

Homework #9 — Chapter 6 — Convolution

Problem 6.2

The convolution duration property implies that $p_2(t) * p_2(t) = 0$ for $t \leq -2 = -1 - 1$ and $t \geq 2 = 1 + 1$.

For $-2 \leq t \leq 0$, the convolution produces

$$p_2(t) * p_2(t) = \int_{-1}^{1+t} 1 d\tau = t + 2$$

For $0 \leq t \leq 2$, the convolution produces

$$p_2(t) * p_2(t) = \int_{-1+t}^1 1 d\tau = 2 - t$$

Problem 6.4

By the convolution duration property, we have that $f_1(t) * f_2(t) = 0$ for $t \leq 0 + 1 = 1$ and $t \geq 1 + 2 = 3$.

In the time interval $1 \leq t \leq 2$, the convolution is given by

$$f_1(t) * f_2(t) = \int_0^{t-1} \tau(-\tau - 1 + t) d\tau = -\frac{(t-1)^3}{3} + \frac{(t-1)^2}{2}(t-1) = \frac{1}{6}(t-1)^3$$

In the time interval $2 \leq t \leq 3$, the convolution is given by

$$f_1(t) * f_2(t) = \int_{t-1}^1 \tau(-\tau - 1 + t) d\tau = -\frac{1}{3} + \frac{1}{2}(t-1) + \frac{(t-2)^3}{3} - \frac{(t-2)^2}{2}(t-1)$$

Problem 6.5

By the convolution duration property, we have that the convolution is equal to zero for $t \leq 0 + 0 = 0$ and $t \geq 1 + 2 = 3$.

In the time interval $0 \leq t \leq 1$, the convolution is given by

$$f_1(t) * f_2(t) = \int_0^t (-\tau + 1) d\tau = -\frac{t^2}{2} + t$$

In the time interval $1 \leq t \leq 2$, the convolution is given by

$$f_1(t) * f_2(t) = \int_{t-1}^1 (-\tau + 1) d\tau + \int_1^t (\tau - 1) d\tau = t^2 - 3t + \frac{5}{2}$$

In the time interval $2 \leq t \leq 3$, the convolution is given by

$$f_1(t) * f_2(t) = \int_{t-1}^2 (\tau - 1) d\tau = \frac{1}{2}(-t^2 + 4t - 3)$$

The continuity test: at $t = 3$ the convolution is equal to zero. Also at $t = 2$ both expressions produce 0.5. Similarly, we can check that the convolution is continuous at $t = 0$ and $t = 1$.

Problem 6.6

By the convolution duration property, the convolution of these two signals is equal to zero for $t \leq 0 + 0 = 0$ and $t \geq 2 + 2 = 4$.

In the time interval $0 \leq t \leq 1$, the convolution is given by

$$f_1(t) * f_2(t) = \int_0^t \tau d\tau = \frac{t^2}{2}$$

In the time interval $1 \leq t \leq 2$, the convolution is given by

$$f_1(t) * f_2(t) = \int_0^1 \tau d\tau + \int_1^t (-\tau + 2) d\tau = -\frac{t^2}{2} + 2t - 1$$

In the time interval $2 \leq t \leq 3$, the convolution is given by

$$f_1(t) * f_2(t) = \int_{t-2}^1 \tau d\tau + \int_1^2 (-\tau + 2) d\tau = -\frac{t^2}{2} + 2t - 1$$

In the time interval $3 \leq t \leq 4$, the convolution is given by

$$f_1(t) * f_2(t) = \int_{t-2}^2 (-\tau + 2) d\tau = \frac{t^2}{2} - 4t + 8$$

Problem 6.9*

The system transfer function and its impulse response are given by

$$H(s) = \frac{1}{s^2 + 5s + 4} = \left(\frac{1}{3}\right)\frac{1}{s+1} - \left(\frac{1}{3}\right)\frac{1}{s+4} \quad \Rightarrow \quad h(t) = \left(\frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}\right)u(t)$$

The signal in Figure 4.13 can be represented in terms of elementary signals as

$$f(t) = u(t) - r(t) + r(t-2) + u(t-3)$$

so that we need to find only via the convolution integral the system step and ramp responses and then to use the linearity principle. The system step response is given by

$$y^{step}(t) = h(t) * u(t) = \int_0^t u(\tau) \left(\frac{1}{3}e^{-(t-\tau)} - \frac{1}{3}e^{-4(t-\tau)} \right) d\tau = \frac{1}{3}(1 - e^{-t}) - \frac{1}{12}(1 - e^{-4t}), \quad t \geq 0$$

The system ramp response, via the convolution procedure, is obtained from

$$y^{ramp}(t) = r(t) * u(t) = \int_0^t \tau \left(\frac{1}{3}e^{-(t-\tau)} - \frac{1}{3}e^{-4(t-\tau)} \right) d\tau = \frac{1}{3}(t-1) + \frac{1}{3}e^{-t} - \frac{1}{12}\left(t + \frac{1}{4}\right) + \frac{1}{48}e^{-4t}, \quad t \geq 0$$

The system zero-state response is given by

$$y_{zs}(t) = y^{step}(t) + y^{step}(t-3) - y^{ramp}(t) + y^{ramp}(t-2)$$

Problem 6.12

The signals $f_1[k]$ and $f_2[k]$ are given by

$$f_1[k] = \begin{cases} 2, & k = 1 \\ 4, & k = 2 \\ 0, & \text{otherwise} \end{cases}, \quad f_2[k] = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

By the convolution duration property, the convolution $f[k] = f_1[k] * f_2[k]$ is equal to zero for $k \leq 0$ and $k \geq 3$.

The sliding tape method produces $f[1] = 2 \times 1 = 2$ and $f[2] = 4 \times 1 = 4$.

In summary, the following convolution result is obtained

$$f[k] = f_1[k] * f_2[k] = \begin{cases} 2, & k = 1 \\ 4, & k = 2 \\ 0, & \text{otherwise} \end{cases}$$

Problem 6.14

The sliding tape method produces $f[k] = f_1[k] * f_2[k] = 0$ for $k \leq 0$ since the signals $f_1[m]$ and $f_2[-m]$ do not overlap. For other values of k , we obtain the following results: $f[1] = 2 \times 1 = 2$, $f[2] = 2 \times 0 + 1 \times 1 = 1$, $f[3] = 2 \times 1 + 1 \times 0 = 2$, $f[4] = 1 \times 1 = 1$, and $f[k] = 0$ for $k \geq 5$.

In summary, the following convolution result is obtained

$$f[k] = f_1[k] * f_2[k] = \begin{cases} 0, & k \leq 0 \\ 2, & k = 1 \\ 1, & k = 2 \\ 2, & k = 3 \\ 1, & k = 4 \\ 0, & k \geq 5 \end{cases}$$

Problem 6.15

Using the sliding tape method, the required convolution result is given by

$$f[k] = f_1[k] * f_2[k] = \begin{cases} 0, & k \leq -1 \\ 1 \times 3 = 3, & k = 0 \\ 1 \times 0 - 1 \times 3 = -3, & k = 1 \\ 1 \times (-2) + (-1) \times 0 + 1 \times 3 = 1, & k = 2 \\ (-1) \times (-2) + 1 \times 0 = 2, & k = 3 \\ 1 \times (-2) = -2, & k = 4 \\ 0, & k \geq 5 \end{cases}$$

Problem 6.16

The sliding tape method produces the following convolution result

$$f[k] = f_1[k] * f_2[k] = \begin{cases} 0, & k \leq -2 \\ (-2) \times 1 = -2, & k = -1 \\ (-2) \times 2 + 2 \times 1 = -2, & k = 0 \\ (-2) \times 3 + 2 \times 2 + 1 \times 1 = -1, & k = 1 \\ (-2) \times 2 + 2 \times 3 + 1 \times 2 + (-1) \times 1 = 3, & k = 2 \\ 2 \times 2 + 1 \times 3 + (-1) \times 2 + 4 \times 1 = 9, & k = 3 \\ 1 \times 2 + (-1) \times 3 + 4 \times 2 = 7, & k = 4 \\ (-1) \times 2 + 4 \times 3 = 10, & k = 5 \\ 2 \times 4 = 8, & k = 6 \\ 0, & k \geq 7 \end{cases}$$