state error due to a unit step input is reduced to

$$e_{ss_{step}} = \frac{1}{1+K_{pc}} = \frac{1}{31} = 0.03226$$

It can be easily checked that the transient response is almost unchanged; in fact, the dominant system poles with this phase-lag compensator are $-1.2026 \pm j2.6119$, which is very close to the dominant poles of the original system (see Example 8.5).

◇

Example 8.7: Consider the following open-loop transfer function

$$G(s)H(s) = \frac{K(s+15)}{s(s+20)(s^2+4s+8)}$$

Let the choice of the static gain $K = 20$ produce a pair of dominant poles on the root locus that guarantees the desired transient specifications. The system closed-loop poles for $K = 20$ are given by

$$\lambda_{1,2} = -0.5327 \pm j2.2024, \quad \lambda_3 = -2.9194, \quad \lambda_4 = -20.0153$$

so that for this value of the static gain K the dominant poles exist, i.e. the absolute value of the real part of the dominant poles (0.5327) is about six times smaller than the absolute value of the real part of the next pole (2.9194), which is in practice sufficient to guarantee poles' dominance. Since we have a type one feedback control system, the steady state error due to a unit step is zero. The velocity constant and the steady state unit ramp error are obtained as

$$K_v = \frac{20 \times 15}{20 \times 8} = \frac{15}{8} \quad \Rightarrow \quad e_{ss_{ramp}} = \frac{1}{K_v} = 0.53$$

Using the phase-lag controller with a zero at -0.1 ($z_c = 0.1$) and a pole at -0.01 ($p_c = 0.01$), we get

$$K_{vc} = K_v \frac{z_c}{p_c} = \frac{150}{8} \quad \Rightarrow \quad e_{ssc_{ramp}} = 0.053$$

It can be easily shown by using MATLAB that the ramp responses of the original and the compensated systems are very close to each other. The same holds for the root loci. Note that even smaller steady state errors can be obtained if we increase the ratio z_c/p_c, e.g. to $z_c/p_c = 100$.

◇

8.5.2 Improvement of Transient Response

The transient response can be improved by using either the PD or phase-lead controllers. In the following, we consider these two controllers independently. However, both of them have the common feature of introducing a positive phase shift, and both of them can be implemented in a similar manner.

PD Controller Design

The PD controller is represented by

$$G_c(s) = s + z_c, \quad z_c > 0 \tag{8.28}$$

which indicates that the compensated system open-loop transfer function will have one additional zero. The effect of this zero is to introduce a positive phase shift. The phase shift and position of the compensator's zero can be determined by using simple geometry. That is, for the chosen dominant complex conjugate poles that produce the desired transient response we apply the root locus angle rule given in formula (7.10) and presented in Table 7.1 as rule number 10. This rule basically says that for a chosen point, s_d, on the root locus the difference of the sum of the angles between the point s_d and the open-loop zeros, and the sum of the angles between the point s_d and the open-loop poles must be $180°$. Applying the root locus angle rule to the compensated system, we get

$$\measuredangle G_c(s_d)G(s_d) = \measuredangle(s_d + z_c) + \sum_{i=1}^{m} \measuredangle(s_d + z_i) - \sum_{i=1}^{n} \measuredangle(s_d + p_i) = 180° \tag{8.29}$$

which implies

$$\measuredangle(s_d + z_c) = 180° - \sum_{i=1}^{m} \measuredangle(s_d + z_i) + \sum_{i=1}^{n} \measuredangle(s_d + p_i) = \alpha_c \tag{8.30}$$

From the obtained angle $\measuredangle(s_d + z_c)$ the location of the compensator's zero is obtained by playing simple geometry as demonstrated in Figure 8.10. Using this figure it can be easily shown that the value of z_c is given by

$$z_c = \frac{\omega_n}{\tan \alpha_c}\left(\zeta \tan \alpha_c + \sqrt{1 - \zeta^2}\right) \tag{8.31}$$

An algorithm for the PD controller design can be formulated as follows.

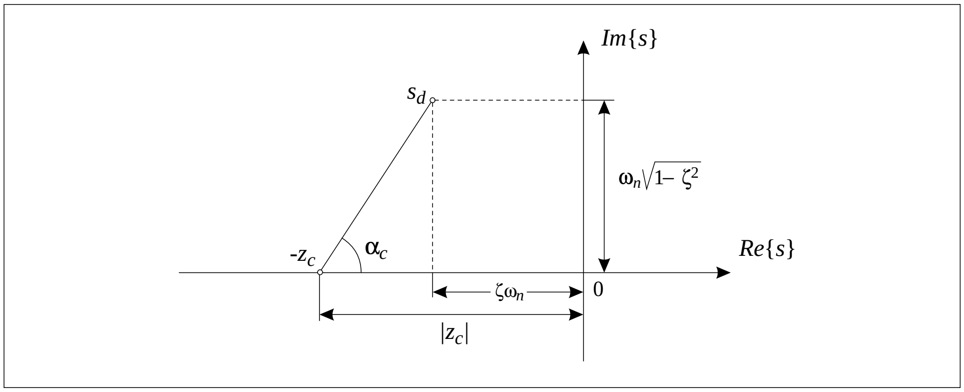

Figure 8:10 Determination of a PD controller's zero location

Design Algorithm 8.3:

1. Choose a pair of complex conjugate dominant poles in the complex plane that produces the desired transient response (damping ratio and natural frequency). Figure 6.2 helps to accomplish this goal.
2. Find the required phase contribution of a PD regulator by using formula (8.30).
3. Find the absolute value of a PD controller's zero by using formula (8.31); see also Figure 8.10.
4. Check that the compensated system has a pair of dominant complex conjugate closed-loop poles.

Example 8.8: Let the design specifications be set such that the desired maximum percent overshoot is less than 20% and the 5%-settling time is $1.5\,\mathrm{s}$. Then, the formula for the maximum percent overshoot given by (6.16) implies

$$-\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = \ln\{OS\} \;\Rightarrow\; \zeta = \sqrt{\frac{\ln^2\{OS\}}{\pi^2+\ln^2\{OS\}}} = 0.456$$

We take $\zeta = 0.46$ so that the expected maximum percent overshoot is less than 20%. In order to have the 5%-settling time of $1.5\,\mathrm{s}$, the natural frequency should satisfy

$$t_s \approx \frac{3}{\zeta\omega_n} \;\;\Rightarrow\;\; \omega_n \approx \frac{3}{\zeta t_s} = 4.348\,\mathrm{rad/s}$$

The desired dominant poles are given by

$$s_d = \lambda_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -2.00 \pm j3.86$$

Consider now the open-loop control system

$$G(s) = \frac{K(s+10)}{(s+1)(s+2)(s+12)}$$

The root locus of this system is represented in Figure 8.11a.

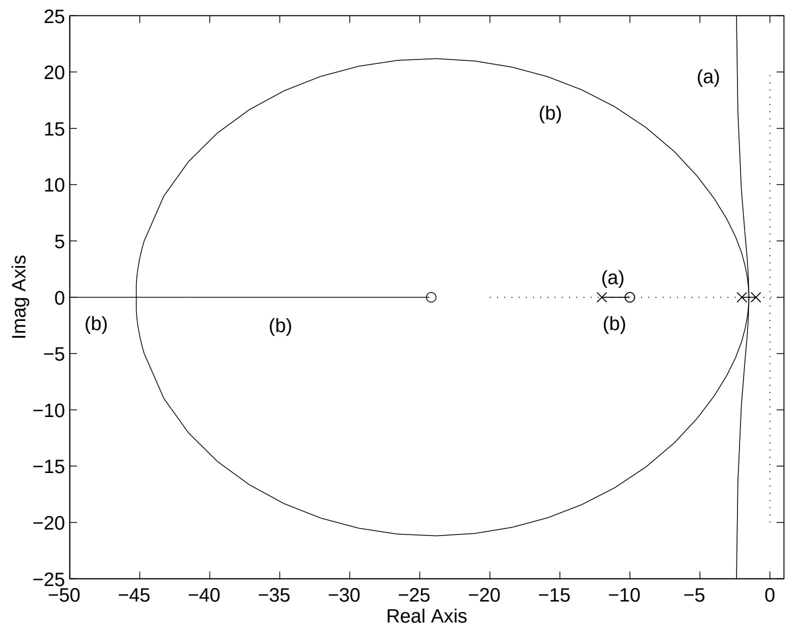

Figure 8.11: Root loci of the original (a) and compensated (b) systems

It is obvious from the above figure that the desired dominant poles do not belong to the original root locus since the breakaway point is almost in the middle of the open-loop poles located at -1 and -2. In order to move the original root locus to the left such that it passes through s_d, we design a PD controller by following Design Algorithm 8.3. Step 1 has been already completed in the previous paragraph. Since we have determined the desired operating point, s_d, we now use formula (8.30) to determine the phase contribution of a PD controller. By

MATLAB function `angle` (or just using a calculator), we can find the following angles

$$\measuredangle(s_d + z_1) = 0.4495\,\mathrm{rad}, \quad \measuredangle(s_d + p_1) = 1.8243\,\mathrm{rad}$$
$$\measuredangle(s_d + p_2) = 1.5708\,\mathrm{rad}, \quad \measuredangle(s_d + p_3) = 0.3684\,\mathrm{rad}$$

Note that MATLAB function `angle` produces results in radians. Using formula (8.30), we get

$$\begin{aligned}\measuredangle(s_d + z_c) &= \pi - 0.4495 + 1.8243 + 1.5708 + 0.3684\\ &= 0.1723\,\mathrm{rad} \ = \ 9.8734^\circ \ = \ \alpha_c\end{aligned}$$

Having obtained the angle α_c, the formula (8.31) produces the location of the controller's zero, i.e. $z_c = 24.1815$, so that the required PD controller is given by

$$G_c(s) = s + 24.1815$$

The root locus of the compensated system is presented in Figures 8.11b and 8.12b. It can be seen from Figure 8.12 that the point $s_d = -2 \pm j3.86$ lies on the root locus of the compensated system.

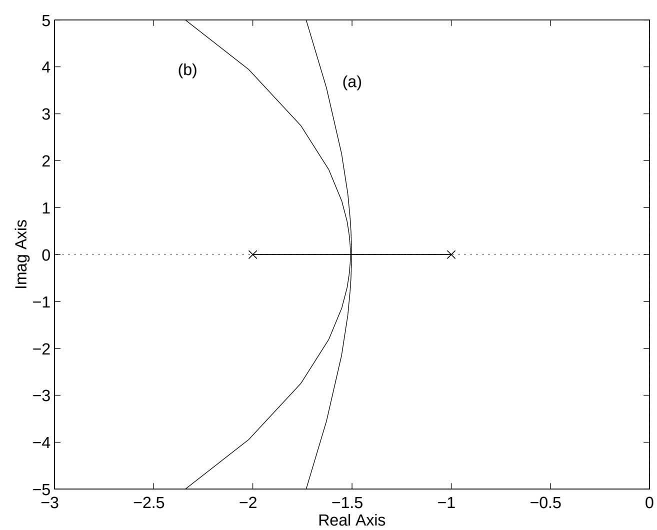

Figure 8.12: Enlarged portion of the root loci in the neighborhood of the desired operating point of the original (a) and compensated (b) systems

At the desired point, s_d, the static gain K, obtained by applying the root locus rule number 9 from Table 7.1, is given by $K = 0.825$. This value can be obtained either by using a calculator or the MATLAB function `abs` as follows:

```
d1=abs(sd+p1);
d2=abs(sd+p2);
d3=abs(sd+p3);
d4=abs(sd+z1);
d5=abs(sd+zc);
K=(d1*d2*d3)/(d4*d5)
```

For this value of the static gain K, the steady state errors for the original and compensated systems are given by $e_{ss} = 0.7442$, $e_{ssc} = 0.1074$. Note that in the case when $z_c > 1$, this controller can also improve the steady state errors. In addition, since the controller's zero will attract one of the system poles for large values of K, it is not advisable to choose small values for z_c since it may damage the transient response dominance by the pair of complex conjugate poles closest to the imaginary axis.

The closed-loop step response for this value of the static gain is presented in Figure 8.13. It can be observed that both the maximum percent overshoot and the settling time are within the specified limits.

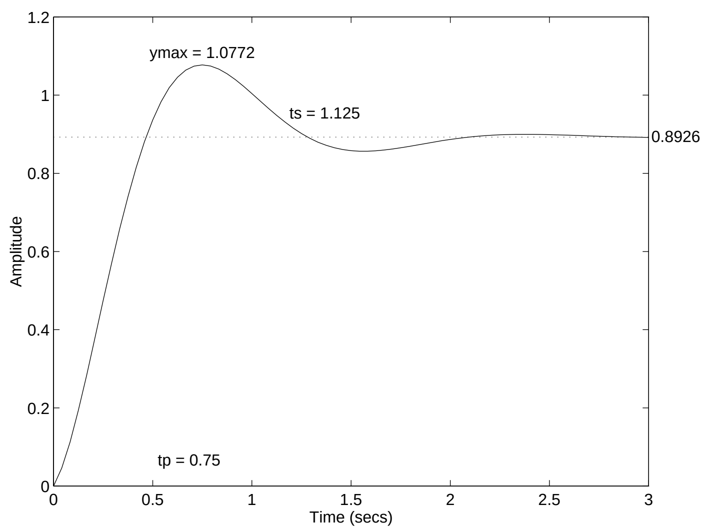

Figure 8.13: Step response of the compensated system for Example 8.8

The values for the overshoot, peak time, and settling time are obtained by the following MATLAB routine:

```
[yc,xc,t]=step(cnumc,cdenc);
% t is a time vector of length i=73;
% cnumc = closed-loop compensated numerator
% cdenc = closed-loop compensated denominator
plot(t,yc);
[ymax,imax]=max(yc);
% ymax is the function maximum;
% imax = time index where maximum occurs;
tp=t(imax)
essc=0.1074;
yss=1–essc;
os=ymax-yss
% procedure for finding the settling time;
delt5=0.05*yss;
i=73;
while abs((yc(i)-yss))<delt5;
i=i-1;
end;
ts=t(i)
```

Using this program, we have found that $t_s = 1.125\,\mathrm{s}$ and $MPOS = 20.68\%$. Our starting assumptions have been based on a model of the second-order system. Since the second-order systems are only approximations for higher-order systems that have dominant poles, the obtained results are satisfactory.

Finally, we have to check that the system response is dominated by a pair of complex conjugate poles. Finding the closed-loop eigenvalues we get $\lambda_1 = -11.8251, \ \ \lambda_{2,3} = -2.000 \pm j3.8600$, which indicates that the presented controller design results are correct since the transient response is dominated by the eigenvalues $\lambda_{2,3}$.

◇

Phase-Lead Controller Design

The phase-lead controller works on the same principle as the PD controller. It uses the argument rule, formula (7.10), of the root locus method, which indicates the phase shift that needs to be introduced by the phase-lead controller such that the desired dominant poles (having the specified transient response characteristics) belong to the root locus.

The general form of this controller is given by (8.19), that is

$$G_c(s) = \frac{s + z_c}{s + p_c}, \quad p_c > z_c > 0$$

By choosing a point s_d for a dominant pole that has the required transient response specifications, the design of a phase-lead controller can be done in similar fashion to that of a PD controller. First, find the angle contributed by a controller such that the point s_d belongs the root locus, which can be obtained from

$$\measuredangle G_c(s_d) = 180^\circ - \measuredangle G(s_d) \tag{8.32}$$

that is

$$\theta_c = \measuredangle(s_d + z_c) - \measuredangle(s_d + p_c) = 180^\circ - \sum_{i=1}^{m} \measuredangle(s_d + z_i) + \sum_{i=1}^{n} \measuredangle(s_d + p_i) \tag{8.33}$$

Second, find locations of controller's pole and zero. This can be done in many ways as demonstrated in Figure 8.14.

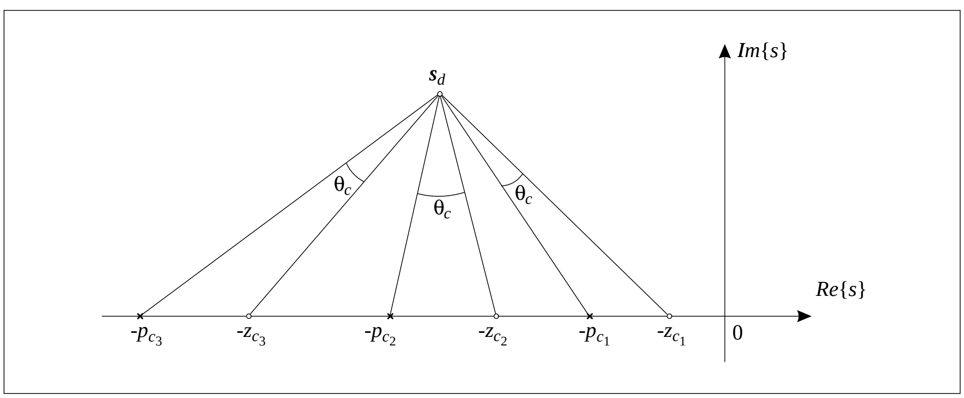

Figure 8.14: Possible locations for poles and zeros of phase-lead controllers that have the same angular contribution

All these controllers introduce the same phase shift and have the same impact on the transient response. However, the impact on the steady state errors is different since it depends on the ratio of z_c/p_c. Since this ratio for a phase-lead controller is less than one, we conclude from formula (8.27) that the corresponding steady state constant is reduced and the steady state error is increased.

Note that if the location of a phase-lead controller zero is chosen, then simple geometry, similar to that used to derive formula (8.31), can be used to find the location of the controller's pole. For example, let $-z_{c3}$ be the required zero, then using Figure 8.14 the pole $-p_{c3}$ is obtained as

$$p_{c3} = \zeta\omega_n + \omega_n\sqrt{1-\zeta^2}\tan\left(\theta_c - \varphi + \pi/2\right) \tag{8.34}$$

where $\varphi = \measuredangle(s_d + z_{c3})$. Note that $\varphi > \theta_c$.

An algorithm for the phase-lead controller design can be formulated as follows.

Design Algorithm 8.4:

1. Choose a pair of complex conjugate poles in the complex plane that produces the desired transient response (damping ratio and natural frequency). Figure 6.2 helps to accomplish this goal.
2. Find the required phase contribution of a phase-lead controller by using formula (8.33).
3. Choose values for the controller's pole and zero by placing them arbitrarily such that the controller will not damage the response dominance of a pair of complex conjugate poles. Some authors (e.g. Van de Verte, 1994) suggest placing the controller zero at $-\zeta\omega_n$.
4. Find the controller's pole by using formula (8.34).
5. Check that the compensated system has a pair of dominant complex conjugate closed-loop poles.

Example 8.9: Consider the following control system represented by its open-loop transfer function

$$G(s) = \frac{K(s+6)}{(s+10)(s^2+2s+2)}$$

It is desired that the closed-loop system have a settling time of $1.5\,\mathrm{s}$ and a maximum percent overshoot of less than 20%. From Example 8.8 we know that the system operating point should be at $s_d = -2 \pm j3.86$. A controller's phase contribution, obtained from formula (8.33) is

$$\begin{aligned}\theta_c &= \pi - 0.7676 + 0.4495 + 1.9072 + 1.7737 \\ &= 6.5044\,\mathrm{rad} = 0.2213\,\mathrm{rad} = 12.6769^\circ\end{aligned}$$

Let us locate a zero at -15 ($z_c = 15$), then by (8.34) the compensator's pole is at $-p_c = -59.2025$. The root loci of the original and compensated systems are given in Figure 8.15, and the corresponding step responses in Figure 8.16.

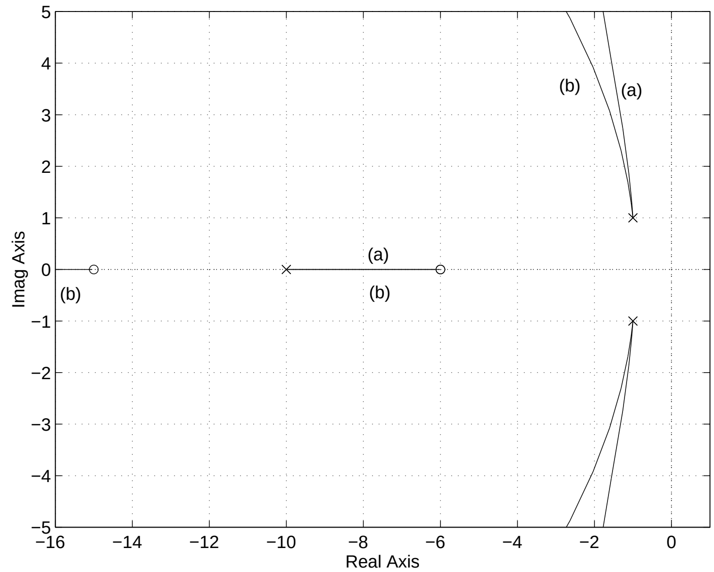

Figure 8.15: Root loci for the original (a) and compensated (b) systems

It can be seen that the root locus indeed passes through the point $-2 \pm j3.86$. For this operating point the static gain is obtained as $K = 101.56$; hence the steady state constants of the original and compensated systems are given by $K_p = 30.468$ and $K_{pc} = K_p(z_c/p_c) = 7.7196$, and the steady state errors are $e_{ss} = 0.0317, e_{ssc} = 0.1147$. Figure 8.16 reveals that for the compensated system both the maximum percent overshoot and settling time are reduced. However, the steady state unit step error is increased, as previously noted analytically.

Consider now another phase-lead compensator with a zero set at -9. From (8.34) we get $p_c = 15.291$. The root locus of the compensated system with a new controller is given in Figure 8.17.

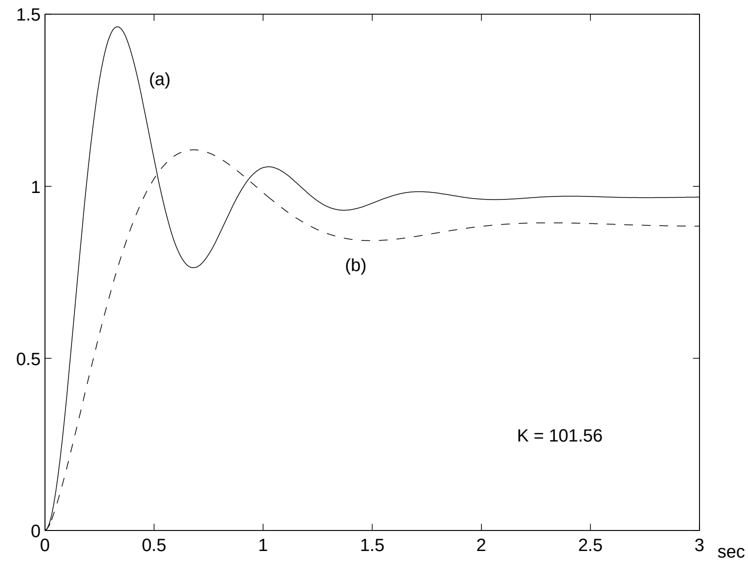

Figure 8.16: Step responses of the original (a) and compensated (b) systems

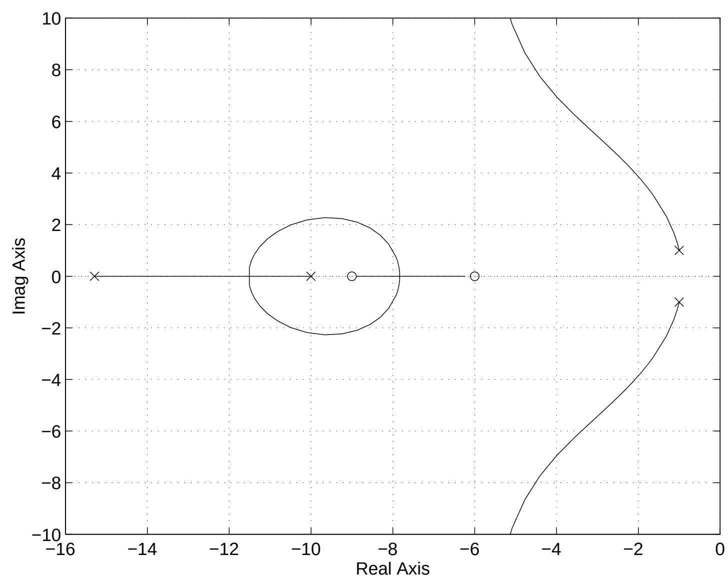

Figure 8.17: Root locus for the compensated
system with the second controller for Example 8.9

The static gain at the desired operating point $-2 \pm j3.86$ is $K = 41.587$, and hence the steady state errors are $e_{ss} = 0.0742$, $e_{ssc} = 0.11986$. The step responses of the original and compensated systems, for $K = 41.587$, are presented in Figure 8.18.

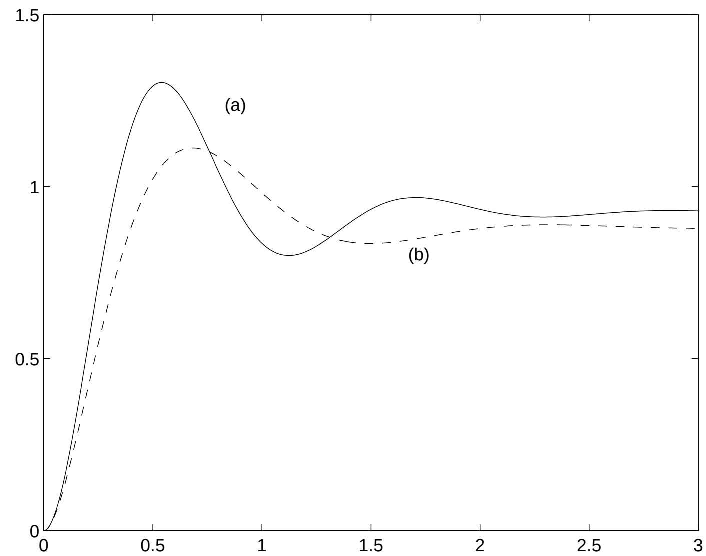

Figure 8.18: Step responses of the original (a) and compensated (b) systems with the second controller for Example 8.9

It can be seen that this controller also reduces both the overshoot and settling time, while the steady state error is slightly increased.

We can conclude that both controllers produce similar transient characteristics and similar steady state errors, but the second one is preferred since the smaller value for the static gain of the compensated system has to be used. The eigenvalues of the closed-loop system for $K = 41.587$ are given by

$$\lambda_{1c} = -12.4165, \ \ \lambda_{2c} = -10.8725, \ \ \lambda_{2c,3c} = -2.000 \pm j3.8600$$

which indicates that the response of this system is still dominated by a pair of complex conjugate poles.

◇

Remark: In some applications for a chosen desired point, s_d, the required phase increase, θ_c, may be very high. In such cases one can use a *multiple phase-lead controller* having the form

$$G_{lead}^{n}(s) = \left(\frac{s + z_c}{s + p_c}\right)^n, \quad p_c > z_c > 0 \tag{8.35}$$

so that each single phase-lead controller has to introduce a phase increase of θ_c/n.

8.5.3 PID and Phase-Lag-Lead Controller Designs

It can be observed from the previous design algorithms that implementation of a PI (phase-lag) controller does not interfere with implementation of a PD (phase-lead) controller. Since these two groups of controllers are used for different purposes—one to improve the transient response and the other to improve the steady state errors—implementing them jointly and independently will take care of both controller design requirements.

Consider first a PID controller. It is represented as

$$\begin{aligned} G_{PID}(s) &= K_p + K_d s + \frac{K_i}{s} = K_d \frac{s^2 + \frac{K_p}{K_d}s + \frac{K_i}{K_d}}{s} \\ &= K_d(s + z_{c_1})\frac{(s + z_{c_2})}{s} = G_{PD}(s)G_{PI}(s) \end{aligned} \tag{8.36}$$

which indicates that the transfer function of a PID controller is the product of transfer functions of PD and PI controllers. Since in Design Algorithms 8.1 and 8.3 there are no conflicting steps, the design algorithm for a PID controller is obtained by combining the design algorithms for PD and PI controllers.

Design Algorithm 8.5: PID Controller

1. Check the transient response and steady state characteristics of the original system.
2. Design a PD controller to meet the transient response requirements.
3. Design a PI controller to satisfy the steady state error requirements.
4. Check that the compensated system has the desired specifications.

Example 8.10: Consider the problem of designing a PID controller for the open-loop control system studied in Example 8.8, that is

$$G(s) = \frac{K(s+10)}{(s+1)(s+2)(s+12)}$$

In fact, in that example, we have designed a PD controller of the form

$$G_{PD}(s) = s + 24.1815$$

such that the transient response has the desired specifications. Now we add a PI controller in order to reduce the steady state error. The corresponding steady state error of the PD compensated system in Example 8.8 is $e_{ssc} = 0.1074$. Since a PI controller is a dipole that has its pole at the origin, we propose the following PI controller

$$G_{PI}(s) = \frac{s+0.1}{s}$$

In comparison to (8.36), we are in fact using a PID controller with $K_d = 1$, $z_{c1} = 24.1815$, $z_{c2} = 0.1$. The corresponding root locus of this system compensated by a PID controller is represented in Figure 8.19.

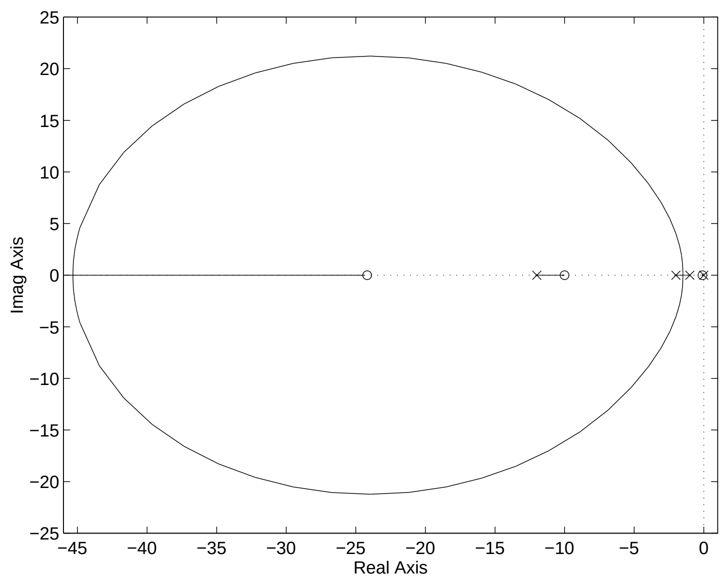

Figure 8.19: Root locus for the system from Example 8.8 compensated by the PID controller

It can be seen that the PI controller does not affect the root locus, and hence Figures 8.11b and 8.19 are almost identical except for a dipole branch.

On the other hand, the step responses of the system compensated by the PD controller and by the PID controller (see Figures 8.13 and 8.20) differ in the steady state parts. In Figure 8.13 the steady state step response tends to $y_{ss} = 0.8926$, and the response from Figure 8.20 tends to 1 since due to the presence of an open-loop pole at the origin, the steady state error is reduced to zero. Thus, we can conclude that the transient response is the same one as that obtained by the PD controller in Example 8.8, but the steady state error is improved due to the presence of the PI controller.

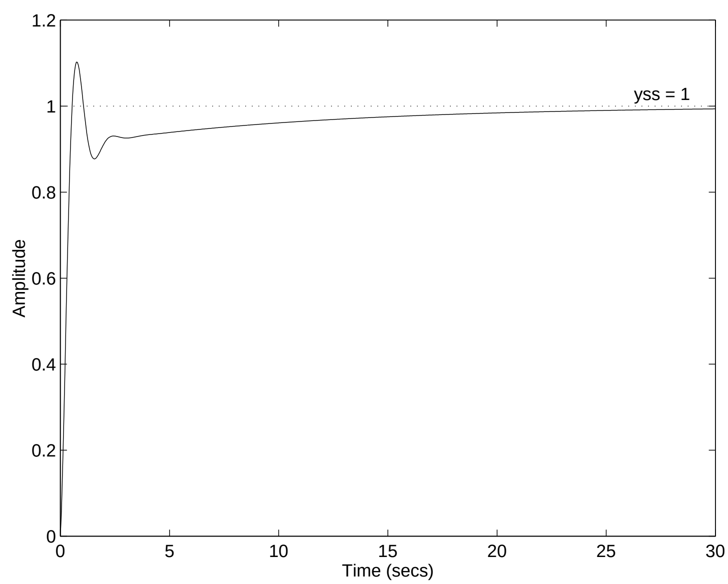

Figure 8.20: Step response of the system from Example 8.8 compensated by the PID controller

◇

Similarly to the PID controller, the design for the phase-lag-lead controller combines Design Algorithms 8.2 and 8.4. Looking at the expression for a phase-lag-lead controller given in formula (8.20), it is easy to conclude that

$$G_{lag/lead}(s) = G_{lag}(s)G_{lead}(s) \tag{8.37}$$

The phase-lag-lead controller design can be implemented by the following algorithm.

Design Algorithm 8.6: Phase-Lag-Lead Controller

1. Check the transient response and steady state characteristics of the original system.
2. Design a phase-lead controller to meet the transient response requirements.
3. Design a phase-lag controller to satisfy the steady state error requirements.
4. Check that the compensated system has the desired specifications.

Example 8.11: In this example we design a phase-lag-lead controller for a control system from Example 8.9, that is

$$G(s) = \frac{K(s+6)}{(s+10)(s^2+2s+2)}$$

such that both the system transient response and steady state errors are improved. We have seen in Example 8.9 that a phase-lead controller of the form

$$G_{lead}(s) = \frac{s+9}{s+15.291}$$

improves the transient response to the desired one. Now we add in series with the phase-lead controller another phase-lag controller, which is in fact a dipole near the origin. For this example we use the following phase-lag controller

$$G_{lag}(s) = \frac{s+0.1}{s+0.01}$$

so that the compensated system becomes

$$G(s) = G(s)G_c(s) = \frac{K(s+6)}{(s+10)(s^2+2s+2)}\frac{(s+9)}{(s+15.291)}\frac{(s+0.1)}{(s+0.01)}$$

The corresponding root locus of the compensated system and its closed-loop step response are represented in Figures 8.21 and 8.22. We can see that the addition of the phase-lag controller does not change the transient response, i.e. the root loci in Figures 8.17 and 8.21 are almost identical. However, the phase-lag controller reduces the steady state error from $e_{ss,lead} = 0.11986$ to $e_{ss,lag/lead} = 0.01344$ since the position constant is increased to

$$K_{p,lag/lead} = K_{p,lead}\frac{0.1}{0.01} = \frac{41.587 \times 9 \times 0.1}{10 \times 2 \times 15 \times 0.01} = 73.432$$

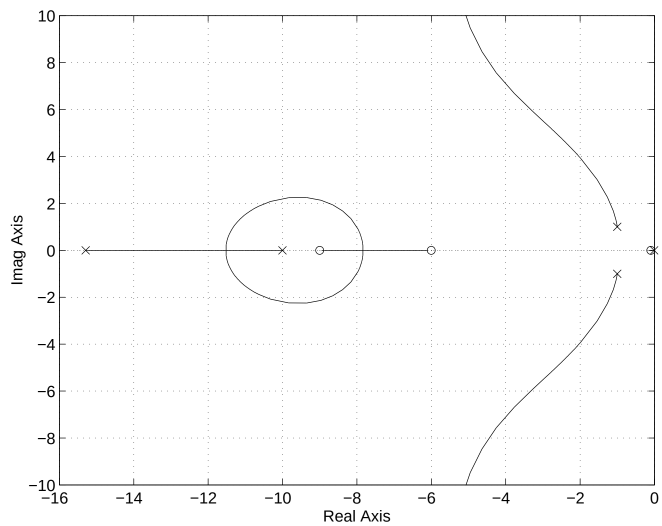

Figure 8.21: Root locus for the system from Example 8.9 compensated by the phase-lag-lead controller

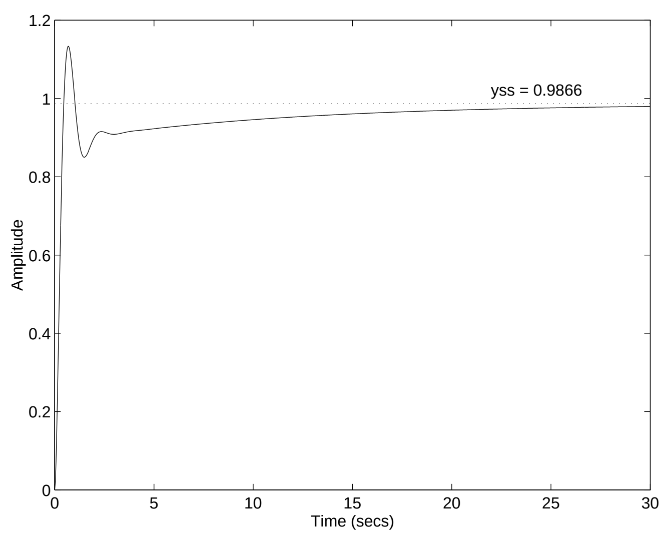

Figure 8.22: Step response of the system from Example 8.9 compensated by the phase-lag-lead controller

so that

$$e_{ss,lag/lead} = \frac{1}{1 + K_{p,lag/lead}} = 0.01344$$

◇

8.6 MATLAB Case Studies

In this section we consider the compensator design for two real control systems: a PD controller designed to stabilize a ship, and a PID controller used to improve the transient response and steady state errors of a voltage regulator control system.

8.6.1 Ship Stabilization by a PD Controller

Consider a ship positioning control system defined in the state space form in Problem 7.5. The open-loop transfer function of this control system is

$$G(s) = \frac{0.8424}{s(s + 0.0546)(s + 1.55)}$$

The root locus of the original system is presented in Figure 8.23a.

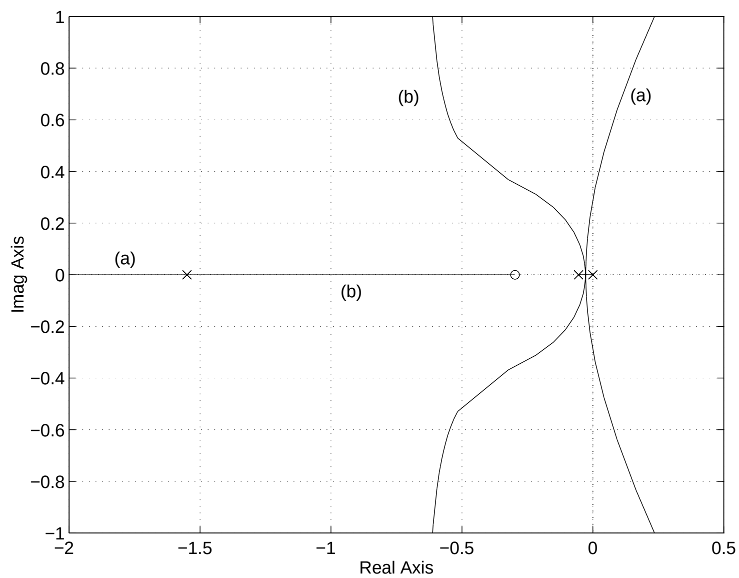

Figure 8.23: Root loci for a ship positioning control problem: (a) original system, (b) compensated system

It can be seen that this system is unstable even for very small values of the static gain. Thus, the system transient response blows up very quickly due to the system's instability. Our goal is to design a PD controller in order to stabilize the system and improve its transient response. Let the desired operating point be located at $s_d = -0.2 \pm j0.3$, which implies $w_n = 0.3606\,\mathrm{rad/s}$ and $\zeta = 0.5547$. We find from (8.30) that the required phase shift is $\alpha_c = 72.0768°$, and from (8.31) the location of the compensator zero is obtained at -0.297. Thus, the PD compensator sought is of the form

$$G_c(s) = s + 0.297$$

It can be seen from Figure 8.23 that the root locus of the compensated system indeed passes through the point $s_d = -0.2 \pm j0.3$ and that the compensated system is stable for all values of the static gain. The static gain at the desired operating point is given by $K_{s_d} = 0.6258$ and the corresponding closed-loop eigenvalues at this operating point are $\lambda_{1c} = -1.2046,\ \lambda_{2c,3c} = -0.2 \pm j0.3$. In Figure 8.24 the unit step response of the compensated system is presented.

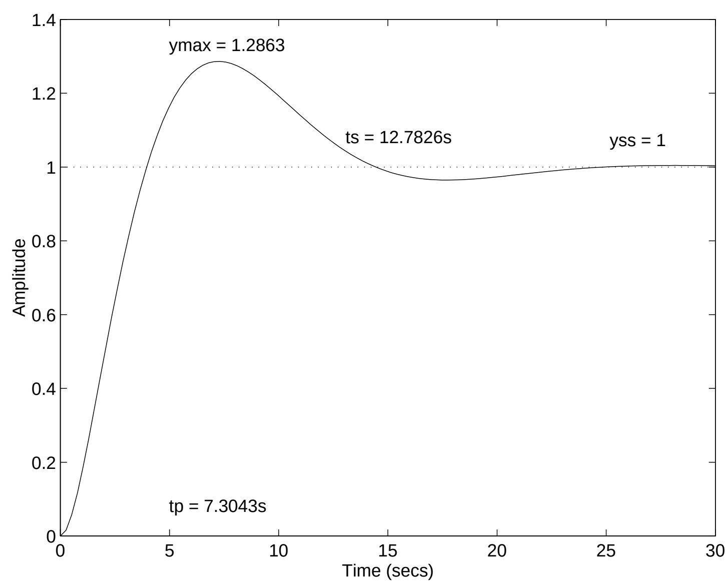

Figure 8.24: Step response of a ship positioning compensated control system

It is found that $y_{max} = 1.2863$, $t_p = 7.3043\,\mathrm{s}$, and $t_s = 12.7826\,\mathrm{s}$. From the same figure we observe that the steady state error for this system is zero, which also follows from the fact that the system open-loop transfer function has one pole at the origin.

8.6.2 PID Controller for a Voltage Regulator Control System

The mathematical model of a voltage regulator control system (Kokotović, 1972) is given in Section 6.7. The open-loop transfer function of this system is

$$G(s) = \frac{154280}{(s+0.2)(s+0.5)(s+10)(s+14.28)(s+25)}$$

The corresponding root locus is presented in Figure 8.25. Since one of the branches goes quite quickly into the instability region, our design goal is to move this branch to the left so that it passes through the operating point selected as $s_d = -1 \pm j1$. For this operating point, we have $w_n = \sqrt{2}\ \mathrm{rad/s}$ and $\zeta = 0.7071$ so that the expected maximum percent overshoot and the 5%-settling time of the compensated system are $MPOS = 4.3214\%,\ t_s = 3\,\mathrm{s}$. In addition, the design objective is to reduce the steady state error due to a unit step to zero.

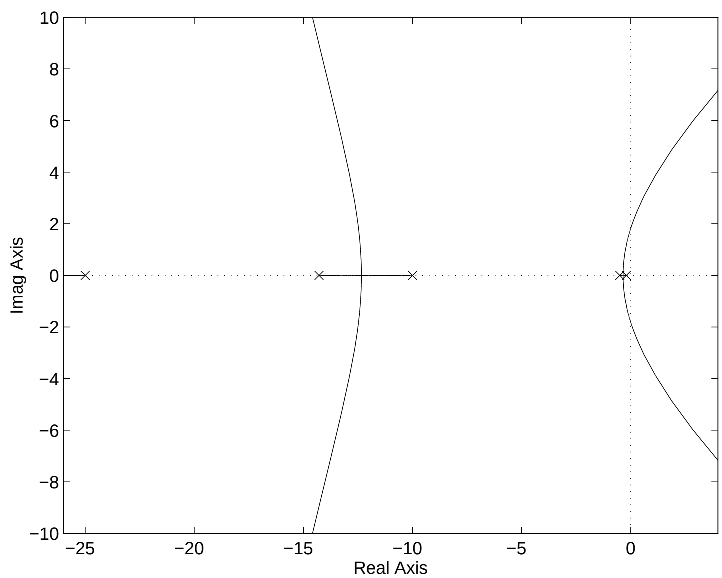

Figure 8.25: Root locus for a voltage regulator system

We use a PID controller to solve the controller design problem defined above. The required phase improvement for the selected operating point is found by using

(8.30) as $\alpha_c = 1.3658\,\mathrm{rad} = 78.2573°$. From formula (8.31) the location of the compensator's zero is obtained as $-z_c = -1.2079$, so that the PD part of a PID compensator is

$$G_{PD}(s) = s + 1.2079$$

The branches of the root loci in the neighborhood of the desired operating point of the original and PD compensated systems are presented in Figure 8.26. It can be seen that the compensated root locus indeed passes through the point $s_d = -1 \pm j1$.

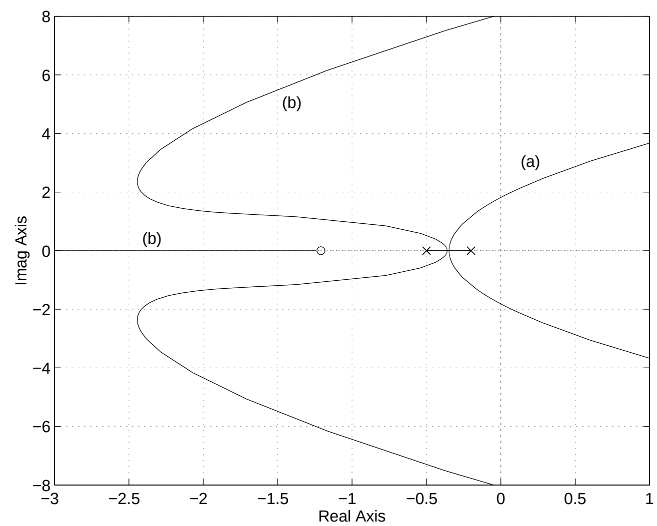

Figure 8:26: Root loci of the original (a) and PD (b) compensated systems

The closed-loop unit step response of the system compensated by the PD controller is represented in Figure 8.27. Using the MATLAB programs given in Example 8.8, gives $MPOS = 6.08\%$, $t_p = 2.1\,\mathrm{s}$, and $t_s = 3.5\,\mathrm{s}$, which is quite satisfactory. However, the steady state unit step error is $e_{ssPD} = 0.0808$. Note that the static gain at the operating point, obtained by applying the root locus rule number 9 from Table 7.1, is $K_{s_d} = 4060.8$. The closed-loop eigenvalues at the operating point are

$$\lambda_{1_{PD}} = -23.7027, \quad \lambda_{2_{PD}} = -18.1675, \quad \lambda_{3_{PD}} = -6.1105$$
$$\lambda_{4,5_{PD}} = -0.997 \pm j1.0011$$

which indicates that the system has preserved a pair of dominant complex conjugate poles.

In order to reduce this steady state error to zero we use a PI controller of the form

$$G_{PI}(s) = \frac{s + 0.1}{s}$$

Since the compensated system open-loop transfer function now has a pole at the origin, we conclude that the steady state error is reduced to zero, which can also be observed from Figure 8.27.

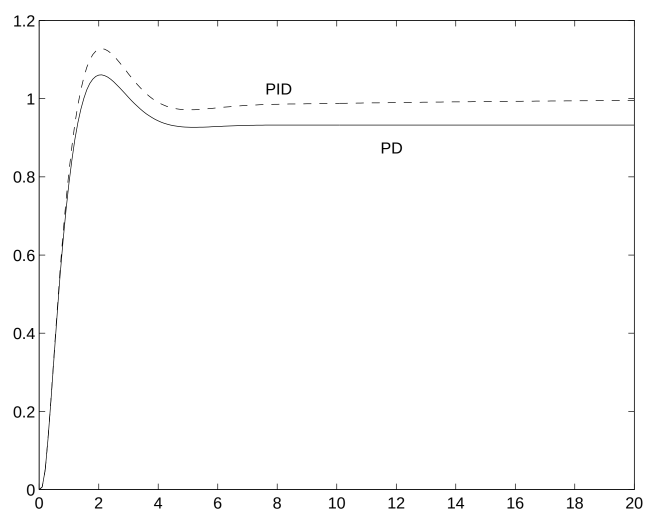

Figure 8.27: Step responses of PD and PID compensated systems

The transient response specifications for the system compensated by the proposed PID controller are $MPOS = 11.277\%$, $t_p = 2.1\,\mathrm{s}$, and $t_s = 3.1\,\mathrm{s}$. Thus, the proposed PI controller has slightly worsened the transient response characteristics. It is left to students, in the form of a MATLAB laboratory experiment, to check that the transient response specifications of the compensated system obtained by using PI controllers that have zeros located at -0.01 and -0.001 are improved.

8.7 Comments on Discrete-Time Controller Design

Similarly to the continuous-time controller design, the root locus method can be used for design of controllers (compensators) in the discrete-time domain. In Section 7.4 we have indicated that the discrete-time root locus method is identical to its continuous-time counterpart. Thus, the results presented in this chapter can be easily extended and used for the controller design of discrete-time systems. For more details, the reader is referred to Kuo (1992) for theoretical aspects, and to Shahian and Hassul (1993) for MATLAB discrete-time controller design.

8.8 MATLAB Laboratory Experiment

Part 1. Consider the control system given in Example 8.8, that is

$$G(s) = \frac{K(s+10)}{(s+1)(s+2)(s+12)}$$

with the transient response requirements determined by a desired operating point located at $s_d = -2 + j3$.

(a) Design a PD controller such that the transient response requirements are met.
(b) Design a PID controller such that the steady state error is reduced to zero.
(c) Design a phase-lead controller for the design problem defined in (a).
(d) Add a phase-lag controller in series with the controller obtained in (c) in order to reduce the steady state error by 50%.

Plot the root loci and unit step responses for (b) and (d) of both the original and compensated systems and compare the results obtained with PID and phase-lag-lead controllers.

Part 2. Use MATLAB to design a phase-lag-lead controller for the control system of Example 8.9 such that $t_s < 2\,\mathrm{s}$, $MPOS < 10\%$, and $e_{ssc} < 0.01$.

Use the MATLAB program from Example 8.8 to find the actual response overshoot and settling time. Find the eigenvalues of the compensated system and check whether the system response is dominated by a pair of complex conjugate poles.

Part 3. For the voltage regulator system considered in Section 8.6.2, use the same PD controller as in Section 8.6.2, but take for a PI controller the following forms:

(a) $G_{PI}(s) = (s + 0.01)/s$.

(b) $G_{PI}(s) = (s + 0.001)/s$.

For both cases, find the unit step responses of the PID compensated systems and determine the transient response parameters. Compare the results obtained with the results from Section 8.6.2.

8.9 References

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8.10 Problems

8.1 Consider a single-input single-output system whose open-loop transfer function is given by

$$G(s) = \frac{10}{s^4 + 3s^3 + 4s^2 + 1}$$

Find the phase variable canonical form for this system and design the full state feedback static controller such that the system closed-loop poles are located at $\lambda_{1,2} = -1 \pm j2, \lambda_3 = -3, \lambda_4 = -10$.

8.2 Find the feedback gain $\mathbf{f}$ such that the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 2 & 3 \\ 0 & -2 & 1 \\ 1 & -1 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} u$$

with $u = -\mathbf{fx}$ has a pair of dominant complex conjugate closed-loop poles at $-1 \pm j2$ and a closed-loop pole at -10.

8.3 Plot the function

$$e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

and comment on the maximum percent overshoot dependence on the damping ratio.

8.4 Design a rate feedback controller such that for the second-order system represented by the closed-loop transfer function

$$\frac{9}{s^2 + 3s + 9}$$

the maximum percent overshoot is 10%. Find the settling time of the compensated system.

8.5 Consider the second-order system from Example 6.6. Its steady state error is $e_{ss} = 0.4$. Design a PI controller to reduce the steady state error to zero. Use MATLAB to plot the root loci and unit step responses of the original and compensated systems.

8.6 For the open-loop control system

$$G(s) = \frac{K(s+10)}{s(s+20)(s^2+2s+10)}$$

draw the root locus. Check that the static gain $K = 3$ produces a pair of complex conjugate dominant poles. Find the corresponding steady state errors and transient response parameters. Design a PI controller such that the steady state error is reduced to zero while the transient response characteristics are preserved.

8.7 Consider the controller design problem for the hydroturbine governors of a power system (Arnautović and Skatarić, 1991), represented by

$$\mathbf{A}=\begin{bmatrix}-0.71 & 0 & 0 & 0 & 0\\ 0 & -2 & 0 & 0 & 0\\ 0.61 & 1.28 & -1.46 & 0.566 & 0\\ -0.18 & -0.37 & 0.56 & -0.594 & -0.23\\ 0 & 0 & 0 & 314.16 & 0\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}0.71\\ 2\\ 0\\ 0\\ 0\end{bmatrix}$$

Assume that the output matrices are given by

$$\mathbf{C}=\begin{bmatrix}0 & 0 & 1 & 0 & 1\end{bmatrix},\quad \mathbf{D}=0$$

Using MATLAB, perform the following:

(a) Find its open-loop transfer function and the steady state error.
(b) Suggest a phase-lag controller to reduce its steady state error ten times.
(c) Locate the system operating point on the root locus approximately at $-1.5+j1$ and find the static gain at that point.

8.8 Design a phase-lag controller for the system represented by

$$G(s)=\frac{K}{(s+1)(s+5)(s+10)}$$

which produces a steady state unit step error of less than 0.01. Take $K=200$.

8.9 Consider the synchronous machine from Section 7.5.2. Design a PI controller to reduce the steady state unit step error by 100%. Choose an operating point on the root locus and find the corresponding static gain. Use MATLAB to check that the closed-loop system is asymptotically stable.

8.10 For the second-order system from Example 6.1, design a PD controller such that the compensated system has $t_s \approx 2\,\mathrm{s}$ and $MPOS < 15\%$.

8.11 Repeat Problem 8.10 with the following requirements $t_s \approx 2\,\mathrm{s}$ and $MPOS < 10\%$.

8.12 Solve Problem 8.11 using a phase-lead controller.

8.13 Consider the problem of a PD controller design. Assume that a controller's zero and desired operating point satisfy $\measuredangle(s_d+z_c) > 90°$. Derive a formula corresponding to (8.31).

8.14 Consider the phase-lead controller design for the system given in Example 8.9. Use different values for the controller's poles and zeros and examine their impact on the steady state errors. Suggest at least five different controllers.

8.15 Derive formula (8.34) for finding the location of the pole of a phase-lead controller assuming that the location of its zero is chosen.

8.16 Design a phase lead-controller such that

$$G(s) = \frac{K(s+10)}{(s+20)(s^2+2s+2)}$$

has $t_s \leq 1.5\,\mathrm{s}$ and $MPOS \leq 20\%$.

8.17 Consider the voltage regulator control system from Section 8.6.2. Use a double PD controller of the form

$$G_{PD}(s) = (s+1.2079)^2$$

to compensate this system.

(a) Draw the root locus of this system compensated with a double PD controller and compare it with the corresponding one from Section 8.6.2 obtained using a single PD controller.

(b) Choose the operating point for a pair of complex conjugate poles such that the damping ratio is the same as in Section 8.6.2, i.e. $\zeta = 0.7071$. Find the static gain at that point and the closed-loop eigenvalues. Does the compensated system preserve the response dominance of a pair of complex conjugate poles?

8.18 Design a phase-lag-lead controller for the voltage regulator system from Section 8.6.2 such that $MPOS < 5\%$, $t_s < 3\,\mathrm{s}$, and $e_{ss} < 1\%$.

8.19 Consider the ship position control system from Section 8.6.1. Design a phase-lead controller such that $MPOS < 10\%$ and $t_s < 10\,\mathrm{s}$.

8.20 Use MATLAB to find and plot the ramp responses of the original and compensated systems studied in Example 8.7.

8.21 Consider the F-15 aircraft under supersonic flight conditions with its state space matrices given in Example 1.4. Note that this aircraft has one input and four outputs, where the outputs represent the state space variables. Thus, we are able to get four transfer functions of the form

$$G_i(s) = \mathbf{C}_i(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}, \qquad i = 1, 2, 3, 4$$

(a) Using MATLAB, find all four transfer functions of the F-15 aircraft.
(b) Plot the root loci for $G_i(s),\ i = 1, 2, 3, 4$. Comment on the stability properties of this aircraft with respect to the values of the static feedback gains $K_i,\ i = 1, 2, 3, 4$.
(c) Find the closed-loop transfer functions with unit feedback and examine their stability.
(d) Propose controllers which will stabilize all outputs of this aircraft.
(e) For the proposed controllers that assure stabilization, find the closed-loop step responses and determine the response steady state errors and transient parameters.
(f) If necessary, design dynamic controllers based on the root locus technique, as discussed in this chapter, such that steady state errors and transient response parameters are improved.

8.22 Repeat Problem 8.21 for the F-15 aircraft under subsonic flight conditions with the state space matrices given in Example 1.4.[1]

[1] Problems 8.21 and 8.22 can be assigned as either term papers or final projects.