Sample Exam 1: Solutions

#1a) The homogeneous solution is obtained from

$$\frac{dy_h(t)}{dt} + 2y_h(t) = 0 \quad \Rightarrow \quad y_h(t) = Ce^{-2t}$$

The particular solution satisfies

$$\frac{dy_p(t)}{dt} + 2y_p(t) = 3 \quad \Rightarrow \quad y_p(t) = \alpha$$

Plugging this solution into the differential equation implies $\alpha = 1.5$ so that $y_p(t) = 1.5$

The system response is given by

$$y(t) = y_h(t) + y_p(t) = Ce^{-2t} + 1.5$$

Its initial condition produces

$$y(0) = 4 = y_h(0) + y_p(0) = C + 1.5 \quad \Rightarrow \quad C = 2.5 \quad \Rightarrow \quad y_h(t) = 2.5e^{-2t}$$

Hence, the system response is

$$y(t) = y_h(t) + y_p(t) = 2.5e^{-2t} + 1.5$$

The system zero-state response satisfies

$$\frac{dy_{zs}(t)}{dt} + 2y_{zs}(t) = 3, \quad y_{zs}(0) = 0, \quad t \geq 0$$

The solution of this differential equation is sought in the form

$$y_{zs}(t) = y_{zs}^h(t) + y_{zs}^p(t)$$

The particular component of the zero-state response is given by

$$\frac{dy_{zs}^p(t)}{dt} + 2y_{zs}^p(t) = 3 \quad \Rightarrow \quad y_{zs}^p(t) = \alpha = 1.5$$

The homogenous component of the system zero-state response is given by $y_{zs}^h(t) = \beta e^{-2t}$. The zero initial condition produces

$$y_{zs}(0) = 0 = y_{zs}^h(0) + y_{zs}^p(0) = \beta + 1.5 \quad \Rightarrow \quad \beta = -1.5 \quad \Rightarrow \quad y_{zs}^h(t) = -1.5e^{-2t} \quad \Rightarrow \quad y_{zs}(t) = 1.5 - 1.5e^{-2t}$$

The zero-input response satisfies

$$\frac{dy_{zi}(t)}{dt} + 2y_{zi}(t) = 0 \quad y_{zi}(0) = 4 \quad \Rightarrow \quad y_{zi}(t) = \gamma e^{-2t}$$

Using the zero-input response initial condition we get $y_{zi}(0) = 4 = \gamma \quad \Rightarrow \quad y_{zi}(t) = 4e^{-2t}$

Note that

$$y_{zi}(t) + y_{zs}(t) = 1.5 + 2.5e^{-2t} = y(t) = y_p(t) + y_h(t)$$

#1b) The linearity principle requires that the initial conditions are set to zero. In that case

$$y_{zs}(t) = \frac{1}{3}(1.5 - 1.5e^{-2t})u(t) + \frac{1}{3}(1.5 - 1.5e^{-2(t-2)})u(t-2) = \frac{1}{2}(1 - e^{-2t})u(t) + \frac{1}{2}(1 - e^{-2(t-2)})u(t-2)$$

#2a)

(i) $e^{-5t}\delta(t-2) = e^{-10}\delta(t-2)$

(ii) $\int_{-\infty}^{+\infty}(t+2)\delta(t-0.5)dt = (t+2)_{|t=1} = 2.5$

(iii) The delta impulse signal is located at $t = 2$, outside of the integration limits, and the integral is equal to 0.

(iv) The delta impulse signal is located exactly at the upper integration bound (need a factor of 0.5)

$$\int_{-5}^{5} e^{-5t}\cos(t-5)\delta(t-5)dt = \frac{1}{2}e^{-5t}\cos(t-5)_{|t=5} = \frac{1}{2}e^{-25}$$

#2b) The signal is given by

$$x(t) = \begin{cases} -t, & t < -2 \\ -t+1, & -2 < t < -1 \\ 2, & -1 < t < 2 \\ 1, & t > 2 \end{cases}$$

The generalized derivative is

$$\frac{Dx(t)}{Dt} = \begin{cases} -1, & t < -2 \\ \delta(t+1), & t = -2 \\ -1, & -2 < t < -1 \\ undefined, & t = -1 \\ 0, & -1 < t < 2 \\ -\delta(t+1), & t = 2 \\ 0, & t > 2 \end{cases}$$

#3a) Note that $T = 2,\ \omega_0 = 2\pi / T = \pi$

$$b_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega_0 t)dt = \frac{1}{2}\int_{-1}^{0} 1\sin(n\omega_0 t)dt + \frac{1}{2}\int_{0}^{1} 2\sin(n\omega_0 t)dt = -\frac{1}{2n\pi}(1-\cos(n\pi)) - \frac{1}{n\pi}(\cos(n\pi)-1)$$
$$= \frac{1}{2n\pi}(1-(-1)^n),$$

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)dt = \frac{1}{2}\int_{-1}^{0} 1\cos(n\omega_0 t)dt + \frac{1}{2}\int_{0}^{1} 2\cos(n\omega_0 t)dt = \frac{1}{2n\pi}(\sin(n\pi)) + \frac{1}{n\pi}(\sin(n\pi)) = 0, \quad n = 1,2,3,...$$
$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{2}{2}\left(\int_{-1}^{0} dt + \int_{0}^{1} 2dt\right) = 3$$

The Fourier series are given by

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{2n\pi}\sin(2\pi n t)$$

#3b)

$$H(j\omega) = \frac{1}{1+j\omega} = |H(j\omega)|\arg\{H(j\omega)\}, \quad |H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}, \quad \arg\{H(j\omega)\} = -\tan^{-1}(\omega)$$

$$X_n(j\omega) = 0.5(a_n - jb_n) = -j0.5b_n = -j\frac{1-(-1)^n}{4n\pi} = |X_n(j\omega)|\arg\{X_n(j\omega)\}, \quad n = 1,2,3,..., \quad X_0 = \frac{a_0}{2} = \frac{3}{2}$$

$$\arg\{X_n(jn\omega_0)\} = \begin{cases} -\frac{\pi}{2}, & n = 1,3,5,... \\ 0, & n = 0,2,4, \end{cases}, \quad |X_n(j\omega)| = \frac{1}{4n\pi}(1-(-1)^n), \quad n = 1,2,3,.., \quad |X_0| = \frac{3}{2}$$

The system output is periodic with the same period as the input signal and represented by the Fourier series with $Y(jn\omega_0) = H(jn\omega_0)X_n(jn\omega_0)$

$$|Y_n(jn\omega_0)| = |H(jn\omega_0)||X_n(jn\omega_0)| = \frac{1}{\sqrt{1+n^2\pi^2}}\frac{1-(-1)^n}{4n\pi}, \quad n = 1,2,..., \quad \omega_0 = \frac{2\pi}{T} = \pi, \quad Y_0 = |H(0)||X_0| = \frac{3}{2}$$

$$\arg\{Y(jn\omega_0\} = \arg\{H(jn\omega_0\} + \arg\{X_n(jn\omega_0\} = -\tan^{-1}(n\pi) + \arg\{X_n(jn\pi\} = \begin{cases} -\frac{\pi}{2}, & n = 1,3,5,... \\ 0, & n = 0,2,4,... \end{cases}$$

The output signal is given by

$$y(t) = Y_0 + 2\sum_{n=1}^{\infty}|Y_n(jn\omega_0)|\cos(n\omega_0 t + \arg\{Y_n(jn\omega_0)\})$$

#3c)

(i) $\quad \mathcal{F}\{te^{-2|t|}\} = j\frac{d}{d\omega}\{\mathcal{F}\{e^{-2|t|}\} = j\frac{d}{d\omega}\left(\frac{4}{4+\omega^2}\right) = -j\frac{8\omega}{(4+\omega^2)^2}$

(ii) $\quad e^{-2t}u_h(t) \leftrightarrow \frac{1}{2+j\omega} \quad \Rightarrow \quad e^{-2t}u_h(t)\cos(t) \leftrightarrow \frac{1}{2}\left(\frac{1}{2+j(\omega+1)} + \frac{1}{2+j(\omega-1)}\right)$

(iii) $\quad p_2(t) \leftrightarrow 2\,\mathrm{sinc}\left(\frac{\omega}{\pi}\right) \quad \Rightarrow \quad p_2(3t) \leftrightarrow \frac{2}{3}\mathrm{sinc}\left(\frac{\omega}{3\pi}\right) = X(j\omega)$

$$\int_{-\infty}^{t} p_2(3\tau)d\tau \leftrightarrow \frac{1}{j\omega}X(j\omega) + \pi\delta(\omega)X(0) = \frac{1}{j\omega}(\frac{2}{3}\mathrm{sinc}\left(\frac{\omega}{3\pi}\right) + \pi\delta(\omega)\frac{2}{3})$$

(iv) $\quad u_h(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega) \quad \Rightarrow \quad u_h(t-2) \leftrightarrow e^{-j2\omega}(\frac{1}{j\omega} + \pi\delta(\omega))$

#3d)

(i)

$$\frac{1}{1+j\omega}\cos(2\omega)e^{-j5\omega} = \frac{0.5}{1+j\omega}(e^{j2\omega} + e^{-j2\omega})e^{-j5\omega} = \frac{0.5}{1+j\omega}e^{-j3\omega} + \frac{0.5}{1+j\omega}e^{-j7\omega}$$

$$\leftrightarrow \frac{1}{2}(e^{-(t-3)}u_h(t-3) + e^{-7t}u_h(t-7))$$

(ii) Using duality we have

$$p_\tau(t) \leftrightarrow \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right) \quad \Rightarrow \quad \frac{2\pi}{\tau}p_\tau(\omega) \quad \Rightarrow \quad \tau\,\mathrm{sinc}\left(\frac{t\tau}{2\pi}\right) \leftrightarrow 2\pi p_\tau(-\omega) = 2\pi p_\tau(\omega)$$

$$\tau\,\mathrm{sinc}\left(\frac{t\tau}{2\pi}\right)_{|\tau=6} \leftrightarrow 2\pi p_\tau(-\omega)_{|\tau=6} \quad \Rightarrow \quad p_6(\omega) \leftrightarrow \frac{3}{\pi}\mathrm{sinc}\left(\frac{3t}{\pi}\right) \quad \Rightarrow \quad p_6(\omega-2) \leftrightarrow \frac{3}{\pi}\mathrm{sinc}\left(\frac{3t}{\pi}\right)e^{j2t}$$

Another way to solve the same problem uses the definition of the inverse Fourier transform

$$\mathcal{F}^{-1}\{p_6(\omega-2)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} p_6(\omega-2)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-1}^{5} 1e^{j\omega t}d\omega = \frac{1}{2j\pi t}(e^{j5t} - e^{-jt}) = \frac{e^{j2t}}{2j\pi t}(e^{j3t} - e^{-j3t})$$

$$= \frac{e^{j2t}}{\pi t}\sin(3t) = \frac{3}{\pi}\mathrm{sinc}\left(\frac{3t}{\pi}\right)e^{j2t}$$

Sample Exam 2: Solutions

#1) We use the time invariance principle and first find the solution to

$$\frac{dy(t)}{dt} + y(t) = \sin(t), \quad y(0) = 0, \quad t \geq 0$$

At the end we will shift the solution obtained by 3 time units. The homogeneous solution is obtained from

$$\frac{dy_h(t)}{dt} + y_h(t) = 0 \quad \Rightarrow \quad y_h(t) = Ce^{-t}$$

The particular solution satisfies

$$\frac{dy_p(t)}{dt} + y_p(t) = \sin(t) \quad \Rightarrow \quad y_p(t) = \alpha\sin(t) + \beta\cos(t)$$

Plugging this solution into the differential equation implies $\alpha = 0.5$ and $\beta = -0.5$ so that the particular solution is given by

$$y_p(t) = 0.5\sin(t) - 0.5\cos(t)$$

The system response is given by

$$y(t) = y_h(t) + y_p(t) = Ce^{-t} + 0.5\sin(t) - 0.5\cos(t)$$

Its initial condition produces

$$y(0) = 0 = y_h(0) + y_p(0) = C - 0.5 \quad \Rightarrow \quad C = 0.5 \quad \Rightarrow \quad y_h(t) = 0.5e^{-t}$$

Hence, the system response due to $\sin(t)$ is

$$y(t) = y_h(t) + y_p(t) = 0.5e^{-t} + 0.5\sin(t) - 0.5\cos(t)$$

The system response due to $\sin(t-3)$, by the time invariance, is given by

$$y(t) = y_h(t) + y_p(t) = (0.5e^{-(t-3)} + 0.5\sin(t-3) - 0.5\cos(t-3))u(t-3)$$

The system zero-state response satisfies

$$\frac{dy_{zs}(t)}{dt} + y_{zs}(t) = \sin(t-3), \quad y_{zs}(0) = 0, \quad t \geq 0$$

Using the previously obtained result, the zero-state response is given by

$$y_{zs}(t) = (0.5e^{-(t-3)} + 0.5\sin(t-3) - 0.5\cos(t-3))u(t-3)$$

The zero-input response satisfies

$$\frac{dy_{zi}(t)}{dt} + y_{zi}(t) = 0, \quad y_{zi}(0) = 0, \quad t \geq 0$$

The zero-input response is given by

$$y_{zi}(t) = Ce^{-t} = 0 \quad t \geq 0$$

The system response in terms of its zero-state and zero-input components, is given by

$$y(t) = y_{zs}(t) + y_{zi}(t) = (0.5e^{-(t-3)} + 0.5\sin(t-3) - 0.5\cos(t-3))u(t-3)$$

The steady state response is practically obtained for large values of time, that is

$$y_{ss}(t) = y(t), \quad for \quad t \quad large \quad \Rightarrow \quad y_{ss}(t) \approx 0.5\sin(t-3) - 0.5\cos(t-3)$$

According to the textbook definition of the transient response, we have

$$y_{tr}(t) = y(t), \quad for \quad t \quad small$$

Using the electrical circuit definition of the transient response we have

$$\bar{y}_{ss}(t) = 0.5(\sin(t-3) - 0.5\cos(t-3))u(t-3), \quad \Rightarrow \quad \bar{y}_{tr}(t) = y(t) - \bar{y}_{ss}(t) = 0.5e^{-(t-3)}u(t-3)$$

#2a)

(i) $$(t-1)^2\delta(t-1) = (1-1)\delta(t-1) = 0\delta(t-1) = 0$$

(ii) $$\int_{-\infty}^{+\infty}\cos(\pi t)\delta^{(3)}(t-1)dt = (-1)^3\frac{d^3}{dt^3}\{\cos(\pi t)\}_{|t=1} = (-1)^3\pi^3\sin(\pi) = 0$$

(iii) The delta impulse signal is located at $t = 2/3$ (within integration limits), hence

$$\int_{-\infty}^{2}\sin(\pi t)\delta(3t-2)dt = \frac{1}{3}\{\sin(\pi t)\}_{|t=2/3} = \frac{1}{3}\sin(\frac{2\pi}{3}) = \frac{\sqrt{3}}{6}$$

(iv) The delta impulse signal is located exactly at the upper integration bound (need a factor of 0.5)

$$\int_{-\infty}^{3}e^{-5t}\sin(3t)\delta(t-3)dt = \frac{1}{2}e^{-5t}\sin(3t)_{|t=3} = \frac{1}{2}e^{-15}\sin(9)$$

(v) The delta impulse signal is outside of integration limits (at $t=-5$), hence

$$\int_{-3}^{4}f(t)\delta^{(1)}(t+5)dt = \int_{-3}^{4}0dt = 0$$

#2b)

$$p_4(2t-5) = \begin{cases}1, & -2 \le 2t-5 \le 2\\ 0, & \text{otherwise}\end{cases} = \begin{cases}1, & 1.5 \le t \le 3.5\\ 0, & \text{otherwise}\end{cases} = p_2(t-2.5)$$

$$u(-4t+2) = \begin{cases}1, & -4t+2 \ge 0\\ 0, & \text{otherwise}\end{cases} = \begin{cases}1, & t < 0.5\\ 0, & \text{otherwise}\end{cases} = u(-t+0.5)$$

The direct method for finding the generalized derivative

$$\frac{Dx(t)}{Dt} = \frac{D}{Dt}(2p_2(t-2.5) - u(-t+0.5) + r(t-3)) = 2\delta(t-1.5) - 2\delta(t-3.5) - (-\delta(t-0.5) + u(t-3)$$

#3a) This is an odd signal so that $a_n = 0, n = 0,1,2,\ldots$. The coefficient b_n is obtained from (using the formula for integration given on the exam sheet)

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2}\frac{2E}{T}t\sin(n\omega_0 t)dt = 2\int_{-1/2}^{1/2}2t\sin(2\pi nt)dt = 8\int_{0}^{1/2}t\sin(2\pi nt)dt = \frac{2(-1)^{n+1}}{n\pi}$$

The Fourier series are given by

$$x(t) = \sum_{n=1}^{\infty}b_n\sin(n\omega_0 t) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin(2\pi nt)$$

#3b)

$$H(j\omega) = \frac{j\omega}{1+j\omega} = |H(j\omega)|\arg\{H(j\omega)\}, \quad |H(j\omega)| = \frac{\omega}{\sqrt{1+\omega^2}}, \quad \arg\{H(j\omega)\} = \frac{\pi}{2} - \tan^{-1}(\omega)$$

$$X_n(j\omega) = 0.5(a_n - jb_n) = -j0.5b_n = j\frac{(-1)^n}{n\pi} = \left|X_n(j\omega)\right| \arg\{X_n(j\omega)\}$$

$$\arg\{X_n(j\omega)\} = (-1)^n \frac{\pi}{2}, \quad \left|X_n(j\omega)\right| = \frac{1}{n\pi}$$

The system output is periodic with the same period as the input signal and represented by the Fourier series with $Y(jn\omega_0) = H(jn\omega_0)X_n(jn\omega_0)$

$$\left|Y_n(jn\omega_0)\right| = \left|H(jn\omega_0)\right|\left|X_n(jn\omega_0)\right| = \frac{n\omega_0}{\sqrt{1+n^2\omega_0^2}}\frac{1}{n\pi} = \frac{2}{\sqrt{1+4n^2\pi^2}}, \quad n = 1,2,\ldots, \quad \omega_0 = \frac{2\pi}{T} = 2\pi, \quad Y_0 = 0$$

$$\arg\{Y(jn\omega_0\} = \arg\{H(jn\omega_0\} + \arg\{X_n(jn\omega_0\} = \frac{\pi}{2} - \tan^{-1}(2n\pi) + (-1)^n \frac{\pi}{2} = \frac{\pi}{2}(1+(-1)^n)$$

The output signal is given by

$$y(t) = 2\sum_{n=1}^{\infty}\left|Y_n(jn\omega_0)\right|\cos(n\omega_0 t + \arg\{Y_n(jn\omega_0)\})$$

#3c)

$$(i) \quad F\{\sin(2\pi t)[u_h(t-2) - u_h(t-1)]\} = -F\{\sin(2\pi t)p_1(t-1.5)\} = -\frac{j}{2}\{F_p(j\omega+2\pi) - F_p(j\omega-2\pi)\}$$

$$F_p(j\omega) = F\{p_1(t-1.5)\} = e^{-j1.5\omega}F\{p_1(t)\} = e^{-j1.5\omega}\sin c(\frac{\omega}{2\pi})$$

$$(ii) \quad F\{t^2e^{-3t}u_h(t)\} = j^2\frac{d^2}{d\omega^2}\{F\{e^{-3t}u_h(t)\} = -\frac{d^2}{d\omega^2}\left(\frac{1}{3+j\omega}\right) = \frac{2}{(3+j\omega)^3}$$

$$(iii) \quad p_\tau(t) \leftrightarrow \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right) \quad \Rightarrow \quad \tau\,\mathrm{sinc}\left(\frac{t\tau}{2\pi}\right) \leftrightarrow 2\pi p_\tau(-\omega) \quad \Rightarrow \quad \mathrm{sinc}\left(\frac{t\tau}{2\pi}\right) \leftrightarrow \frac{2\pi}{\tau}p_\tau(\omega)$$

$$\Rightarrow \quad \mathrm{sinc}(t) \leftrightarrow p_\tau\left(\frac{\tau\omega}{2\pi}\right) = p_{2\pi}(\omega) = p_1(f) \quad \Rightarrow \quad \mathrm{sinc}(3t-4) \leftrightarrow \frac{e^{-j\frac{4}{3}\omega}}{3}p_{2\pi}\left(\frac{\tau\omega}{3}\right)$$

#3d)

$$\mathrm{sinc}\left(\frac{t\tau}{2\pi}\right) \leftrightarrow \frac{2\pi}{\tau}p_\tau(\omega) \quad \Rightarrow \quad p_\tau(\omega) \leftrightarrow \frac{\tau}{2\pi}\mathrm{sinc}\left(\frac{t\tau}{2\pi}\right) \quad \Rightarrow \quad p_2(\omega) \leftrightarrow \frac{1}{\pi}\mathrm{sinc}\left(\frac{t}{\pi}\right)$$

$$p_4(\omega) \leftrightarrow \frac{2}{\pi}\mathrm{sinc}\left(\frac{2t}{\pi}\right)$$

$$X(j\omega) = 2p_4(\omega) - p_2(\omega-1) \leftrightarrow \frac{4}{\pi}\mathrm{sinc}\left(\frac{2t}{\pi}\right) - e^{jt}\frac{1}{\pi}\mathrm{sinc}\left(\frac{t}{\pi}\right)$$

#3e)

$$y(t) = 5\left|H(j10)\right|\cos(10t + \frac{\pi}{3} + \arg\{H(j10)\}$$

$$H(j\omega) = \frac{j\omega}{1+j\omega} \quad \Rightarrow \quad \left|H(j10)\right| = \frac{10}{\sqrt{1+10^2}}, \qquad \arg\{H(j10)\} = \frac{\pi}{2} - \tan^{-1}(10)$$