4.3 Lyapunov Stability of Linear Systems

In this section we present the Lyapunov stability method specialized for the linear time invariant systems studied in this book. The method has more theoretical importance than practical value and can be used to derive and prove other stability results. Its final statement for linear time invariant systems is elegant and easily tested using MATLAB. However, it is computationally more involved than the other methods for examining the stability of linear systems. Its importance lies in its generality since it can be applied to all nonlinear and linear systems without taking into account whether or not these systems are time invariant or time varying. More about the general study of Lyapunov stability can be found in several books on nonlinear systems (see for example Khalil, 1992). Here, we study the Lyapunov stability theory for time invariant continuous and discrete linear systems only.

In 1892 the Russian mathematician Alexander Mikhailovitch Lyapunov introduced his famous stability theory for nonlinear and linear systems. A complete English translation of Lyapunov's doctoral dissertation was published in the *International Journal of Control* in March 1992. The stability definition given in Section 4.1, Definition 4.1, in fact corresponds to the Lyapunov stability definition, so that "stable" used in this book also means "stable in the sense of Lyapunov". According to Lyapunov, one can check stability of a system by finding some function $V(\mathbf{x})$, called the Lyapunov function, which for time invariant systems satisfies

$$V(\mathbf{x}) > 0, \qquad V(\mathbf{0}) = 0 \tag{4.26a}$$

$$\dot{V}(\mathbf{x}) = \frac{dV}{dt} = \frac{\partial V}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} \leq 0 \tag{4.26b}$$

There is no general procedure for finding the Lyapunov functions for nonlinear systems, but for linear time invariant systems, the procedure comes down to the problem of solving a linear algebraic equation, called the Lyapunov algebraic equation.

In view of (4.26a) and (4.26b), a linear time invariant system is *stable* if one is able to find a scalar function $V(\mathbf{x})$ such that when this function is associated with the system, both conditions given in (4.26) are satisfied. If the condition (4.26b) is a strict inequality, then the result is *asymptotic* stability. It can be shown that for a linear system (4.1) the Lyapunov function can be chosen to be quadratic, that is

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}, \qquad \mathbf{P} = \mathbf{P}^T > \mathbf{0} \tag{4.27}$$

which with the use of (4.1) leads to

$$\dot{V}(\mathbf{x}) = \mathbf{x}^T\left(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}\right)\mathbf{x}$$

i.e. the system is asymptotically stable if the following condition is satisfied

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} < \mathbf{0}$$

or, equivalently

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}, \qquad \mathbf{Q} = \mathbf{Q}^T > \mathbf{0} \tag{4.28}$$

where $\mathbf{Q}$ is any positive definite matrix. Recall that positive definite matrices have all eigenvalues in the closed right-hand half of the complex plane (see Appendix C). The matrix algebraic equation (4.28) is known as the Lyapunov algebraic equation. More about this important equation and its role in system stability and control can be found in Gajić and Qureshi (1995). Now we are able to formulate the Lyapunov stability theory for linear continuous time invariant systems.

Theorem 4.7 *The linear time invariant system (4.1) is asymptotically stable if and only if for any* $\mathbf{Q} = \mathbf{Q}^T > 0$ *there exists a unique* $\mathbf{P} = \mathbf{P}^T > 0$ *such that (4.28) is satisfied.*

Example 4.9: In this example we demonstrate the necessary steps required in applying the Lyapunov stability test. Consider the following continuous time invariant system represented by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}$$

It is easy to check by MATLAB function `eig` that the eigenvalues of this system are $\lambda = -2.3247,\ -0.3376 \pm j0.5623$, and hence this system is asymptotically stable. In order to apply the Lyapunov method, we first choose a positive definite matrix $\mathbf{Q}$. The standard "initial guess" for $\mathbf{Q}$ is identity, i.e. $\mathbf{Q} = \mathbf{I}_3$. With the help of the MATLAB function `lyap` (used for solving the algebraic Lyapunov equation), we can execute the following statement `P=lyap(A',Q)` and obtain the solution for $\mathbf{P}$ as

$$\mathbf{P} = \begin{bmatrix} 2.3 & 2.1 & 0.5 \\ 2.1 & 4.6 & 1.3 \\ 0.5 & 1.3 & 0.6 \end{bmatrix}$$

Note that we have used a transpose for the system matrix in the MATLAB function `lyap`, i.e. $\left(\mathbf{A}' = \mathbf{A}^T\right)$, since that function solves the equation that represents the transpose of the algebraic Lyapunov equation (4.28). Examining the positive definiteness of the matrix $\mathbf{P}$ (all eigenvalues of $\mathbf{P}$ must be in the closed right half plane), we get that the eigenvalues of this matrix are given by $6.1827,\ 1.1149,\ 0.2024$; hence $\mathbf{P}$ is positive definite and the Lyapunov test indicates that the system under consideration is stable.

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It can be seen from this particular example that the Lyapunov stability test is not numerically very efficient since we have first to solve the linear algebraic Lyapunov equation and then to test the positive definiteness of the matrix $\mathbf{P}$, which requires finding its eigenvalues. Of course, we can find the eigenvalue of the matrix $\mathbf{A}$ immediately and from that information determine the system stability. It is true that the Lyapunov stability test is not the right method to test the stability of linear systems when the system matrix is given by numerical entries. However, it can be used as a *useful concept in theoretical considerations*, e.g. to prove some other stability results. This will be demonstrated in Section 4.4 where we will give a very simple and elegant proof of the very well-known Routh–Hurwitz stability criterion.

Note that Theorem 4.7 can be generalized to include the case when the matrix $\mathbf{Q}$ is positive semidefinite, $\mathbf{Q} = \mathbf{C}^T\mathbf{C} \geq 0$. Recall that positive semidefinite matrices have eigenvalues in the open right half of the complex plane (see Appendix C). Another form of Theorem 4.7 can be formulated as follows (Chen, 1984).

Theorem 4.8 *The time invariant linear system (4.1) is asymptotically stable if and only if the pair* $(\mathbf{A}, \mathbf{C})$ *is observable*[4] *and the algebraic Lyapunov equation (4.28) has a unique positive definite solution.*

4 For a definition of observability, see Section 5.2.

The observability of the pair $(\mathbf{A}, \mathbf{C})$ can be relaxed to its detectability.[5] This is natural since the detectability implies the observability of the modes which are not asymptotically stable.

Example 4.10: Consider the same system matrix $\mathbf{A}$ as in Example 4.9 with the matrix $\mathbf{Q}$ obtained from

$$\mathbf{Q}_1 = \mathbf{C}^T\mathbf{C} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the pair $(\mathbf{A}, \mathbf{C})$ is observable since $\mathtt{rank}\{\mathcal{O}(\mathbf{A}, \mathbf{C})\} = 3$. The algebraic Lyapunov equation

$$\mathbf{A}^T\mathbf{P}_1 + \mathbf{P}_1\mathbf{A} = \mathbf{Q}_1$$

has the positive definite solution

$$\mathbf{P}_1 = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0 & 0.1 & 0.2 \end{bmatrix} > 0$$

which can be confirmed by finding the eigenvalues of $\mathbf{P}_1$, so that the considered linear system is asymptotically stable.

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Theorems corresponding to Theorems 4.7 and 4.8 can be stated for stability of discrete-time systems. For a linear discrete-time system (4.2) the Lyapunov function has a quadratic form, which, according to the Lyapunov stability theory, must satisfy (Ogata, 1987)

$$\begin{aligned} V(k) &= \mathbf{x}^T(k)\mathbf{P}\mathbf{x}(k) > 0 \\ \Delta V(k) &= V(k+1) - V(k) \le 0 \end{aligned} \tag{4.29}$$

Since

$$\begin{aligned} V(k+1) - V(k) &= \mathbf{x}^T(k+1)\mathbf{P}\mathbf{x}(k+1) - \mathbf{x}^T(k)\mathbf{P}\mathbf{x}(k) \\ &= \mathbf{x}^T(k)\big(\mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{P}\big)\mathbf{x}(k) \le 0 \end{aligned}$$

5 For a definition of detectability, see Section 5.5, Definition 5.2.

the stability requirement imposed in (4.29) leads (similarly to the continuous-time argument) to the discrete-time algebraic Lyapunov equation

$$\mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{P} = -\mathbf{Q} \tag{4.30}$$

which for asymptotic stability, according to the Lyapunov stability theory (dual result to Theorem 4.7), must have a unique positive definite solution for some positive definite matrix $\mathbf{Q}$. Thus, we have the following theorem.

Theorem 4.9 *The linear time invariant discrete system (4.2) is asymptotically stable if and only if for any $\mathbf{Q} = \mathbf{Q}^T > 0$ there exists a unique $\mathbf{P} = \mathbf{P}^T > 0$ such that (4.30) is satisfied.*

The relaxed form of this theorem, valid for a positive semidefinite matrix $\mathbf{Q}$, requires the observability of the pair $\left(\mathbf{A}, \sqrt{\mathbf{Q}}\right)$. Note that by the very well-known theorem from linear algebra every symmetric positive semidefinite matrix can be written in the form $\mathbf{Q} = \mathbf{C}^T\mathbf{C}$ with matrix $\mathbf{C}$ being known as the square root of $\mathbf{Q}$, i.e. $\mathbf{C} = \sqrt{\mathbf{Q}}$.

A theorem corresponding to Theorem 4.8 is given as follows.

Theorem 4.10 *The time invariant linear discrete system (4.2) is asymptotically stable if and only if the pair $(\mathbf{A}, \mathbf{C})$ is observable, $\mathbf{Q} = \mathbf{Q}^T \geq 0$, and the algebraic Lyapunov equation (4.30) has a unique positive definite solution.*

Example 4.11: Consider the following discrete-time linear system represented by

$$\mathbf{A} = \begin{bmatrix} 0.1 & 0.2 & -0.1 \\ 0.2 & 0.4 & 0 \\ 0 & -0.1 & 0.7 \end{bmatrix}$$

The eigenvalues of this matrix obtained by using MATLAB function `eig` are $0.0058,\ 0.4810,\ 0.7132$, which indicates that this system is asymptotically stable in the discrete-time domain. If we want to check the stability of this system by using the Lyapunov theory, we have to choose a positive definite matrix $\mathbf{Q}$, say $\mathbf{Q} = \mathbf{I}_3$, and to solve the discrete-time algebraic Lyapunov equation (4.30). Using the MATLAB function `dlyap` and the statement `P=dlyap(A',Q)`, we get the following solution for $\mathbf{P}$

$$\mathbf{P} = \begin{bmatrix} 1.0692 & 0.1437 & -0.0511 \\ 0.1437 & 1.3180 & -0.2424 \\ -0.0511 & -0.2424 & 1.9958 \end{bmatrix} > 0$$