

Sample Exam 3: Solutions

#1)

$$\begin{aligned}
\mathcal{L}\{f_1(t)\} &= \mathcal{L}\{(t+2)e^{-t}u(t-1)\} = \mathcal{L}\left\{(t-1+1+2)e^{-(t-1+1)}u(t-1)\right\} \\
&= e^{-1}\mathcal{L}\left\{(t-1+3)e^{-(t-1)}u(t-1)\right\} = e^{-1}\mathcal{L}\left\{(t-1)e^{-(t-1)}u(t-1)\right\} + 3e^{-1}\mathcal{L}\left\{e^{-(t-1)}u(t-1)\right\} \\
&= e^{-1}e^{-s}\mathcal{L}\{te^{-t}u(t)\} + 3e^{-1}e^{-s}\mathcal{L}\{e^{-t}u(t)\} = e^{-(s+1)}\frac{1}{(s+1)^2} + 3e^{-(s+1)}\frac{1}{s+1}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{f_2(t)\} &= \mathcal{L}\{\cos(\pi t)u(t-2)\} = \mathcal{L}\{\cos(\pi(t-2)+2\pi)u(t-2)\} \\
&= \mathcal{L}\{[\cos(\pi(t-2))\cos(2\pi)-\sin(\pi(t-2))\sin(2\pi)]u(t-2)\} \\
&= \mathcal{L}\{[\cos(\pi(t-2))-0]u(t-2)\} = e^{-2s}\mathcal{L}\{\cos(\pi t)u(t)\} = e^{-2s}\frac{s}{s^2+\pi^2}
\end{aligned}$$

For the third signal the Laplace transform can be obtained directly from the table as

$$\mathcal{L}\{f_3(t)\} = \mathcal{L}\{6te^{-2t}\sin(3t)u(t)\} = \frac{6(s+2)}{\left((s+2)^2+9\right)^2}$$

#2)

$$\begin{aligned}
F(s) &= \frac{2s+e^{-s}}{s^2(s+1)} = \frac{2}{s(s+1)} + \frac{e^{-s}}{s^2(s+1)} = F_1(s) + F_2(s)e^{-s} \\
F_1(s) &= \frac{k_1}{s} + \frac{k_2}{s+1} = \frac{2}{s} - \frac{2}{s+1} \leftrightarrow (2-2e^{-t})u(t) = f_1(t) \\
F_2(s) &= \frac{c_{12}}{s} + \frac{c_{11}}{s^2} + \frac{c_2}{s+1} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1} \leftrightarrow (-1+t+e^{-t})u(t) = f_2(t) \\
f(t) &= (2-2e^{-t})u(t) + (-1+t-1+e^{-(t-1)})u(t-1) = (2-2e^{-t})u(t) + (-2+t+e^{-(t-1)})u(t-1)
\end{aligned}$$

#3)

$$\begin{aligned}
f_1[k] &= 3^{k+2}(k+2)u[k-2] = 3^{k-2+2+2}(k-2+2+2)u[k] = 81 \times 3^{k-2}(k-2)u[k-2] \\
&= 81 \times 3^{k-2}(k-2)u[k-2] + 324 \times 3^{k-2}u[k-2] \leftrightarrow 81\frac{3z}{(z-3)^2}z^{-2} + 324\frac{z}{z-3}z^{-2}
\end{aligned}$$

Using the table of the \mathcal{Z} -transform common pairs we have

$$f[k] = \cos\left[k\frac{\pi}{2}\right]u[k] \leftrightarrow \frac{z^2 - z\cos\left(\frac{\pi}{2}\right)}{z^2 - 2z\cos\left(\frac{\pi}{2}\right) + 1} = \frac{z^2}{z^2 + 1} = F(z)$$

By the time multiplication property we obtain

$$kf[k] \leftrightarrow -z\frac{dF(z)}{dz} = -z\frac{d}{dz}\left\{\frac{z^2}{z^2+1}\right\} = -\frac{2z^2}{(z^2+1)^2}$$

$$f_3[k] = \begin{cases} 2, & k = 5 \\ 4, & k = 9 \\ 0, & \text{otherwise} \end{cases} \Rightarrow \mathcal{Z}\{f_3[k]\} \triangleq \frac{2}{z^5} + \frac{4}{z^9}$$

#4)

$$\begin{aligned} F(z) &= \frac{4(z+2)}{(z+1)(z+3)(z+5)} \times \frac{z}{z} \Rightarrow \frac{1}{z}F(z) = \frac{k_1}{z} + \frac{k_2}{z+1} + \frac{k_3}{z+3} + \frac{k_4}{z+5} \\ k_1 &= \frac{4(z+2)}{(z+1)(z+3)(z+5)}|_{z=0} = \frac{8}{15}, \quad k_2 = \frac{4(z+2)}{z(z+3)(z+5)}|_{z=-1} = -\frac{1}{2} \\ k_3 &= \frac{4(z+2)}{z(z+1)(z+5)}|_{z=-3} = -\frac{1}{3}, \quad k_4 = \frac{4(z+2)}{z(z+1)(z+3)}|_{z=-1} = \frac{3}{10} \\ F(z) &= \frac{8}{15} - \frac{1}{2} \frac{z}{(z+1)} - \frac{1}{3} \frac{z}{(z+3)} + \frac{3}{10} \frac{z}{(z+5)} \leftrightarrow \frac{8}{15} \delta[k] + \left(-\frac{1}{2}(-1)^k - \frac{1}{3}(-3)^k + \frac{3}{10}(-5)^k \right) u[k] \end{aligned}$$

The initial value theorem is always applicable, hence

$$f[0] = \lim_{z \rightarrow \infty} \{F(z)\} = \lim_{z \rightarrow \infty} \left\{ \frac{4(z+2)}{(z+1)(z+3)(z+5)} \right\} = 0$$

This can be verified from the expression obtained for $f[k]$

$$f[0] = \frac{8}{15} - \frac{1}{2} - \frac{1}{3} + \frac{3}{10}$$

The final value theorem is not applicable in this case since the function $(z-1)F(z)$ has poles outside the unit circle.

#5)

$$\begin{aligned} H(s) &= \frac{3s-1}{s^2+6s+9} \\ h(t) &= \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{3s-1}{(s+3)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+3} - \frac{10}{(s+3)^2}\right\} = (3e^{-3t} - 10te^{-3t})u(t) \\ y_{step}(t) &= \int_0^t h(\tau)d\tau = \int_0^t (3e^{-3\tau} - 10\tau e^{-3\tau})d\tau = \left(-\frac{1}{9} + \frac{1}{9}e^{-3t} + \frac{10}{3}te^{-3t}\right)u(t) \end{aligned}$$

Another way to find the step response

$$Y_{step}(s) = H(s) \frac{1}{s} = \frac{k_1}{s} + \frac{k_{21}}{s+3} + \frac{k_{22}}{(s+3)^2} = -\frac{1/9}{s} + \frac{1/9}{s+3} + \frac{10/3}{(s+3)^2} \leftrightarrow \left(-\frac{1}{9} + \frac{1}{9}e^{-3t} + \frac{10}{3}te^{-3t}\right)u(t)$$

$$Y_{zs}(s) = H(s)F(s) = \frac{3s-1}{(s+3)^2(s+1)} = \frac{1}{s+3} + \frac{5}{(s+3)^2} - \frac{1}{s+1} \leftrightarrow (e^{-3t} + 5te^{-3t} - e^{-t})u(t)$$

$$s^2 Y_{zi}(s) - s y_{zi}(0^-) - y_{zi}^{(1)}(0^-) + 6(sY_{zi}(s) - y_{zi}(0^-)) + 9Y_{zi}(s) = 0, \quad y_{zi}(0^-) = 1, \quad y_{zi}^{(1)}(0^-) = 0$$

$$(s^2 + 6s + 9)Y_{zi}(s) = s + 6 \Rightarrow Y_{zi}(s) = \frac{s+6}{(s+3)^2} = \frac{1}{s+3} + \frac{3}{(s+3)^2}$$

$$\Rightarrow y_{zi}(t) = \mathcal{L}^{-1}\{Y_{zi}(s)\} = (e^{-t} + 3e^{-3t})u(t)$$

#6) The system transfer function is

$$H(z) = \frac{z+1}{z^2 - \frac{1}{6}z - \frac{1}{6}}$$

The impulse response is obtained from

$$\begin{aligned} h[k] &= \mathcal{Z}^{-1}\{H(z)\}, \quad \frac{1}{z}H(z) = -\frac{6}{z} + \frac{18/5}{z - \frac{1}{2}} + \frac{12/5}{z + \frac{1}{3}} \\ h[k] &= \mathcal{Z}^{-1}\left\{-6 + \frac{18}{5}\frac{z}{(z - \frac{1}{2})} + \frac{12}{5}\frac{z}{(z + \frac{1}{3})}\right\} = -6\delta[k] + \left(\frac{18}{5}\left(\frac{1}{2}\right)^k + \frac{12}{5}\left(-\frac{1}{3}\right)^k\right)u[k] \end{aligned}$$

The zero-input response is given by

$$\begin{aligned} \mathcal{Z}\left\{y_{zi}[k] - \frac{1}{6}y_{zi}[k-1] - \frac{1}{6}y_{zi}[k-2] = 0\right\} \\ \Rightarrow Y_{zi}(z) - \frac{1}{6}\left(\frac{1}{z}Y_{zi}(z) + y[-1]\right) - \frac{1}{6}\left(\frac{1}{z^2}Y_{zi}(z) + \frac{1}{z}y_{zi}[-1] + y_{zi}[-2]\right) = 0 \\ Y_{zi}(z)\left(1 - \frac{1}{6z} - \frac{1}{6z}\right) = \frac{1}{6}y_{zi}[-1] + \frac{1}{6z}y_{zi}[-1] + \frac{1}{6}y_{zi}[-2] \\ Y_{zi}(z) = -\frac{1}{3}\frac{z^2}{(z^2 - \frac{1}{6}z - \frac{1}{6})} = -\frac{1}{5}\frac{z}{(z - \frac{1}{2})} - \frac{2}{15}\frac{z}{(z + \frac{1}{3})} \leftrightarrow \left(-\frac{1}{5}\left(\frac{1}{2}\right)^k - \frac{2}{15}\left(-\frac{1}{3}\right)^k\right)u[k] = y_{zi}[k] \end{aligned}$$

The zero-state response is

$$\begin{aligned} \Rightarrow Y_{zs}(z) &= H(z)F(z) = \frac{z+1}{(z^2 - \frac{1}{6}z - \frac{1}{6})(z+1)} = \frac{z}{z^2 - \frac{1}{6}z - \frac{1}{6}} \\ \frac{1}{z}Y_{zs}(z) &= \frac{k_1}{z - \frac{1}{2}} + \frac{k_2}{z + \frac{1}{3}} = \frac{6/5}{z - \frac{1}{2}} - \frac{6/5}{z + \frac{1}{3}} \\ y_{zs}[k] &= \mathcal{Z}^{-1}\left\{\frac{6}{5}\frac{z}{(z - \frac{1}{2})} - \frac{6}{5}\frac{z}{(z + \frac{1}{3})}\right\} = \left(\frac{6}{5}\left(\frac{1}{2}\right)^k - \frac{6}{5}\left(-\frac{1}{3}\right)^k\right)u[k] \end{aligned}$$

The steady state response due to $f[k] = 5u[k]$ can be obtained using the known formula

$$y_{ss} = 5H(1) = 5 \times 12 = 60$$

Sample Exam 4: Solutions

#1a)

$$\begin{aligned}
u(3t - 4) &= \begin{cases} 1, & t \geq 4/3 \\ 0, & t < 4/3 \end{cases} = u\left(t - \frac{4}{3}\right) \Rightarrow e^{-t}u(3t - 4) = e^{-t}u\left(t - \frac{4}{3}\right) \\
\mathcal{L}\{f_1(t)\} &= \mathcal{L}\left\{e^{-t}u\left(t - \frac{4}{3}\right)\right\} = \mathcal{L}\left\{e^{-(t-\frac{4}{3}+\frac{4}{3})}u\left(t - \frac{4}{3}\right)\right\} = e^{\frac{4}{3}}\mathcal{L}\left\{e^{-(t-\frac{4}{3})}u\left(t - \frac{4}{3}\right)\right\} \\
&= e^{\frac{4}{3}}e^{-s\frac{4}{3}}\mathcal{L}\{e^{-t}u(t)\} = \frac{1}{s+1}e^{\frac{4}{3}}e^{-s\frac{4}{3}}
\end{aligned}$$

Another way to find the Laplace transform of $f_1(t)$ is to use the Laplace transform definition integral

$$\begin{aligned}
\mathcal{L}\{f_1(t)\} &= \mathcal{L}\left\{e^{-t}u\left(t - \frac{4}{3}\right)\right\} \triangleq \mathcal{L}\left\{e^{-t}u\left(t - \frac{4}{3}\right)\right\} = \int_{0^-}^{\infty} e^{-t}u\left(t - \frac{4}{3}\right)e^{-st}dt \\
&= \int_{\frac{4}{3}}^{\infty} e^{-(s+1)t}dt = -\frac{1}{(s+1)}e^{-(s+1)t}|_{t=\frac{4}{3}}^{\infty} = \frac{1}{(s+1)}e^{-(s+1)\frac{4}{3}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{f_2(t)\} &= \mathcal{L}\{\cos(\pi t)[u(t) - u(t-2)]\} = \mathcal{L}\{\cos(\pi t)u(t)\} - \mathcal{L}\{\cos(\pi t)u(t-2)\} \\
&= \frac{s}{s^2 + \pi^2} - \mathcal{L}\{\cos(\pi(t-2) + 2\pi)u(t-2)\} \\
&= \frac{s}{s^2 + \pi^2} - \mathcal{L}\{[\cos(\pi(t-2))\cos(2\pi) - \sin(\pi(t-2))\sin(2\pi)]u(t-2)\} \\
&= \frac{s}{s^2 + \pi^2} - \mathcal{L}\{[\cos(\pi(t-2)) - 0]u(t-2)\} = \frac{s}{s^2 + \pi^2} - e^{-2s}\mathcal{L}\{\cos(\pi t)u(t)\} \\
&\quad \frac{s}{s^2 + \pi^2} - e^{-2s}\frac{s}{s^2 + \pi^2} = \frac{s(1 - e^{-2s})}{s^2 + \pi^2}
\end{aligned}$$

Another way to solve the same problem is to use the modulation property

$$\begin{aligned}
\mathcal{L}\{[u(t) - u(t-2)]\} &= \frac{1}{s} - \frac{1}{s}e^{-2s} = \frac{1 - e^{-2s}}{s} = F(s) \\
[u(t) - u(t-2)]\cos(\pi t) &\leftrightarrow \frac{1}{2}\{F(s+j\pi) + F(s-j\pi)\} = \frac{1}{2}\left(\frac{1 - e^{-2(s+j\pi)}}{s+j\pi} + \frac{1 - e^{-2(s-j\pi)}}{s-j\pi}\right) \\
&\quad \frac{(1 - e^{-2(s+j\pi)})(s-j\pi) + (1 - e^{-2(s-j\pi)})(s+j\pi)}{2(s^2 + \pi^2)} \\
&= \frac{(1 - e^{-2s})(s-j\pi) + (1 - e^{-2s})(s+j\pi)}{2(s^2 + \pi^2)} = \frac{s - se^{-2s}}{s^2 + \pi^2}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{f_3(t)\} &= \mathcal{L}\{(t^2 + 1)u(t-1)\} = \mathcal{L}\{(t^2 + 2t - 2t + 1)u(t-1)\} \\
&= \mathcal{L}\left\{\left[(t-1)^2 + 2t\right]u(t-1)\right\} = \mathcal{L}\{(t-1)^2u(t-1)\} + \mathcal{L}\{2tu(t-1)\}
\end{aligned}$$

$$\begin{aligned}
&= e^{-s} \mathcal{L}\{t^2 u(t)\} + \mathcal{L}\{2(t-1+1)u(t-1)\} \\
&= \frac{2e^{-s}}{s^3} + \mathcal{L}\{2(t-1)u(t-1)\} + \mathcal{L}\{2u(t-1)\} = \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{2e^{-s}}{s}
\end{aligned}$$

Another way to solve this problem

$$\begin{aligned}
\mathcal{L}\{u(t-1)\} &= \frac{1}{s}e^{-s}, \quad t^2 u(t-1) \leftrightarrow (-1)^2 \frac{d^2}{ds^2} \left(\frac{e^{-s}}{s} \right) = e^{-s} \left(\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3} \right) \\
\mathcal{L}\{f_3(t)\} &= \mathcal{L}\{t^2 u(t-1)\} + \mathcal{L}\{u(t-1)\} = e^{-s} \left(\frac{2}{s} + \frac{2}{s^2} + \frac{2}{s^3} \right) = \frac{2e^{-s}}{s} \left(1 + \frac{1}{s} + \frac{1}{s^2} \right)
\end{aligned}$$

#1b1)

$$F_1(s) = \frac{e^{-2s}}{s(s+1)(s+2)(s+3)(s+4)(s+5)} = F_0(s)e^{-2s}$$

The time domain component corresponding to the pole at the origin is given by

$$k_1 u(t-2), \quad k_1 = \lim_{s \rightarrow 0} \{s F_0(s)\} = \frac{1}{1 \times 2 \times 3 \times 4 \times 5} = \frac{1}{120}$$

#1b2)

$$\begin{aligned}
F_2(s) &= \frac{e^{-3s}}{s(s^2+1)} = F_3(s)e^{-3s}, \quad F_3(s) = \frac{k_1}{s} + \frac{k_2}{s+j} + \frac{k_2^*}{s-j} \\
k_1 &= \frac{1}{s^2+1}|_{s=0} = 1, \quad k_2 = \frac{1}{s(s-j)}|_{s=-j} = \frac{1}{(-j)(-2j)} = -\frac{1}{2} = |k_2| \angle k_2 = \frac{1}{2} \angle \pi \\
F_3(s) &= \frac{1}{s} + \frac{-1/2}{s+j} + \frac{-1/2}{s-j} \leftrightarrow u(t) + 2|k_2|e^{0} \cos(t + \angle k_2)u(t) \\
&= u(t) + \cos(t + \pi)u(t) = (1 + \cos(t))u(t) = f_3(t) \Rightarrow f_2(t) = (1 + \cos(t-3))u(t-3)
\end{aligned}$$

Another way to solve this problem

$$\begin{aligned}
F_3(s) &= \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{As^2 + A + Bs^2 + Cs}{(s^2+1)} \Rightarrow A = 1 = -B, C = 0 \\
F_3(s) &= \frac{1}{s} - \frac{s}{s^2+1} \leftrightarrow u(t) - \cos(t)u(t) \Rightarrow f_2(t) = (1 - \cos(t-3))u(t-3)
\end{aligned}$$

#2a)

$$f_1[k] = 3^{k+2}(k+2)u[k] = 9 \times 3^k k u[k] + 18 \times 3^k u[k] \leftrightarrow 9 \frac{3z}{(z-3)^2} + 18 \frac{z}{z-3} = \frac{18z^2 - 27z}{(z-3)^2}$$

Another way to find the \mathcal{Z} -transform for $f_1[k]$

$$\begin{aligned}
f_1[k] &= 3^{k+2}(k+2)u[k] = f[k+2]u[k], \quad f[k]u[k] = 3^k k u[k] \leftrightarrow \frac{3z}{(z-3)^2} = F(z) \\
\mathcal{Z}\{f_1[k]\} &= z^2 F(z) - z^2 f[0] - z f[1] = \frac{3z^3}{(z-3)^2} - 0 + 3z = \frac{18z^2 - 27z}{(z-3)^2}
\end{aligned}$$

$$\begin{aligned}
f_2[k] &= \cos \left[k \frac{\pi}{2} \right] u[k-1] = \cos \left[(k-1+1) \frac{\pi}{2} \right] u[k-1] \\
&= \left(\cos \left[(k-1) \frac{\pi}{2} \right] \cos \left[\frac{\pi}{2} \right] - \sin \left[(k-1) \frac{\pi}{2} \right] \sin \left[\frac{\pi}{2} \right] \right) u[k-1] = -\sin \left[(k-1) \frac{\pi}{2} \right] u[k-1] \\
&\leftrightarrow -z^{-1} \mathcal{Z} \left\{ \sin \left[k \frac{\pi}{2} \right] u[k] \right\} = -z^{-1} \frac{z \sin \left(\frac{\pi}{2} \right)}{z^2 - 2z \cos \left(\frac{\pi}{2} \right) + 1} = -\frac{1}{z^2 + 1}
\end{aligned}$$

$$f_3[k] = \begin{cases} 3, & k = 15 \\ -4, & k = 29 \\ 0, & \text{otherwise} \end{cases} \Rightarrow \mathcal{Z}\{f_3[k]\} \triangleq \frac{3}{z^{15}} - \frac{4}{z^{29}}$$

#2b)

$$\begin{aligned}
F(z) &= \frac{5(z+2)}{(z+1)^2(z-1)} \times \frac{z}{z} \Rightarrow \frac{1}{z} F(z) = \frac{k_1}{z} + \frac{k_2}{z-1} + \frac{k_{31}}{z+1} + \frac{k_{32}}{(z+1)^2} \\
k_1 &= \frac{5(z+2)}{(z-1)(z+1)}|_{z=0} = -10, \quad k_2 = \frac{5(z+2)}{z(z+1)}|_{z=1} = \frac{15}{4} \\
k_{32} &= \frac{5(z+2)}{z(z-1)}|_{z=-1} = \frac{5}{2}, \quad k_{31} = \frac{d}{dz} \left\{ \frac{5(z+2)}{z(z-1)} \right\}|_{z=-1} = \frac{25}{4} \\
F(z) &= -10 + \frac{(15/4)z}{z-1} + \frac{(25/4)z}{z+1} + \frac{(5/2)z}{(z+1)^2} \leftrightarrow -10\delta[k] + \left(\frac{15}{4} + \frac{25}{4}(-1)^k - \frac{5}{2}k(-1)^k \right) u[k]
\end{aligned}$$

The final value theorem is not applicable in this case since the function $(z-1)F(z)$ has a double pole on the unit circle at $z = -1$. The initial value theorem is always applicable, hence

$$f[0] = \lim_{z \rightarrow \infty} \{F(z)\} = \lim_{z \rightarrow \infty} \left\{ \frac{5(z+2)}{(z+1)^2(z-1)} \right\} = 0$$

#3a)

$$\begin{aligned}
H(s) &= \frac{1}{s^2 + 5s + 4} \\
h(t) &= \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)(s+4)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1/3}{(s+1)} - \frac{1/3}{s+4} \right\} = \frac{1}{3} (e^{-t} - e^{-4t}) u(t) \\
y_{step}(t) &= \int_0^t h(\tau) d\tau = \frac{1}{3} \int_0^t (e^{-\tau} - e^{-4\tau}) d\tau = \left(\frac{1}{4} - \frac{1}{3}e^{-t} + \frac{1}{12}e^{-4t} \right) u(t)
\end{aligned}$$

Another way to find the step response

$$\begin{aligned}
Y_{step}(s) &= H(s) \frac{1}{s} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+4} = \frac{1/4}{s} + \frac{1/12}{s+1} + \frac{-1/3}{s+4} \leftrightarrow \left(\frac{1}{4} - \frac{1}{12}e^{-t} - \frac{1}{3}e^{-4t} \right) u(t) \\
Y_{zs}(s) &= H(s)F(s) = \frac{1}{(s+1)(s+4)} \times \frac{5}{s} = \frac{5/4}{s} - \frac{5/3}{s+1} + \frac{5/12}{s+4} \leftrightarrow \left(\frac{5}{4} - \frac{5}{3}e^{-t} + \frac{5}{12}e^{-4t} \right) u(t) = 5y_{step}(t)
\end{aligned}$$

$$\begin{aligned}
s^2 Y_{zi}(s) - s y_{zi}(0^-) - y_{zi}^{(1)}(0^-) + 5(sY_{zi}(s) - y_{zi}(0^-)) + 4Y_{zi}(s) &= 0, \quad y_{zi}(0^-) = 2, \quad y_{zi}^{(1)}(0^-) = 0 \\
(s^2 + 5s + 4) Y_{zi}(s) &= 2s + 10 \Rightarrow Y_{zi}(s) = \frac{2s + 10}{(s+1)(s+4)} = \frac{8/3}{s+1} - \frac{2/3}{s+4} \\
&\Rightarrow y_{zi}(t) = \mathcal{L}^{-1}\{Y_{zi}(s)\} = \left(\frac{8}{3}e^{-t} - \frac{2}{3}e^{-4t} \right) u(t)
\end{aligned}$$

The steady state response is

$$y_{ss} = \lim_{t \rightarrow \infty} \{y(t)\} = \lim_{t \rightarrow \infty} \{y_{zs}(t) + y_{zi}(t)\} = \frac{5}{4} = H(0) \times 5 = \frac{5}{4}$$

The system can reach its steady state value since both poles are in the left half complex plane.

#3b)

$$H(z) = \frac{z+1}{z^2+z+\frac{1}{4}}$$

$$h[k] = \mathcal{Z}^{-1}\{H(z)\}, \quad \frac{1}{z}H(z) = \frac{4}{z} - \frac{4}{z+\frac{1}{2}} - \frac{1}{(z+\frac{1}{2})^2}$$

$$h[k] = \mathcal{Z}^{-1}\left\{4 - \frac{4z}{z+\frac{1}{2}} - \frac{z}{(z+\frac{1}{2})^2}\right\} = \delta[k] + \left(-4\left(-\frac{1}{2}\right)^k + 2k\left(-\frac{1}{2}\right)\right)u[k]$$

$$\mathcal{Z}\left\{y_{zi}[k] + y_{zi}[k-1] + \frac{1}{4}y_{zi}[k-2] = 0\right\}$$

$$\Rightarrow Y_{zi}(z) + \frac{1}{z}Y_{zi}(z) + y[-1] + \frac{1}{4}\left(\frac{1}{z^2}Y_{zi}(z) + \frac{1}{z}y_{zi}[-1] + y_{zi}[-2]\right) = 0$$

$$Y_{zi}(z)\left(1 + \frac{1}{z} + \frac{1}{4z}\right) = -y_{zi}[-1] - \frac{1}{z}y_{zi}[-1] - y_{zi}[-2] = -\frac{3}{2} - \frac{1}{4z}$$

$$Y_{zi}(z) = -\frac{1}{2}(3z+1)\frac{z}{(z^2+z+\frac{1}{4})}$$

$$\frac{1}{z}Y_{zi}(z) = -0.5\frac{3z+1}{(z^2+z+\frac{1}{4})} = \frac{k_1}{z+\frac{1}{2}} + \frac{k_2}{(z+\frac{1}{2})^2} = \frac{-3/2}{z+\frac{1}{2}} + \frac{1/4}{(z+\frac{1}{2})^2}$$

$$y_{zi}[k] = \mathcal{Z}^{-1}\left\{-\frac{3}{2}\frac{z}{(z+\frac{1}{2})} + \frac{1}{4}\frac{z}{(z+\frac{1}{2})^2} \times \frac{(-\frac{1}{2})}{(-\frac{1}{2})}\right\} = -\frac{3}{2}\left(-\frac{1}{2}\right)^k u[k] - \frac{1}{2}k\left(-\frac{1}{2}\right)^k u[k]$$

$$\Rightarrow Y_{zs}(z) = H(z)F(z) = \frac{z+1}{(z^2+z+\frac{1}{4})} \frac{z}{(z+\frac{1}{4})}$$

$$\frac{1}{z}Y_{zs}(z) = \frac{k_1}{z+\frac{1}{4}} + \frac{k_{21}}{z+\frac{1}{2}} + \frac{k_{22}}{(z+\frac{1}{2})^2} = \frac{12}{z+\frac{1}{4}} - \frac{12}{z+\frac{1}{2}} - \frac{2}{(z+\frac{1}{2})^2}$$

$$y_{zs}[k] = \mathcal{Z}^{-1}\left\{\frac{12}{z+\frac{1}{4}} - \frac{12z}{z+\frac{1}{2}} - \frac{2z}{(z+\frac{1}{2})^2} \times \frac{(-\frac{1}{2})}{(-\frac{1}{2})}\right\} = 12\left(-\frac{1}{4}\right)^k u[k] - 12\left(-\frac{1}{2}\right)^k u[k] + 4k\left(-\frac{1}{2}\right)^k u[k]$$