

Sample Exam 5 Solutions

#1)

The duration property implies $f(t) = 0$, $t > 1 + 2 = 3$ and $t < -1 + 0 = -1$.

$$f(t) = \int_0^{1+t} \tau d\tau = \frac{1}{2}(t+1)^2, \quad -1 \leq t \leq 0$$

$$f(t) = \frac{1}{2} + \int_1^{1+t} (-\tau + 2) d\tau = \frac{1}{2} + \frac{1}{2}t(2-t), \quad 0 \leq t \leq 1$$

$$f(t) = \int_{t-1}^1 \tau d\tau + \frac{1}{2} = \frac{1}{2} - \frac{1}{2}(t-1)^2, \quad 1 \leq t \leq 2$$

$$f(t) = \int_{t-1}^2 (-\tau + 2) d\tau = \frac{1}{2}(3-t)^2, \quad 2 \leq t \leq 3$$

#2)

$$f[k] = \begin{cases} 1, & k = 0 \\ 0, & k = 1 \\ -4, & k = 2 \\ 6, & k = 3 \\ 2, & k = 4 \\ -16, & k = 5 \\ 13, & k = 6 \\ 2, & k = 7 \\ -11, & k = 8 \\ 4, & k = 9 \\ 4, & k = 10 \\ 0, & \text{otherwise} \end{cases}$$

#3a)

$$H(s) = C(sI - A)^{-1}B = C\Phi(s)B = [0 \quad 1] \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{s}{s^2 + 4s + 3}$$

#3b)

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s+3)}\right\} = \left(-\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t}\right)u(t)$$

#3c)

$$\Phi(s) = (sI - A)^{-1} = \frac{1}{s(s+4)+3} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}$$

$$\Phi(t) = \mathcal{L}^{-1}\{\Phi(s)\} = \begin{bmatrix} \mathcal{L}^{-1}\left\{\frac{s+4}{(s+1)(s+3)}\right\} & \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+3)}\right\} \\ \mathcal{L}^{-1}\left\{\frac{-3}{(s+1)(s+3)}\right\} & \mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s+3)}\right\} \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{bmatrix}$$

Comment: You are supposed to perform the Laplace inverse and find $\Phi_{ij}(t)$, $i = 1, 2$, $j = 1, 2$.

#3d)

$$\begin{aligned}
Y(s) &= C\Phi(s)x(0) + H(s)F(s) = [0 \quad 1] \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{s}{(s+1)(s+3)} \times \frac{1}{s+3} \\
&= -\Phi_{22}(s) + \frac{s}{(s+1)(s+3)^2} = -\frac{s}{(s+1)(s+3)} + \frac{s}{(s+1)(s+3)^2} \\
&= \frac{1/2}{s+1} - \frac{3/2}{s+3} + \frac{-1/4}{s+1} + \frac{1/4}{s+3} + \frac{3/2}{(s+3)^2} = \frac{1/4}{s+1} - \frac{5/4}{s+3} + \frac{3/2}{(s+3)^2} \\
&\leftrightarrow y(t) = \left(\frac{1}{4}e^{-t} - \frac{5}{4}e^{-3t} + \frac{3}{2}te^{-3t} \right) u(t)
\end{aligned}$$

#4a)

$$\begin{aligned}
H(z) &= C(zI - A)^{-1}B = [1 \quad 0] \begin{bmatrix} z & -1 \\ 1 & z+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [1 \quad 0] \frac{1}{z(z+2)+1} \begin{bmatrix} z+2 & 1 \\ -1 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{z(z+2)+1} [z+2 \quad 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{z^2+2z+1} = \frac{1}{(z+1)^2}
\end{aligned}$$

#4b)

$$\begin{aligned}
\Phi(z) &= z(sI - A)^{-1} = z \begin{bmatrix} z & -1 \\ 1 & z+2 \end{bmatrix}^{-1} = \frac{z}{z(z+2)+1} \begin{bmatrix} z+2 & 1 \\ -1 & z \end{bmatrix} = \begin{bmatrix} \frac{z(z+2)}{(z+1)^2} & \frac{z}{(z+1)^2} \\ \frac{-z}{(z+1)^2} & \frac{z^2}{(z+1)^2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{z}{z+1} + \frac{z}{(z+1)^2} & \frac{z}{z+1} - \frac{z}{(z+1)^2} \\ -\frac{z}{(z+1)^2} & \frac{z}{z+1} - \frac{z}{(z+1)^2} \end{bmatrix} \leftrightarrow \Phi[k] = \begin{bmatrix} -(-1)^k - k(-1)^k & -k(-1)^k \\ k(-1)^k & -1 + k(-1)^k \end{bmatrix} u[k]
\end{aligned}$$

#4c)

$$\begin{aligned}
Y(z) &= C\Phi(z)x(0) + H(z)F(z) = [1 \quad 0] \begin{bmatrix} \Phi_{11}(z) & \Phi_{12}(z) \\ \Phi_{21}(z) & \Phi_{22}(z) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \frac{1}{(z+1)^2} \frac{z}{z-0.5} \\
&= 2\Phi_{11}(z) + \frac{z}{(z+1)^2(z-0.5)} = \frac{2z(z+2)}{(z+1)^2} + \frac{z}{(z+1)^2(z-0.5)} \\
y[k] &= \mathcal{Z}^{-1}\{Y(z)\} = \frac{2z}{z+1} + \frac{2z}{(z+1)^2} + \mathcal{Z}^{-1}\left\{ \frac{\frac{4}{9}z}{z-\frac{1}{2}} - \frac{\frac{4}{9}z}{z+1} - \frac{\frac{2}{3}}{(z+1)^2} \right\} \\
&= 2\left((-1)^k - k(-1)^k\right)u[k] + \left(\frac{4}{9}\left(\frac{1}{2}\right)^k - \frac{4}{9}(-1)^k + \frac{2}{3}k(-1)^k\right)u[k]
\end{aligned}$$

#5)

$$\begin{aligned}
H(s) &= \frac{1}{s+\alpha} \Rightarrow M(s) = \frac{H(s)}{1+H(s)} = \frac{1}{s+\alpha+1}, \quad S^\alpha(s) = \frac{\Delta M(s)/M(s)}{\Delta\alpha/\alpha} \\
&= \frac{\alpha}{\Delta\alpha M(s)} \left(\frac{1}{s+\alpha+\Delta\alpha+1} - \frac{1}{s+\alpha+1} \right) = -\frac{\alpha}{s+\alpha+\Delta\alpha+1}
\end{aligned}$$

Sample Exam 6 Solutions

#1a)

The duration property implies $f(t) = 0$, $t > 1+3 = 4$ and $t < -1+0 = -1$. Flip $p_3(t - 1.5)$, then

$$f(t) = \int_{-1}^t (2 + \tau) d\tau = \frac{1}{2}t^2 + 2t + \frac{3}{2}, \quad -1 \leq t \leq 0$$

$$f(t) = \int_{-1}^0 (2 + \tau) d\tau + \int_0^t (2 - \tau) d\tau = -\frac{1}{2}t^2 + 2t + \frac{3}{2}, \quad 0 \leq t \leq 1$$

$$f(t) = \int_{-1}^0 (2 + \tau) d\tau + \int_0^1 (2 - \tau) d\tau = 3, \quad 1 \leq t \leq 2$$

$$f(t) = \int_{t-3}^0 (2 + \tau) d\tau + \int_0^1 (2 - \tau) d\tau = -\frac{1}{2}t^2 + t + 3, \quad 2 \leq t \leq 3$$

$$f(t) = \int_{t-3}^1 (2 - \tau) d\tau = \frac{1}{2}t^2 - 5t + 12, \quad 3 \leq t \leq 4$$

#1b)

$$f[k] = \begin{cases} 0, & k \leq 1 \\ 2, & k = 2 \\ 3, & k = 3 \\ 2, & k = 4 \\ 11, & k = 5 \\ 9, & k = 6 \\ -2, & k = 7 \\ -7, & k = 8 \\ -2, & k = 9 \\ 0, & k \geq 10 \end{cases}$$

#2Aa)

$$\Phi(s) = (sI - A)^{-1} = \frac{1}{s(s+3)+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\Phi(t) = \mathcal{L}^{-1}\{\Phi(s)\} = \begin{bmatrix} \mathcal{L}^{-1}\left\{\frac{s+3}{(s+1)(s+2)}\right\} & \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} \\ \mathcal{L}^{-1}\left\{\frac{-2}{(s+1)(s+2)}\right\} & \mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s+2)}\right\} \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{bmatrix}$$

$$\Phi_{11}(t) = \mathcal{L}^{-1}\left\{\frac{s+3}{(s+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{1}{s+2}\right\} = (2e^{-t} - e^{-2t})u(t)$$

$$\Phi_{12}(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} = (e^{-t} - e^{-2t})u(t)$$

$$\Phi_{21}(t) = \mathcal{L}^{-1}\left\{\frac{-2}{(s+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{-\frac{2}{s+1} + \frac{2}{s+2}\right\} = (-2e^{-t} + e^{-2t})u(t)$$

$$\Phi_{22}(t) = \mathcal{L}^{-1} \left\{ \frac{s}{(s+1)(s+2)} \right\} = \mathcal{L}^{-1} \left\{ -\frac{1}{s+1} + \frac{2}{s+2} \right\} = (-e^{-t} + 2e^{-2t})u(t)$$

#2Ab)

$$H(s) = C(sI - A)^{-1}B = C\Phi(s)B = [3 \ 0] \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3}{s^2 + 3s + 2}$$

#2Ac)

$$\begin{aligned} X(s) &= \Phi(s)x(0) + \Phi(s)BF(s) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times 1 \\ &= \begin{bmatrix} 2\Phi_{11}(s) \\ 2\Phi_{21}(s) \end{bmatrix} + \begin{bmatrix} \Phi_{12}(s) \\ \Phi_{22}(s) \end{bmatrix} = 2 \begin{bmatrix} \frac{2}{s+1} - \frac{2}{s+2} \\ -\frac{2}{s+1} + \frac{2}{s+2} \end{bmatrix} + \begin{bmatrix} \frac{1}{s+1} - \frac{1}{s+2} \\ -\frac{1}{s+1} + \frac{2}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{5}{s+1} - \frac{3}{s+2} \\ -\frac{5}{s+1} + \frac{6}{s+2} \end{bmatrix} \\ &\leftrightarrow \begin{bmatrix} 5e^{-t} - 3e^{-2t} \\ -5e^{-t} + 6e^{-2t} \end{bmatrix} u(t) = x(t) \end{aligned}$$

#2Ad)

$$\begin{aligned} Y(s) &= C\Phi(s)x(0) + H(s)F(s) = [3 \ 0] \begin{bmatrix} \Phi_{11}(s) & \Phi_{12}(s) \\ \Phi_{21}(s) & \Phi_{22}(s) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \frac{3}{(s+1)(s+2)} \times \frac{1}{s+3} \\ &= 6\Phi_{11}(s) + \frac{3}{(s+1)(s+2)(s+3)} = \frac{6(s+3)}{(s+1)(s+2)} + \frac{3}{(s+1)(s+2)(s+3)} \\ &= \frac{12}{s+1} - \frac{6}{s+2} + \frac{3/2}{s+1} - \frac{3}{s+2} + \frac{3/2}{s+3} = \frac{27/2}{s+1} - \frac{3}{s+2} + \frac{3/2}{s+3} \\ &\leftrightarrow y(t) = \left(\frac{27}{2}e^{-t} - 9e^{-2t} + \frac{3}{2}e^{-3t} \right) u(t) \end{aligned}$$

#2Ba)

$$\begin{aligned} \Phi(z) &= z(sI - A)^{-1} = z \begin{bmatrix} z & -1 \\ \frac{1}{6} & z + \frac{5}{6} \end{bmatrix}^{-1} = \frac{z}{z(z + \frac{5}{6}) + \frac{1}{6}} \begin{bmatrix} z + \frac{5}{6} & 1 \\ -\frac{1}{6} & z \end{bmatrix} \\ &= \frac{z}{(z + \frac{1}{2})(z + \frac{1}{3})} \begin{bmatrix} z + \frac{5}{6} & 1 \\ -\frac{1}{6} & z \end{bmatrix} = \begin{bmatrix} -\frac{2z}{z + \frac{1}{2}} + \frac{3z}{z + \frac{1}{3}} & -\frac{6z}{z + \frac{1}{2}} + \frac{6z}{z + \frac{1}{3}} \\ \frac{z}{z + \frac{1}{2}} - \frac{z}{z + \frac{1}{3}} & \frac{3z}{z + \frac{1}{2}} - \frac{2z}{z + \frac{1}{3}} \end{bmatrix} \\ &\leftrightarrow \Phi[k] = \begin{bmatrix} -2(-\frac{1}{2})^k + 3(-\frac{1}{3})^k & -6(-\frac{1}{2})^k + 6(-\frac{1}{3})^k \\ (-\frac{1}{2})^k - (-\frac{1}{3})^k & 3(-\frac{1}{2})^k - 2(-\frac{1}{3})^k \end{bmatrix} u[k] \end{aligned}$$

#2Bb)

$$\begin{aligned} H(z) &= C(zI - A)^{-1}B = [0 \ 2] \begin{bmatrix} z & -1 \\ \frac{1}{6} & z + \frac{5}{6} \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = [0 \ 2] \frac{1}{z^2 + \frac{5}{6}z + \frac{1}{6}} \begin{bmatrix} z + \frac{5}{6} & 1 \\ -\frac{1}{6} & z \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} \\ &= \frac{1}{z^2 + \frac{5}{6}z + \frac{1}{6}} [-\frac{1}{3} \ 2z] \begin{bmatrix} 3 \\ 0 \end{bmatrix} = -\frac{1}{z^2 + \frac{5}{6}z + \frac{1}{6}} \end{aligned}$$

#2Bc)

$$\begin{aligned} Y(z) &= C\Phi(z)x(0) + H(z)F(z) = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} \Phi_{11}(z) & \Phi_{12}(z) \\ \Phi_{21}(z) & \Phi_{22}(z) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{(z^2 + \frac{5}{6}z + \frac{1}{6})} \frac{z}{(z + \frac{1}{4})} \\ &= 2\Phi_{21}(z) + 4\Phi_{22}(z) - \frac{z}{(z + \frac{1}{2})(z + \frac{1}{3})(z + \frac{1}{4})} \\ &= -\frac{1}{3} \frac{z}{(z + \frac{1}{2})(z + \frac{1}{3})} + \frac{4z^2}{(z + \frac{1}{2})(z + \frac{1}{3})} - \frac{z}{(z + \frac{1}{2})(z + \frac{1}{3})(z + \frac{1}{4})} \\ y[k] &= \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{ \frac{z(4z^2 + \frac{2}{3}z - \frac{13}{12})}{(z + \frac{1}{2})(z + \frac{1}{3})(z + \frac{1}{4})} \right\} = \mathcal{Z}^{-1}\left\{ -\frac{10z}{z + \frac{1}{2}} + \frac{62z}{z + \frac{1}{3}} - \frac{48z}{z + \frac{1}{4}} \right\} \\ &= \left(-10\left(-\frac{1}{2}\right)^k + 6\left(-\frac{1}{3}\right)^k - 48\left(-\frac{1}{4}\right)^k \right) u[k] \end{aligned}$$

#2Bd)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & -2 & 1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [4 \quad 0 \quad 0 \quad 1 \quad 0], \quad D = 0$$

#3)

$$\begin{aligned} K_p &= \lim_{s \rightarrow 0} \{H(s)\} = \lim_{s \rightarrow 0} \left\{ \frac{K}{s(s+1)(s+2)(s+5)} \right\} = \infty \Rightarrow e_{ss}^{step} = \frac{1}{1 + K_p} = 0 \\ K_v &= \lim_{s \rightarrow 0} \{sH(s)\} = \lim_{s \rightarrow 0} \left\{ \frac{sK}{s(s+1)(s+2)(s+5)} \right\} = \frac{K}{10} \Rightarrow e_{ss}^{ramp} = \frac{1}{K_v} = \frac{10}{K} \\ K_a &= \lim_{s \rightarrow 0} \{s^2H(s)\} = \lim_{s \rightarrow 0} \left\{ \frac{s^2K}{s(s+1)(s+2)(s+5)} \right\} = 0 \Rightarrow e_{ss}^{par} = \frac{1}{K_a} = \infty \end{aligned}$$