Chapter Ten

Control System Theory Overview

In this book we have presented results mostly for continuous-time, time-invariant, deterministic control systems. We have also, to some extent, given the corresponding results for discrete-time, time-invariant, deterministic control systems. However, in control theory and its applications several other types of system appear. If the coefficients (matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$) of a linear control system change in time, one is faced with *time-varying control systems*. If a system has some parameters or variables of a random nature, such a system is classified as a stochastic system. Systems containing variables delayed in time are known as *systems with time delays*.

In applying control theory results to real-world systems, it is very important to minimize both the amount of energy to be spent while controlling a system and the difference (error) between the actual and desired system trajectories. Sometimes a control action has to be performed as fast as possible, i.e. in a minimal time interval. These problems are addressed in modern *optimal control theory*. The most recent approach to optimal control theory emerged in the early eighties. This approach is called the H_∞ *optimal control theory*, and deals simultaneously with the optimization of certain performance criteria and minimization of the norm of the system transfer function(s) from undesired quantities in the system (disturbances, modeling errors) to the system's outputs.

Obtaining mathematical models of real physical systems can be done either by applying known physical laws and using the corresponding mathematical equations, or through an experimental technique known as *system identification*. In the latter case, a system is subjected to a set of standard known input functions and by measuring the system outputs, under certain conditions, it is possible to obtain a mathematical model of the system under consideration.

In some applications, systems change their structures so that one has first to perform on-line estimation of system parameters and then to design a control law that will produce the desired characteristics for the system. These systems are known as *adaptive control systems*. Even though the original system may be linear, by using the closed-loop adaptive control scheme one is faced, in general, with a nonlinear control system problem.

Nonlinear control systems are described by nonlinear differential equations. One way to control such systems is to use the linearization procedure described in Section 1.6. In that case one has to know the system nominal trajectories and inputs. Furthermore, we have seen that the linearization procedure is valid only if deviations from nominal trajectories and inputs are small. In the general case, one has to be able to solve nonlinear control system problems. Nonlinear control systems have been a "hot" area of research since the middle of the eighties, since when many valuable nonlinear control theory results have been obtained. In the late eighties and early nineties, *neural networks, which are in fact nonlinear systems with many inputs and many outputs*, emerged as a universal technological tool of the future. However, many questions remain to be answered due to the high level of complexity encountered in the study of nonlinear systems.

In the last section of this chapter, we comment on other important areas of control theory such as algebraic methods in control systems, discrete events systems, intelligent control, fuzzy control, large scale systems, and so on.

10.1 Time-Varying Systems

A time-varying, continuous-time, linear control system in the state space form is represented by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_o \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \end{aligned} \tag{10.1}$$

Its coefficient matrices are time functions, which makes these systems much more challenging for analytical studies than the corresponding time-invariant ones.

It can be shown that the solution of (10.1) is given by (Chen, 1984)

$$\mathbf{x}(t) = \Phi(t,t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau \tag{10.2}$$

where $\Phi(t,t_0)$ is the state transition matrix. For an *input-free system*, the transition matrix relates the state of the system at the initial time and the state of the system at any given time, that is

$$\mathbf{x}(t) = \Phi(t,t_0)\mathbf{x}(t_0) \tag{10.3}$$

It is easy to establish from (10.3) that the state transition matrix has the following properties:

(1) $\Phi(t,t_0)$ satisfies the system differential equation

$$\frac{\partial}{dt}\Phi(t,t_0) = \mathbf{A}(t)\Phi(t,t_0), \qquad \Phi(t_0,t_0) = \mathbf{I} \tag{10.4}$$

(2) $\Phi(t,t_0)$ is nonsingular, which follows from

$$\Phi^{-1}(t,t_0) = \Phi(t_0,t) \tag{10.5}$$

(3) $\Phi(t,t_0)$ satisfies

$$\Phi(t_2,t_0) = \Phi(t_2,t_1)\Phi(t_1,t_0) \tag{10.6}$$

Due to the fact that the system matrix, $\mathbf{A}(t)$, is a function of time, *it is not possible, in general, to find the analytical expression for the system state transition matrix* so that the state response equation (10.2) can be solved only numerically.

Since the coefficient matrices $\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t), \mathbf{D}(t)$ are time functions, three essential system concepts presented in Chapters 4 and 5—stability, controllability, and observability—have to be redefined for the case of time-varying systems.

The stability of time-varying systems cannot be defined in terms of the system eigenvalues as for time-invariant systems. Furthermore, several stability definitions have to be introduced for time-varying systems, such as bounded-input bounded-output stability, stability of the system's equilibrium points, global stability, uniform stability, and so on (Chen, 1984). The stability in the sense of Lyapunov, applied to the system equilibrium points, $\mathbf{x}_e(t)$, defined by

$$\mathbf{A}(t)\mathbf{x}_e(t) = 0 \tag{10.7}$$

indicates that the corresponding equilibrium points are stable if the system state transition matrix is bounded, that is

$$\|\Phi(t, t_0)\| \leq \mathrm{const} < \infty, \qquad \forall t > t_0 \tag{10.8}$$

Since the system transition matrix has no analytical expression, it is hard, in general, to test the stability condition (10.8).

Similarly, the controllability and observability of time-varying systems are tested differently to the corresponding ones of time-invariant systems. It is necessary to use the notions of controllability and observability Grammians of time-varying systems, respectively, defined by

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\Phi(t_0, \tau)d\tau \tag{10.9}$$

and

$$W_o(t_0, t_1) = \int_{t_0}^{t_1} \Phi^T(\tau, t_0)\mathbf{C}^T(\tau)\mathbf{C}(\tau)\Phi(\tau, t_0)d\tau \tag{10.10}$$

The controllability and observability tests are defined in terms of these Grammians and the state transition matrix. Since the system state transition matrix is not known in its analytical form, we conclude that it is very hard to test the controllability and observability of time-varying systems. The reader interested in this topic is referred to Chen (1984), Klamka (1991), and Rugh (1993).

Corresponding results can be presented for discrete-time, time-varying, linear systems defined by

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k), \qquad \mathbf{x}(k_0) = \mathbf{x}_o \\ \mathbf{y}(k) &= \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{u}(k) \end{aligned} \tag{10.11}$$

The state transition matrix for the system (10.11) is given by

$$\Phi(k, k_0) = \mathbf{A}(k)\mathbf{A}(k-1)\cdots\mathbf{A}(k_0+1)\mathbf{A}(k_0) \tag{10.12}$$

It is interesting to point out that, unlike the continuous-time result, *the discrete-time transition matrix of time-varying systems is in general singular*. It is nonsingular if and only if the matrix $\mathbf{A}(i)$ is nonsingular for $\forall i = k_0, k_0+1, ..., k$.

Similarly to the stability study of continuous-time, time-varying systems, in the discrete-time domain one has to consider several stability definitions. The eigenvalues are no longer indicators of system stability. The stability of the system equilibrium points can be tested in terms of the bounds imposed on the system state transition matrix. The system controllability and observability conditions are given in terms of the discrete-time controllability and observability Grammians. The interested reader can find more about discrete-time, time-invariant and time-varying linear systems in Ogata (1987).

10.2 Stochastic Linear Control Systems

Stochastic linear control systems can be defined in several frameworks, such as jump linear systems, Markov chains, systems driven by white noise, to name a few. From the control theory point of view, linear control systems driven by white noise are the most interesting. Such systems are described by the following equations

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{G}\mathbf{w}(t), \quad E\{\mathbf{x}(t_0)\} = \overline{\mathbf{x}}_\mathrm{o} \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t)\end{aligned} \tag{10.13}$$

where $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are white noise stochastic processes, which represent the system noise (disturbance) and the measurement noise (inaccuracy of sensors or their inability to measure the state variables perfectly). White noise stochastic processes are mathematical fictions that represent real stochastic processes that have a large frequency bandwidth. They are good approximate mathematical models for many real physical processes such as wind, white light, thermal noise, unevenness of roads, and so on. *The spectrum of white noise is constant at all frequencies*. The corresponding constant is called the white noise intensity. Since the spectrum of a signal is the Fourier transform of its covariance, the covariance matrices of the system (plant) white noise and measurement white noise are given by

$$E\left\{\mathbf{w}(t)\mathbf{w}^T(\tau)\right\} = \mathbf{W}\delta(t-\tau), \quad E\left\{\mathbf{v}(t)\mathbf{v}^T(\tau)\right\} = \mathbf{V}\delta(t-\tau) \tag{10.14}$$

It is also assumed that the mean values of these white noise processes are equal to zero, that is

$$E\{\mathbf{w}(t)\} = \mathbf{0}, \qquad E\{\mathbf{v}(t)\} = \mathbf{0} \tag{10.15}$$

even though this assumption is not crucial.

The problem of finding the optimal control for (10.13) such that a given performance criterion is optimized will be studied in Section 10.3. Here we consider only an input-free system, $\mathbf{u}(t) = 0$, subjected to (i.e. corrupted by) white noise

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{G}\mathbf{w}(t), \quad E\{\mathbf{x}(t_0)\} = \overline{\mathbf{x}}_\text{o} \tag{10.16}$$

and present results for the mean and variance of the state space variables. In order to obtain a valid result, i.e. in order that the mean and variance describe completely the stochastic nature of the system, we have to assume that white noise stochastic processes are Gaussian. It is well known that *Gaussian stochastic processes are completely described by their mean and variance* (Kwakernaak and Sivan, 1972).

Applying the expected value (mean) operator to equation (10.16), we get

$$E\{\dot{\mathbf{x}}(t)\} = \mathbf{A}E\{\mathbf{x}(t)\} + \mathbf{G}E\{\mathbf{w}(t)\}, \quad E\{\mathbf{x}(t_0)\} = \overline{\mathbf{x}}_\text{o} \tag{10.17}$$

Denoting the expected value $E\{\mathbf{x}(t)\} = \mathbf{m}(t)$, and using the fact that the mean value of white noise is zero, we get

$$\dot{\mathbf{m}}(t) = \mathbf{A}\mathbf{m}(t), \quad \mathbf{m}(t_0) = E\{\mathbf{x}(t_0)\} = \overline{\mathbf{x}}_\text{o} \tag{10.18}$$

which implies that the mean of the state variables of a linear stochastic system driven by white noise is represented by a pure deterministic input-free system, the solution of which has the known simple form

$$\mathbf{m}(t) = E\{\mathbf{x}(t)\} = e^{\mathbf{A}(t-t_0)}\mathbf{m}(t_0) = e^{\mathbf{A}(t-t_0)}\overline{\mathbf{x}}_\text{o} \tag{10.19}$$

In order to be able to find the expression for the variance of state trajectories for the system defined in (10.16), we also need to know a value for the initial variance of $\mathbf{x}(t_0)$. Let us assume that the initial variance is $\mathbf{Q}(t_0) = \mathbf{Q}_\text{o}$. The variance is defined by

$$Var\{\mathbf{x}(t)\} = E\Big\{[\mathbf{x}(t) - \mathbf{m}(t)][\mathbf{x}(t) - \mathbf{m}(t)]^T\Big\} = \mathbf{Q}(t) \tag{10.20}$$

It is not hard to show (Kwakernaak and Sivan, 1972; Sage and White, 1977) that *the variance of the state variables of a continuous-time linear system driven by white noise satisfies the famous continuous-time matrix Lyapunov equation*

$$\dot{\mathbf{Q}}(t) = \mathbf{Q}(t)\mathbf{A}^T + \mathbf{A}\mathbf{Q}(t) + \mathbf{G}\mathbf{W}\mathbf{G}^T, \quad \mathbf{Q}(t_0) = \mathbf{Q}_{\mathrm{o}} \tag{10.21}$$

Note that if the system matrix $\mathbf{A}$ is stable, the system reaches the steady state and the corresponding state variance is given by the algebraic Lyapunov equation of the form

$$\mathbf{0} = \mathbf{Q}\mathbf{A}^T + \mathbf{A}\mathbf{Q} + \mathbf{G}\mathbf{W}\mathbf{G}^T \tag{10.22}$$

which is, in fact, the steady state counterpart to (10.21).

Example 10.1: MATLAB and its function `lyap` can be used to solve the algebraic Lyapunov equation (10.22). In this example, we find the variance of the state variables of the F-8 aircraft, considered in Section 5.7, under wind disturbances. The matrix $\mathbf{A}$ is given in Section 5.7. Matrices $\mathbf{G}$ and $\mathbf{W}$ are given by Teneketzis and Sandell (1977)

$$\mathbf{G} = \begin{bmatrix} -46.3 & 1.214 & -1.214 & -9.01 \end{bmatrix}^T, \quad \mathbf{W} = 0.000315$$

Note that the wind can be quite accurately modeled as a white noise stochastic process with the intensity matrix $\mathbf{W}$ (Teneketzis and Sandell, 1977). The MATLAB statement

```
Q=lyap(A,G*W*G');
```

produces the unique solution (since the matrix $\mathbf{A}$ is stable) for (10.22) as

$$\mathbf{Q} = \begin{bmatrix} 0.4731 & -0.0050 & 0.0106 & 0.0326 \\ -0.0050 & 0.0001 & -0.0002 & -0.0009 \\ 0.0106 & -0.0002 & 0.0009 & 0.0009 \\ 0.0326 & -0.0009 & 0.0009 & 0.0068 \end{bmatrix}$$

◇

Similarly, one is able to obtain corresponding results for discrete-time systems, i.e. the state trajectories of a linear stochastic discrete-time system driven by Gaussian white noise

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{G}\mathbf{w}(k), \;\; E\{\mathbf{x}(0)\} = \overline{\mathbf{x}}_{\mathrm{o}}, \;\; Var\{\mathbf{x}(0)\} = \mathbf{Q}_{\mathrm{o}} \tag{10.23}$$

satisfy the following mean and variance equations

$$\mathbf{m}(k+1) = \mathbf{A}\mathbf{m}(k), \quad \mathbf{m}(0) = \mathbf{m}_\mathrm{o} \quad \Rightarrow \mathbf{m}(k) = \mathbf{A}^k\mathbf{m}_\mathrm{o} \tag{10.24}$$

$$\mathbf{Q}(k+1) = \mathbf{A}\mathbf{Q}(k)\mathbf{A}^T + \mathbf{G}\mathbf{W}\mathbf{G}^T, \quad \mathbf{Q}(0) = \mathbf{Q}_\mathrm{o} \tag{10.25}$$

Equation (10.25) is known as the *discrete-time matrix difference Lyapunov equation*. If the matrix $\mathbf{A}$ is stable, then one is able to define the steady state system variance in terms of the solution of the algebraic discrete-time Lyapunov equation. This equation is obtained by setting $\mathbf{Q}(k+1) = \mathbf{Q}(k) = \mathbf{Q}$ in (10.25), that is

$$\mathbf{Q} = \mathbf{A}\mathbf{Q}\mathbf{A}^T + \mathbf{G}\mathbf{W}\mathbf{G}^T \tag{10.26}$$

The MATLAB function `dlyap` can be used to solve (10.26). The interested reader can find more about the continuous- and discrete-time Lyapunov matrix equations, and their roles in system stability and control, in Gajić and Qureshi (1995).

10.3 Optimal Linear Control Systems

In Chapters 8 and 9 of this book we have designed dynamic controllers such that the closed-loop systems display the desired transient response and steady state characteristics. The design techniques presented in those chapters have sometimes been limited to trial and error methods while searching for controllers that meet the best given specifications. Furthermore, we have seen that in some cases it has been impossible to satisfy all the desired specifications, due to contradictory requirements, and to find the corresponding controller.

Controller design can also be done through rigorous mathematical optimization techniques. One of these, which originated in the sixties (Kalman, 1960)—called modern optimal control theory in this book—is a time domain technique. During the sixties and seventies, the main contributor to modern optimal control theory was Michael Athans, a professor at the Massachusetts Institute of Technology (Athans and Falb, 1966). Another optimal control theory, known as H_∞, is a trend of the eighties and nineties. H_∞ optimal control theory started with the work of Zames (1981). It combines both the time and frequency domain optimization techniques to give a unified answer, which is optimal from both the time domain and frequency domain points of view (Francis, 1987). Similarly to

H_∞ optimal control theory, the so-called H_2 optimal control theory optimizes systems in both time and frequency domains (Doyle *et al*., 1989, 1992; Saberi *et al*., 1995) and is the trend of the nineties. Since H_∞ optimal control theory is mathematically quite involved, in this section we will present results only for the modern optimal linear control theory due to Kalman. It is worth mentioning that very recently a new philosophy has been introduced for system optimization based on linear matrix inequalities (Boyd *et al*., 1994).

In the context of the modern optimal linear control theory, we present results for the deterministic optimal linear regulator problem, the optimal Kalman filter, and the optimal stochastic linear regulator. Only the main results are given without derivations. This is done for both continuous- and discrete-time domains with emphasis on the infinite time optimization (steady state) and continuous-time problems. In some places, we also present the corresponding finite time optimization results. In addition, several examples are provided to show how to use MATLAB to solve the corresponding optimal linear control theory problems.

10.3.1 Optimal Deterministic Regulator Problem

In modern optimal control theory of linear deterministic dynamic systems, represented in continuous-time by

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{10.27}$$

we use linear state feedback, that is

$$\mathbf{u}(t) = -\mathbf{F}(t)\mathbf{x}(t) \tag{10.28}$$

and optimize the value for the feedback gain, $\mathbf{F}(t)$, such that the following performance criterion is minimized

$$J = \min_{\mathbf{u}(t)} \left\{ \frac{1}{2} \int_{t_0}^{t_f} \left[\mathbf{x}^T(t)\mathbf{R}_1\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}_2\mathbf{u}(t)\right] dt \right\}, \quad \mathbf{R}_1 \geq 0, \ \ \mathbf{R}_2 > 0 \tag{10.29}$$

This choice for the performance criterion is quite logical. It requires minimization of the "square" of input, which means, in general, minimization of the input energy required to control a given system, and minimization of the "square" of the state variables. Since the state variables—in the case when a linear system is obtained through linearization of a nonlinear system—represent deviations from the nominal system trajectories, control engineers are interested in minimizing the "square" of this difference, i.e. the "square" of $\mathbf{x}(t)$. In the case when the linear mathematical model (10.27) represents the "pure" linear system, the minimization of (10.29) can be interpreted as the goal of bringing the system as close as possible to the origin ($\mathbf{x}(t) = \mathbf{0}$) while optimizing the energy. This regulation to zero can easily be modified (by shifting the origin) to regulate state variables to any constant values.

It is shown in Kalman (1960) that the linear feedback law (10.28) produces the global minimum of the performance criterion (10.29). The solution to this optimization problem, obtained by using one of two mathematical techniques for dynamic optimization—dynamic programming (Bellman, 1957) and calculus of variations—is given in terms of the solution of the famous Riccati equation (Bittanti *et al.*, 1991; Lancaster and Rodman, 1995). It can be shown (Kalman, 1960; Kirk, 1970; Sage and White, 1977) that the required optimal solution for the feedback gain is given by

$$\mathbf{F}_{opt}(t) = -\mathbf{R}_2^{-1}\mathbf{B}^T(t)\mathbf{P}(t) \tag{10.30}$$

where $\mathbf{P}(t)$ is the positive semidefinite solution of the matrix differential Riccati equation

$$-\dot{\mathbf{P}}(t) = \mathbf{A}^T(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{R}_1 - \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}_2^{-1}\mathbf{B}^T(t)\mathbf{P}(t), \quad \mathbf{P}(t_f) = 0 \tag{10.31}$$

In the case of time invariant systems and for an infinite time optimization period, i.e. for $t_f \to \infty$, the differential Riccati equation becomes an algebraic one

$$\mathbf{0} = \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{R}_1 - \mathbf{P}\mathbf{B}\mathbf{R}_2^{-1}\mathbf{B}^T\mathbf{P} \tag{10.32}$$

If the original system is both controllable and observable (or only stabilizable and detectable) the unique positive definite (semidefinite) solution of (10.32) exists, such that the closed-loop system

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} - \mathbf{B}\mathbf{R}_2^{-1}\mathbf{B}^T\mathbf{P}\right)\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_\text{o} \tag{10.33}$$

is asymptotically stable. In addition, the optimal (minimal) value of the performance criterion is given by (Kirk, 1970; Kwakernaak and Sivan, 1972; Sage and White, 1977)

$$J_{opt} = J_{min} = \frac{1}{2}\mathbf{x}_\text{o}^T\mathbf{P}\mathbf{x}_\text{o} \tag{10.34}$$

Example 10.2: Consider the linear deterministic regulator for the F-8 aircraft whose matrices $\mathbf{A}$ and $\mathbf{B}$ are given in Section 5.7. The matrices in the performance criterion together with the system initial condition are taken from Teneketzis and Sandell (1977)

$$\mathbf{R}_1 = diag\{0.01 \;\; 0 \;\; 3260 \;\; 3260\}, \quad \mathbf{R}_2 = 3260$$

$$\mathbf{x}_\mathrm{o} = [100 \quad 0 \quad 0.2 \quad 0]^T$$

The linear deterministic regulator problem is also known as the linear-quadratic optimal control problem since the system is linear and the performance criterion is quadratic. The MATLAB function `lqr` and the corresponding instruction

```
[F,P,ev]=lqr(A,B,R1,R2);
```

produce values for optimal gain $\mathbf{F}$, solution of the algebraic Riccati equation $\mathbf{P}$, and closed-loop eigenvalues. These quantities are obtained as

$$\mathbf{F} = [-0.004 \quad 0.5557 \quad -0.2521 \quad 0.0590]$$

$$\mathbf{P} = 10^4 \begin{bmatrix} 0.0000 & -0.0016 & -0.0003 & -0.0003 \\ -0.0016 & 1.6934 & 0.1499 & 0.2199 \\ -0.0003 & 0.1499 & 0.8211 & -0.0713 \\ -0.0003 & 0.2199 & -0.0713 & 0.1361 \end{bmatrix}$$

$$\lambda_{1,2} = -0.9631 \pm j3.0061, \quad \lambda_{3,4} = -0.0373 \pm j0.0837$$

The optimal value for the performance criterion can be found from (10.34), which produces $J_{opt} = 743.9707$.

$\diamond$

For linear *discrete-time* control systems, a corresponding optimal control theory result can be obtained. Let the discrete-time performance criterion for an infinite time optimization problem of a time-invariant, discrete-time, linear system

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \quad \mathbf{x}(0) = \mathbf{x}_\mathrm{o} \tag{10.35}$$

be defined by

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left[\mathbf{x}^T(k)\mathbf{R}_1\mathbf{x}(k) + \mathbf{u}^T(k)\mathbf{R}_2\mathbf{u}(k)\right], \quad \mathbf{R}_1 \geq 0, \;\; \mathbf{R}_2 > 0 \tag{10.36}$$

then the optimal control is given by

$$\mathbf{u}(k) = -\left(\mathbf{R}_2 + \mathbf{B}^T\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A}\mathbf{x}(k) = -\mathbf{F}_{opt}\mathbf{x}(k) \tag{10.37}$$

where $\mathbf{P}$ satisfies the discrete-time algebraic Riccati equation

$$\mathbf{P} = \mathbf{A}^T\mathbf{P}\mathbf{A}+\mathbf{R}_1 - \mathbf{A}^T\mathbf{P}\mathbf{B}\left(\mathbf{B}^T\mathbf{P}\mathbf{B} + \mathbf{R}_2\right)^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A} \tag{10.38}$$

It can be shown that the standard controllability–observability conditions imply the existence of a unique stabilizing solution of (10.38) such that the closed-loop system

$$\mathbf{x}(k+1) = (\mathbf{A} - \mathbf{B}\mathbf{F}_{opt})\mathbf{x}(k) \tag{10.39}$$

is asymptotically stable. The optimal performance criterion in this case is also given by (10.34) (Kwakernaak and Sivan, 1972; Sage and White, 1977; Ogata, 1987).

10.3.2 Optimal Kalman Filter

Consider a stochastic continuous-time system disturbed by white Gaussian noise with the corresponding measurements also corrupted by white Gaussian noise, that is

$$\begin{gathered}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{G}\mathbf{w}(t), \quad E\{\mathbf{x}(t_0)\} = \overline{\mathbf{x}}_\circ, \quad Var\{\mathbf{x}(t_0)\} = \mathbf{Q}_\circ \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t)\end{gathered} \tag{10.40}$$

Since the system is disturbed by white noise, the state space variables are also stochastic quantities (processes). Under the assumption that both the system noise and measurement noise are Gaussian stochastic processes, then so too are the state variables. Thus, the state variables are stochastically completely determined by their mean and variance values.

Since the system measurements are corrupted by white noise, exact information about state variables is not available. The Kalman filtering problem can be formulated as follows: find a dynamical system that produces as its output the best estimates, $\hat{\mathbf{x}}(t)$, of the state variables $\mathbf{x}(t)$. The term "the best estimates" means those estimates for which the variance of the estimation error

$$\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t) \tag{10.41}$$

is minimized. This problem was originally solved in a paper by Kalman and Bucy (1961). However, in the literature it is mostly known simply as the Kalman filtering problem.

The Kalman filter is a stochastic counterpart to the deterministic observer considered in Section 5.6. It is a dynamical system built by control engineers and driven by the outputs (measurements) of the original system. In addition, it has the same order as the original physical system.

The optimal Kalman filter is given by (Kwakernaak and Sivan, 1972)

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{K}_{opt}(t)(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)) \tag{10.42}$$

where the optimal filter gain satisfies

$$\mathbf{K}_{opt}(t) = \mathbf{Q}(t)\mathbf{C}^T\mathbf{V}^{-1} \tag{10.43}$$

The matrix $\mathbf{Q}(t)$ represents the minimal value for the variance of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, and is given by the solution to the filter differential Riccati equation

$$\dot{\mathbf{Q}}(\mathbf{t}) = \mathbf{A}\mathbf{Q}(t) + \mathbf{Q}(t)\mathbf{A}^T + \mathbf{G}\mathbf{W}\mathbf{G}^T - \mathbf{Q}(t)\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{Q}(t), \qquad \mathbf{Q}(t_0) = \mathbf{Q}_\circ \tag{10.44}$$

Assuming that the filter reaches steady state, the differential Riccati equation becomes the algebraic one, that is

$$\mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A}^T + \mathbf{G}\mathbf{W}\mathbf{G}^T - \mathbf{Q}\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{Q} = \mathbf{0} \tag{10.45}$$

so that the optimal Kalman filter gain $\mathbf{K}_{opt}$ as given by (10.43) and (10.45) is constant at steady state.

Note that in the case when an input is present in the state equation, as in (10.13), the Kalman filter has to be driven by the same input as the original system, that is

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{K}_{opt}(t)(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)) \tag{10.46}$$

The expression for the optimal filter gain stays the same, and is given by (10.43).

Example 10.3: Consider the F-8 aircraft example. Its matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are given in Section 5.7, and matrices $\mathbf{G}$ and $\mathbf{W}$ in Example 10.1. From the paper by Teneketzis and Sandell (1977) we have the value for the intensity matrix of the measurement noise

$$\mathbf{V} = \begin{bmatrix} 0.000686 & 0 \\ 0 & 40 \end{bmatrix}$$

The optimal filter gain at steady state can be obtained by using the MATLAB function `lqe`, which stands for the linear-quadratic estimator (filter) design. This name is justified by the fact that we consider a linear system and intend to minimize the variance ("square") of the estimation error. Thus, by using

```
[K,Q,ev]=lqe(A,G,C,W,V);
```

we get *steady state values* for optimal Kalman filter gain $\mathbf{K}$, minimal (optimal) error variance $\mathbf{Q}$, and closed-loop filter eigenvalues. For the F-8 aircraft, these are given by

$$\mathbf{K} = \begin{bmatrix} 21.6433 & -0.6020 & 0.6021 & 4.5056 \\ 0.0081 & -0.0001 & 0.0001 & 0.0004 \end{bmatrix}^T$$

$$\mathbf{Q} = \begin{bmatrix} 0.3229 & -0.0024 & 0.0053 & 0.0148 \\ -0.0024 & 0.0001 & -0.0001 & -0.0004 \\ 0.0053 & -0.0001 & 0.0004 & 0.0004 \\ 0.0148 & -0.0004 & 0.0004 & 0.0031 \end{bmatrix}$$

$$\lambda_1 = -3.7491, \quad \lambda_2 = -2.6410, \quad \lambda_{3,4} = -0.0104 \pm j0.0760$$

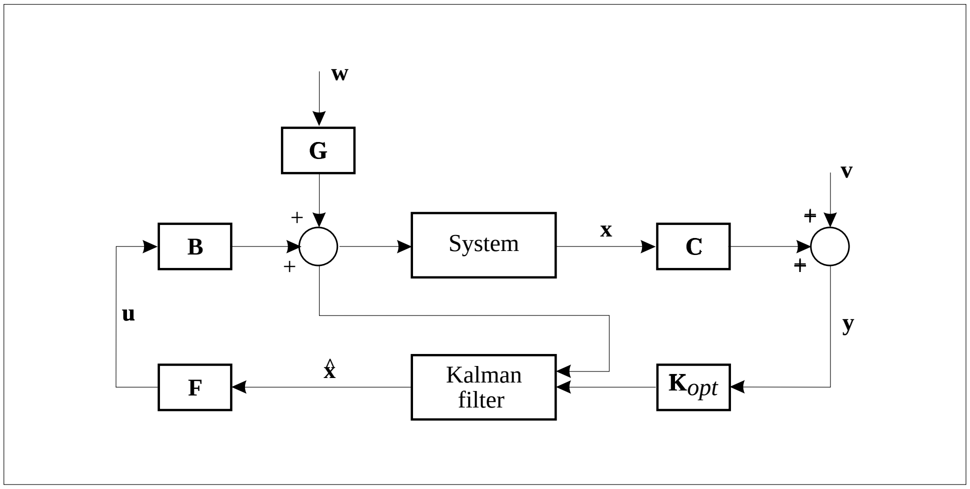
◇

Note that the closed-loop filter in Example 10.3 is asymptotically stable, where the closed-loop structure is obtained by rearranging (10.42) or (10.46) as

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{K}_{opt}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{K}_{opt}\mathbf{y}(t) \tag{10.47}$$

It can be seen from this structure that the optimal closed-loop Kalman filter is driven by both the system measurements and control. A block diagram for this system-filter configuration is given in Figure 10.1.

Similarly to the continuous-time Kalman filter, it is possible to develop corresponding results for the *discrete-time Kalman filter*. The interested reader can find the corresponding results in several books (e.g. Kwakernaak and Sivan, 1972; Ogata, 1987).



10.1: System-filter configuration

10.3.3 Optimal Stochastic Regulator Problem

In the optimal control problem of linear stochastic systems, represented in continuous time by (10.13), the following stochastic performance criterion is defined

$$J = E\left\{ \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} \left[\mathbf{x}^T(t)\mathbf{R}_1\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}_2\mathbf{u}(t) \right] dt \right\}, \quad \mathbf{R}_1 \geq 0, \quad \mathbf{R}_2 > 0 \tag{10.48}$$

The goal is to find the optimal value for $\mathbf{u}(t)$ such that the performance measure (10.48) is minimized along the trajectories of dynamical system (10.13).

The solution to the above problem is greatly simplified by the application of the so-called *separation principle*. *The separation principle says: perform first the optimal estimation, i.e. construct the Kalman filter, and then do the optimal regulation*, i.e. find the optimal regulator as outlined in Subsection 10.3.1 with $\mathbf{x}(t)$ replaced by $\hat{\mathbf{x}}(t)$. For more details about the separation principle and its complete proof see Kwakernaak and Sivan (1972). By the separation principle the optimal control for the above problem is given by

$$\mathbf{u}_{opt}(t) = -\mathbf{F}_{opt}\hat{\mathbf{x}}(t) \tag{10.49}$$

where $\mathbf{F}_{opt}$ is obtained from (10.30) and (10.32), and $\hat{\mathbf{x}}(t)$ is obtained from the Kalman filter (10.46). The optimal performance criterion at steady state can be obtained by using either one of the following two expressions (Kwakernaak and Sivan, 1972)

$$J_{opt} = \mathrm{trace}\{\mathbf{PKVK}^T + \mathbf{QR}_1\} = \mathrm{trace}\{\mathbf{PGWG}^T + \mathbf{QF}^T\mathbf{R}_2\mathbf{F}\} \tag{10.50}$$

Example 10.4: Consider again the F-8 aircraft example. The matrices $\mathbf{F}_{opt}$ and $\mathbf{P}$ are obtained in Example 10.2 and the matrices $\mathbf{K}_{opt}$ and $\mathbf{Q}$ are known from Example 10.3. Using any of the formulas given in (10.50), we obtain the optimal value for the performance criterion as $J_{opt} = 25.0425$. The optimal control is given by (10.49) with the optimal estimates $\hat{\mathbf{x}}(t)$ obtained from the Kalman filter (10.46).

◇

The solution to the *discrete-time* optimal stochastic regulator is also obtained by using the separation principle, i.e. by combining the results of optimal filtering and optimal regulation. For details, the reader is referred to Kwakernaak and Sivan (1972) and Ogata (1987).

10.4 Linear Time-Delay Systems

The dynamics of linear systems containing time-delays is described by delay-differential equations (Driver, 1977). The state space form of a time-delay linear control system is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}x(t) + \mathbf{A}_D x(t-T) + \mathbf{B}u(t) \tag{10.51}$$

where T represents the time-delay. This form can be generalized to include state variables delayed by $2T, 3T$, ..., time-delay periods. For the purpose of this introduction to linear time-delay systems it is sufficient to consider only models given by (10.51).

Taking the Laplace transform of (10.51) we get

$$s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) - \mathbf{A}_D\mathbf{X}(s)e^{-sT} = \mathbf{x}(0^-) + \mathbf{B}\mathbf{U}(s) \tag{10.52}$$

which produces the *characteristic equation for linear time-delay systems* in the form

$$\begin{aligned} &\det\left(s\mathbf{I} - \mathbf{A} - \mathbf{A}_D e^{-sT}\right) = 0 \\ &= s^n + a_{n-1}\left(e^{-sT}\right)s^{n-1} + \cdots + a_1\left(e^{-sT}\right)s + a_0\left(e^{-Ts}\right) = 0 \end{aligned} \tag{10.53}$$

Note that the coefficients $a_i, i = 0, 1, 2, ..., n-1$, are functions of e^{-sT}, i.e. of the complex frequency s, and therefore the characteristic equation is not in the polynomial form as in the case of continuous-time, time-invariant linear systems without time-delay. This implies that the transfer function for time-delay linear systems is not a rational function. Note that the rational functions can be represented by a ratio of two polynomials.

An important feature of the characteristic equation (10.53) is that it has in general, infinitely many solutions. Due to this fact the study of time-delay linear systems is much more mathematically involved than the study of linear systems without time-delay.

It is interesting to point out that the stability theory of time-delay systems comes to the conclusion that asymptotic stability is guaranteed if and only if all roots of the characteristic equations (10.53) are strictly in the left half of the complex plane, even though the number of roots may in general be infinite (Mori and Kokame, 1989; Su *et al.*, 1994). In practice, stability of time-delay linear time-invariant systems can be examined by using the Nyquist stability test (Kuo, 1991), as well as by employing Bode diagrams and finding the corresponding phase and gain stability margins.

Studying the controllability of time-delay linear systems is mathematically very complex as demonstrated in the corresponding chapter in the book by Klamka (1991) on system controllability. Analysis, optimization, and applications of time-delayed systems are presented in Malek-Zavarei and Jamshidi (1987). For control problems associated with these systems the reader is referred to Marshall (1977).

Note that in some cases linear time-delay control systems can be related to the sampled data control systems (Ackermann, 1985), which are introduced in Chapter 2, and to the discrete-time linear control systems whose state space form is considered in detail in Chapter 3 and some of their properties are studied throughout this book. Approximations of the time-delay element e^{-Ts} for small values of T are considered in Section 9.7.

10.5 System Identification and Adaptive Control

System identification is an experimental technique used to determine mathematical models of dynamical systems. Adaptive control is applied for systems that change their mathematical models or some parameters with time. Since system identification is included in every adaptive control scheme, in this section we present some essential results for both system identification and adaptive control.

10.5.1 System Identification

The identification procedure is based on data collected by applying known inputs to a system and measuring the corresponding outputs. Using this method, a family of pairs $(\mathbf{y}(t_i), \mathbf{u}(t_i))$, $i = 1, 2, 3, ...$ is obtained, where t_i stand for the time instants at which the results are recorded (measured). In this section, we will present only one identification technique, known as the *least-square estimation method*, which is relevant to this book since it can be used to identify the system transfer function of time-invariant linear systems. Many other identification techniques applicable either to deterministic or stochastic systems are presented in several standard books on identification (see for example Ljung, 1987; Soderstrom and Stoica, 1989). For simplicity, in this section we study the transfer function identification problem of single-input single-output systems in the discrete-time domain.

Consider an nth-order time-invariant, discrete-time, linear system described by the corresponding difference equation as presented in Section 3.3.1, that is

$$\begin{aligned} &y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_0 y(k) \\ &= b_{n-1}u(k+n-1) + b_{n-2}u(k+n-2) + \cdots + b_0 u(k) \end{aligned} \tag{10.54}$$

It is assumed that parameters

$$\mathbf{a} = [a_{n-1} \quad a_{n-2} \quad \cdots \quad a_0], \quad \mathbf{b} = [b_{n-1} \quad b_{n-2} \quad \cdots \quad b_0] \tag{10.55}$$

are not known and ought to be determined using the least-square identification technique.

Equation (10.54) can be rewritten as

$$\begin{aligned} y(k+n) = &-a_{n-1}y(k+n-1) - \cdots - a_0 y(k) \\ &+ b_{n-1}u(k+n-1) + b_{n-2}u(k+n-2) + \cdots + b_0 u(k) \end{aligned} \tag{10.56}$$

and put in the vector form

$$y(k+n) = \mathbf{f}(k+n-1)\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} \tag{10.57}$$

where $\mathbf{a}$ and $\mathbf{b}$ are defined in (10.55), and $\mathbf{f}(k+n-1)$ is given by

$$\mathbf{f}(k+n-1) = [-y(k+n-1) \quad \cdots \quad -y(k) \quad u(k+n-1) \quad \cdots \quad u(k)] \tag{10.58}$$

Assume that for the given input, we perform N measurements and that the actual (measured) system outputs are known, that is

$$\mathbf{Y_a}(k,N) = \begin{bmatrix} y_a(k+n) \\ y_a(k+n-1) \\ \cdots \\ y_a(k+n-N+1) \end{bmatrix} \tag{10.59}$$

The problem now is how to determine $2n$ parameters in $\mathbf{a}$ and $\mathbf{b}$ such that the actual system outputs $\mathbf{Y_a}(k,N)$ are as close as possible to the mathematically computed system outputs that are represented by formula (10.57).

We can easily generate N equations from (10.57) as

$$\mathbf{Y}(k,N) = \begin{bmatrix} y(k+n) \\ y(k+n-1) \\ \cdots \\ y(k+n-N+1) \end{bmatrix} = \begin{bmatrix} \mathbf{f}(k+n-1) \\ \mathbf{f}(k+n-2) \\ \cdots \\ \mathbf{f}(k+n-N+1) \\ \mathbf{f}(k+n-N) \end{bmatrix} \begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} = \boldsymbol{\Psi}(k,N)\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} \tag{10.60}$$

Define the estimation (identification) error as

$$\mathbf{E}(k,N) = \mathbf{Y_a}(k,N) - \mathbf{Y}(k,N) \tag{10.61}$$

The least-square estimation method requires that the choice of the unknown parameters $\mathbf{a}$ and $\mathbf{b}$ minimizes the "square" of the estimation error, that is

$$\min_{\mathbf{a},\mathbf{b}} J = \frac{1}{2} \min_{\mathbf{a},\mathbf{b}} \left\{ \mathbf{E}^T(k,N)\mathbf{E}(k,N) \right\} \tag{10.62}$$

Using expressions for the vector derivatives (see Appendix C) and (10.60), we can show that

$$\min_{\mathbf{a},\mathbf{b}} \{J\} \Rightarrow \frac{\partial J}{\partial \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}} = 0 \Rightarrow \Psi^T(k,N)\Psi(k,N) \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} - \Psi^T(k,N)\mathbf{Y}_a(k,N) = 0 \tag{10.63}$$

which produces the least-square optimal estimates for the unknown parameters as

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \left\{ \Psi^T(k,N)\Psi(k,N) \right\}^{-1} \Psi^T(k,N)\mathbf{Y}_a(k,N) \tag{10.64}$$

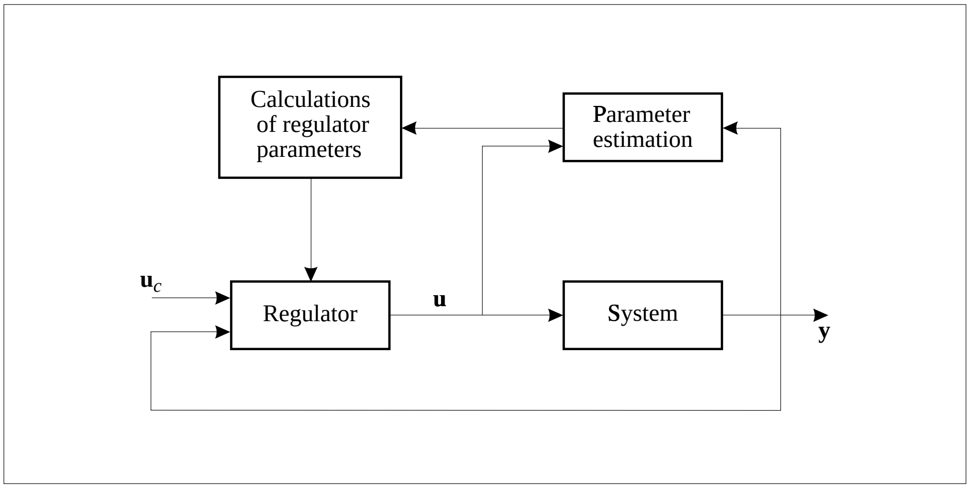
Note that the input signal has to be chosen such that the matrix inversion defined in (10.64) exists.

Sometimes it is sufficient to estimate (identify) only some parameters in a system or in a problem under consideration in order to obtain a complete insight into its dynamical behavior. Very often the identification (estimation) process is combined with known physical laws which describe some, but not all, of the system variables and parameters. It is interesting to point out that MATLAB contains a special toolbox for system identification.

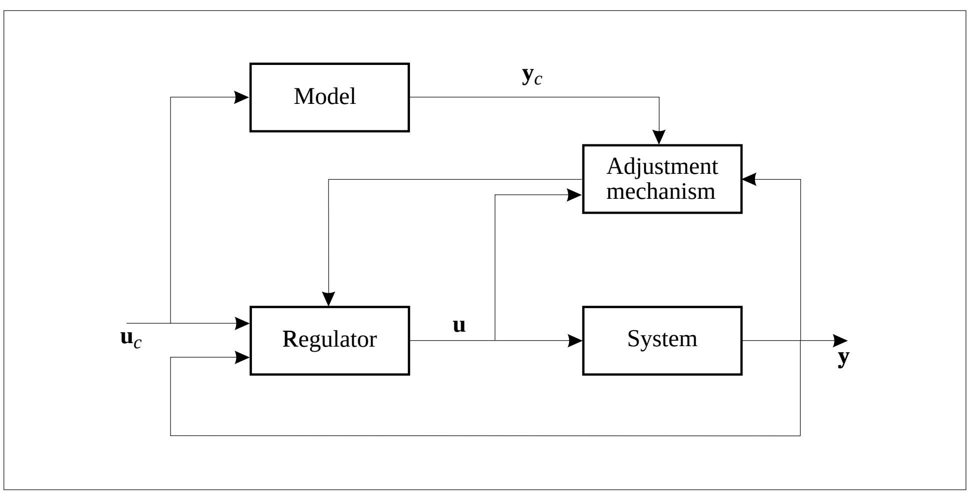
10.5.2 Adaptive Control

Adaptive control schemes in closed-loop configurations represent nonlinear control systems even in those cases when the systems under consideration are linear. Due to this fact, it is not easy to study adaptive control systems analytically. However, due to their practical importance, adaptive controllers are widely used nowadays in industry since they produce satisfactory results despite the fact that many theoretical questions remain unsolved.

Two major configurations in adaptive control theory and practice are *self-tuning regulators* and *model-reference adaptive schemes*. These configurations are represented in Figures 10.2 and 10.3. For self-tuning regulators, it is assumed that the system parameters are constant, but unknown. On the other hand, for the model-reference adaptive scheme, it is assumed that the system parameters change over time.



10.2: Self-tuning regulator



10.3: Model-reference adaptive control scheme

It can be seen from Figure 10.2 that for self-tuning regulators the "separation principle" is used, i.e. the problem is divided into independent estimation and regulation tasks. In the regulation problem, the estimates are used as the true values of the unknown parameters. The command signal $\mathbf{u}_c$ must be chosen such that unknown system parameters can be estimated. The stability of the closed-loop systems and the convergence of the proposed schemes for self-tuning regulators are very challenging and interesting research areas (Wellstead and Zarrop, 1991).

In the model-reference adaptive scheme, a desired response is specified by using the corresponding mathematical model (see Figure 10.3). The error signal generated as a difference between desired and actual outputs is used to adjust system parameters that change over time. It is assumed that system parameters change much slower than system state variables.

The adjusted parameters are used to design a controller, a feedback regulator. There are several ways to adjust parameters; one commonly used method is known as the MIT rule (Astrom and Wittermark, 1989). As in the case of self-tuning regulators, model-reference adaptive schemes still have many theoretically unresolved stability and convergence questions, even though they do perform very well in practice.

For detailed study of self-tuning regulators, model-reference adaptive systems, and other adaptive control schemes and techniques applicable to both deterministic and stochastic systems, the reader is referred to Astrom and Wittenmark (1989), Wellstead and Zarrop (1991), Isermann *et al*. (1992), and Krstić *et al*. (1995).

10.6 Nonlinear Control Systems

Nonlinear control systems are introduced in this book in Section 1.6, where we have presented a method for their linearization. Mathematical models of time-invariant nonlinear control systems are given by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathcal{F}(\mathbf{x}(t), \mathbf{u}(t)), \qquad \mathbf{x}(t_0) = \mathbf{x}_o \\ \mathbf{y}(t) &= \mathcal{G}(\mathbf{x}(t), \mathbf{u}(t)) \end{aligned} \tag{10.65}$$

where $\mathbf{x}(t)$ is a state vector, $\mathbf{u}(t)$ is an input vector, $\mathbf{y}(t)$ is an output vector, and $\mathcal{F}$ and $\mathcal{G}$ are nonlinear matrix functions. These control systems are in general very difficult to study analytically. Most of the analytical results come from the mathematical theory of classic nonlinear differential equations, the theory of which has been developing for more than a hundred years (Coddington and Levinson, 1955). In contrast to classic differential equations, due to the presence of the input vector in (10.65) one is faced with the even more challenging so-called controlled differential equations. The mathematical and engineering theories of controlled nonlinear differential equations are the modern trend of the eighties and nineties (Sontag, 1990).

Many interesting phenomena not encountered in linear systems appear in nonlinear systems, e.g. hysteresis, limit cycles, subharmonic oscillations, finite escape time, self-excitation, multiple isolated equilibria, and chaos. For more details about these nonlinear phenomena see Siljak (1969) and Khalil (1992).

It is very hard to give a brief presentation of any result and/or any concept of nonlinear control theory since almost all of them take quite complex forms. Familiar notions such as system stability and controllability for nonlinear systems have to be described by using several definitions (Klamka, 1991; Khalil, 1992).

One of the most interesting results of nonlinear theory is the so-called *stability concept in the sense of Lyapunov*. This concept deals with the stability of system equilibrium points. The equilibrium points of nonlinear systems are defined by

$$\mathbf{0} = \mathcal{F}(\mathbf{x}_e(t), \mathbf{u}(t)) \;\Rightarrow\; \mathbf{x}_e(t) \tag{10.66}$$

Roughly speaking, an equilibrium point is stable in the sense of Lyapunov if a small perturbation in the system initial condition does not cause the system trajectory to leave a bounded neighborhood of the system equilibrium point. The Lyapunov stability can be formulated for time invariant nonlinear systems (10.65) as follows (Slotine and Li, 1991; Khalil, 1992; Vidyasagar, 1993).

Theorem 10.1 *The equilibrium point $\mathbf{x}_e = 0$ of a time invariant nonlinear system is stable in the sense of Lyapunov if there exists a continuously differentiable scalar function $V(\mathbf{x})$ such that along the system trajectories the following is satisfied*

$$\begin{gathered} V(\mathbf{x}) > 0, \quad V(0) = 0 \\ \dot{V}(\mathbf{x}) = \frac{dV}{dt} = \frac{\partial V}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} \leq 0 \end{gathered} \tag{10.67}$$

Thus, the problem of examining system stability in the sense of Lyapunov requires finding a scalar function known as the *Lyapunov function* $V(\mathbf{x})$. Again, this is a very hard task in general, and one is able to find the Lyapunov function $V(\mathbf{x})$ for only a few real physical nonlinear control systems.

In this book we have presented in Section 1.6 the procedure for linearization of nonlinear control systems. Another classic method, known as the describing function method, has proved very popular for analyzing nonlinear control systems (Slotine and Li, 1991; Khalil, 1992; Vidyasagar, 1993).

10.7 Comments

In addition to the classes of control systems extensively studied in this book, and those introduced in Chapter 10, many other theoretical and practical control areas have emerged during the last thirty years. For example, decentralized control (Siljak, 1991), learning systems (Narendra, 1986), algebraic methods for multivariable control systems (Callier and Desoer, 1982; Maciejowski, 1989), robust control (Morari and Zafiriou, 1989; Chiang and Safonov, 1992; Grimble, 1994; Green and Limebeer, 1995), control of robots (Vukobratović and Stokić, 1982; Spong and Vidyasagar, 1989; Spong *et al*., 1993), differential games (Isaacs, 1965; Basar and Olsder, 1982; Basar and Bernhard, 1991), neural network control (Gupta and Rao, 1994), variable structure control (Itkis, 1976; Utkin, 1992), hierarchical and multilevel systems (Mesarović *et al.*, 1970), control of systems with slow and fast modes (singular perturbations) (Kokotović and Khalil, 1986; Kokotović *et al.*, 1986; Gajić and Shen, 1993), predictive control (Soeterboek, 1992), distributed parameter control, large-scale systems (Siljak, 1978; Gajić and Shen, 1993), fuzzy control systems (Kandel and Langholz, 1994; Yen *et al.*, 1995), discrete event systems (Ho, 1991; Ho and Cao, 1991), intelligent vehicles and highway control systems, intelligent control systems (Gupta and Sinha, 1995; de Silva, 1995), control in manufacturing (Zhou and DiCesare, 1993), control of flexible structures, power systems control (Anderson and Fouad, 1984), control of aircraft (McLean, 1991), linear algebra and numerical analysis control algorithms (Laub, 1985; Bittanti *et al*., 1991; Petkov *et al*., 1991; Bingulac and Vanlandingham, 1993; Patel *et al*., 1994), and computer-controlled systems (Astrom and Wittenmark, 1990).

Finally, it should be emphasized that control theory and its applications are studied within all engineering disciplines, and as well as in applied mathematics (Kalman *et al*., 1969; Sontag, 1990) and computer science.

10.8 References

Ackermann, J., *Sampled-Data Control Systems*, Springer-Verlag, London, 1985.

Anderson, P. and A. Fouad, *Power System Control and Stability*, Iowa State University Press, Ames, Iowa, 1984.

Astrom, K. and B. Wittenmark, *Adaptive Control*, Wiley, New York, 1989.

Astrom, K. and B. Wittenmark, *Computer-Controlled Systems: Theory and Design*, Prentice Hall, Englewood Cliffs, New Jersey, 1990.

Athans, M. and P. Falb, *Optimal Control: An Introduction to the Theory and its Applications*, McGraw-Hill, New York, 1966.

Basar, T. and P. Bernhard, H^∞*—Optimal Control and Related Minimax Design Problems*, Birkhauser, Boston, Massachusetts, 1991.

Basar, T. and G. Olsder, *Dynamic Non-Cooperative Game Theory*, Academic Press, New York, 1982.

Bellman, R., *Dynamic Programming*, Princeton University Press, Princeton, New Jersey, 1957.

Bingulac, S. and H. Vanlandingham, *Algorithms for Computer-Aided Design of Multivariable Control Systems*, Marcel Dekker, New York, 1993.

Bittanti, S., A. Laub, and J. Willems, (eds.) *The Riccati Equation*, Springer-Verlag, Berlin, 1991.

Boyd, S., L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

Callier, F. and C. Desoer, *Multivariable Feedback Systems*, Springer-Verlag, Stroudsburg, Pennsylvania, 1982.

Chen, C., *Linear System Theory and Design*, Holt, Rinehart and Winston, New York, 1984.

Chiang, R. and M. Safonov, *Robust Control Tool Box User's Guide*, The Math Works, Inc., Natick, Massachusetts, 1992.

Coddington, E. and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

de Silva, C., *Intelligent Control: Fuzzy Logic Applications*, CRC Press, Boca Raton, Florida, 1995.

Doyle, J., B. Francis, and A. Tannenbaum, *Feedback Control Theory*, Macmillan, New York, 1992.

Doyle, J., K. Glover, P. Khargonekar, and B. Francis, "State space solutions to standard H_2 and H_∞ control problems," *IEEE Transactions on Automatic Control*, vol. AC-34, 831–847, 1989.

Driver, R., *Ordinary and Delay Differential Equations*, Springer-Verlag, New York, 1977.

Francis, B., *A Course in H_∞ Control Theory*, Springer-Verlag, Berlin, 1987.

Gajić, Z. and X. Shen, *Parallel Algorithms for Optimal Control of Large Scale Linear Systems*, Springer-Verlag, London, 1993.

Gajić, Z. and M. Qureshi, *Lyapunov Matrix Equation in System Stability and Control*, Academic Press, San Diego, California, 1995.

Green, M. and D. Limebeer, *Linear Robust Control*, Prentice Hall, Englewood Cliffs, New Jersey, 1995.

Grimble, M., *Robust Industrial Control*, Prentice Hall International, Hemel Hempstead, 1994.

Gupta, M. and D. Rao, (eds.), *Neuro-Control Systems*, IEEE Press, New York, 1994.

Gupta, M. and N. Sinha, (eds.), *Intelligent Control Systems: Concepts and Algorithms*, IEEE Press, New York, 1995.

Ho, Y., (ed.), *Discrete Event Dynamic Systems*, IEEE Press, New York, 1991.

Ho, Y. and X. Cao, *Perturbation Analysis of Discrete Event Dynamic Systems*, Kluwer, Dordrecht, 1991.

Isaacs, R., *Differential Games*, Wiley, New York, 1965.

Isermann, R., K. Lachmann, and D. Matko, *Adaptive Control Systems*, Prentice Hall International, Hemel Hempstead, 1992.

Itkis, U., *Control Systems of Variable Structure*, Wiley, New York, 1976.

Kailath, T., *Linear Systems*, Prentice Hall, Englewood Cliffs, New Jersey, 1980.

Kalman, R., "Contribution to the theory of optimal control," *Boletin Sociedad Matematica Mexicana*, vol. 5, 102–119, 1960.

Kalman, R. and R. Bucy, “New results in linear filtering and prediction theory,” *Journal of Basic Engineering, Transactions of ASME, Ser. D.*, vol. 83, 95–108, 1961.

Kalman, R., P. Falb, and M. Arbib, *Topics in Mathematical System Theory*, McGraw-Hill, New York, 1969.

Kandel A. and G. Langholz (eds.), *Fuzzy Control Systems*, CRC Press, Boca Raton, Florida, 1994.

Kirk, D., *Optimal Control Theory*, Prentice Hall, Englewood Cliffs, New Jersey, 1970.

Khalil, H., *Nonlinear Systems*, Macmillan, New York, 1992.

Klamka, J., *Controllability of Dynamical Systems*, Kluwer, Warszawa, 1991.

Kokotović P. and H. Khalil, *Singular Perturbations in Systems and Control*, IEEE Press, New York, 1986.

Kokotović, P., H. Khalil, and J. O’Reilly, *Singular Perturbation Methods in Control: Analysis and Design*, Academic Press, Orlando, Florida, 1986.

Krstić, M., I. Kanellakopoulos, and P. Kokotović, *Nonlinear and Adaptive Control Design*, Wiley, New York, 1995.

Kuo, B., *Automatic Control Systems*, Prentice Hall, Englewood Cliffs, New Jersey, 1991.

Kwakernaak, H. and R. Sivan, *Linear Optimal Control Systems*, Wiley, New York, 1972.

Lancaster, P. and L. Rodman, *Algebraic Riccati Equation*, Clarendon Press, Oxford, 1995.

Laub, A., “Numerical linear algebra aspects of control design computations,” *IEEE Transactions on Automatic Control*, vol. AC-30, 97–108, 1985.

Lewis, F., *Applied Optimal Control and Estimation*, Prentice Hall, Englewood Cliffs, New Jersey, 1992.

Ljung, L., *System Identification: Theory for the User*, Prentice Hall, Englewood Cliffs, New Jersey, 1987.

Maciejowski, J., *Multivariable Feedback Design*, Addison-Wesley, Wokingham, 1989.

Malek-Zavarei, M. and M. Jamshidi, *Time-Delay Systems: Analysis, Optimization and Applications*, North-Holland, Amsterdam, 1987.

Marsall, J., *Control of Time-Delay Systems*, IEE Peter Peregrinus, New York, 1977.

McLean, D., *Automatic Flight Control Systems*, Prentice Hall International, Hemel Hempstead, 1991.

Mesarović, M., D. Macko, and Y. Takahara, *Theory of Hierarchical, Multilevel, Systems*, Academic Press, New York, 1970.

Morari, M. and E. Zafiriou, *Robust Process Control*, Prentice Hall, Englewood Cliffs, New Jersey, 1989.

Mori, T. and H. Kokame, "Stability of $\dot{x}(t) = Ax(t) + Bx(t - \tau)$," *IEEE Transactions on Automatic Control*, vol. AC-34, 460–462, 1989.

Narendra, K. (ed.), *Adaptive and Learning Systems—Theory and Applications*, Plenum Press, New York, 1986.

Ogata, K., *Discrete-Time Control Systems*, Prentice Hall, Englewood Cliffs, New Jersey, 1987.

Patel, R., A. Laub, and P. Van Dooren, *Numerical Linear Algebra Techniques for Systems and Control*, IEEE Press, New York, 1994.

Petkov, P., N. Christov, and M. Konstantinov, *Computational Methods for Linear Control Systems*, Prentice Hall International, Hemel Hempstead, 1991.

Rugh, W., *Linear System Theory*, Prentice Hall, Englewood Cliffs, New Jersey, 1993.

Saberi, A., P. Sannuti, and B. Chen, H_2*—Optimal Control*, Prentice Hall International, Hemel Hempstead, 1995.

Sage, A. and C. White, *Optimum Systems Control*, Prentice Hall, Englewood Cliffs, New Jersey, 1977.

Siljak, D., *Nonlinear Systems*, Wiley, New York, 1969.

Siljak, D., *Large Scale Dynamic Systems: Stability and Structure*, North-Holland, New York, 1978.

Siljak, D., *Decentralized Control of Complex Systems*, Academic Press, San Diego, California, 1991.

Slotine, J. and W. Li, *Applied Nonlinear Control*, Prentice Hall, Englewood Cliffs, New Jersey, 1991.

Soderstrom, T. and P. Stoica, *System Identification*, Prentice Hall International, Hemel Hempstead, 1989.

Soeterboek, R., *Predictive Control—A Unified Approach*, Prentice Hall, Englewood Cliffs, New Jersey, 1992.

Sontag, E., *Mathematical Control Theory*, Springer-Verlag, New York, 1990.

Spong, M. and M. Vidyasagar, *Robot Dynamics and Control*, Wiley, New York, 1989.

Spong, M., F. Lewis, and C. Abdallah (eds.), *Robot Control: Dynamics, Motion Planing, and Analysis*, IEEE Press, New York, 1993.

Su, J., I. Fong, and C. Tseng, “Stability analysis of linear systems with time delay,” *IEEE Transactions on Automatic Control*, vol. AC-39, 1341–1344, 1994.

Teneketzis, D. and N. Sandell, “Linear regulator design for stochastic systems by multiple time-scale method,” *IEEE Transactions on Automatic Control*, vol. AC-22, 615–621, 1977.

Utkin, V., *Sliding Modes in Control Optimization*, Springer-Verlag, Berlin, 1992.

Vukobratović, M. and D. Stokić, *Control of Manipulation Robots: Theory and Applications*, Springer-Verlag, Berlin, 1982.

Vidyasagar, M., *Nonlinear Systems Analysis*, Prentice Hall, Englewood Cliffs, New Jersey, 1993.

Wellstead, P. and M. Zarrop, *Self-Tuning Systems—Control and Signal Processing*, Wiley, Chichester, 1991.

Yen, J., R. Langari, and L. Zadeh, (eds.), *Industrial Applications of Fuzzy Control and Intelligent Systems*, IEEE Press, New York, 1995.

Zames, G., “Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximative inverses,” *IEEE Transactions on Automatic Control*, vol. AC-26, 301–320, 1981.

Zhou, M. and F. DiCesare, *Petri Net Synthesis for Discrete Event Control of Manufacturing Systems*, Kluwer, Boston, Massachusetts, 1993.