

## Homework #8 — Chapter 5 — Z-Transform in System Analysis

### Problem 5.21

By shifting the original difference equation

$$y[k+2] + \frac{1}{2}y[k+1] + y[k] = 0, \quad y[0] = 1, \quad y[1] = 2$$

two steps backward in time, we obtain its integral formulation as

$$y[k] + \frac{1}{2}y[k-1] + y[k-2] = 0, \quad y[-1] = ?, \quad y[-2] = ?$$

The initial conditions for the integral formulation are obtained as follows. For  $k = 1$ , we have

$$y[1] + \frac{1}{2}y[0] + y[-1] = 0 \Rightarrow y[-1] = -y[1] - \frac{1}{2}y[0] = -2.5$$

For  $k = 0$ , we obtain the second initial condition

$$y[0] + \frac{1}{2}y[-1] + y[-2] = 0 \Rightarrow y[-2] = -y[0] - \frac{1}{2}y[-1] = 0.25$$

### Problem 5.23

By shifting the integral formulation

$$y[k] + \frac{1}{3}y[k-1] + \frac{1}{2}y[k-2] = f[k-1] + \frac{1}{2}f[k-2], \quad f[k] = u[k], \quad y[-2] = -2, \quad y[-1] = 0$$

two steps forward in time, we obtain the derivative formulation as

$$y[k+2] + \frac{1}{3}y[k+1] + \frac{1}{2}y[k] = f[k+1] + \frac{1}{2}f[k], \quad y[0] = ?, \quad y[1] = ?$$

The initial conditions for the derivative formulation are obtained as follows. For  $k = -2$ , we have

$$y[0] + \frac{1}{3}y[-1] + \frac{1}{2}y[-2] = u[-1] + \frac{1}{2}u[-2] = 0 \Rightarrow y[0] = -\frac{1}{3}y[-1] - \frac{1}{2}y[-2] = 0 + 1 = 1$$

For  $k = -1$ , we have

$$y[1] + \frac{1}{3}y[0] + \frac{1}{2}y[-1] = u[0] + \frac{1}{2}u[-1] = 1 \Rightarrow y[1] = 1 - \frac{1}{3}y[0] - \frac{1}{2}y[-1] = 1 - \frac{1}{3} - 0 = \frac{2}{3}$$

### Problem 5.24

The system poles are  $p_{1,2} = -1/6 \pm j\sqrt{17}/6$ . The required partial fraction expansion is

$$\frac{1}{z}H(z) = \frac{z + \frac{1}{2}}{z^2 + \frac{1}{3}z + \frac{1}{2}} = \frac{c_1}{z + \frac{1}{6} + j\frac{\sqrt{17}}{6}} + \frac{c_1^*}{z + \frac{1}{6} - j\frac{\sqrt{17}}{6}} + \frac{c_2}{z}$$

with

$$c_2 = \frac{z + \frac{1}{2}}{z^2 + \frac{1}{3}z + \frac{1}{2}}|_{z=0} = 1, \quad c_1 = \frac{z + \frac{1}{2}}{z(z + \frac{1}{6} - j\frac{\sqrt{17}}{6})}|_{z=-\frac{1}{6}-j\frac{\sqrt{17}}{6}} = \frac{6 - j3\sqrt{17}}{-17 + j\sqrt{17}} = -\frac{1}{2} + j\frac{45\sqrt{17}}{306}$$

The magnitudes and angles for the coefficient  $c_1$  and the corresponding pole  $p_1 = -1/6 - j\sqrt{17}/6$  are

$$|c_1| = 0.7859, \quad \angle c_1 = 129.5^\circ, \quad |p_1| = \frac{\sqrt{2}}{2}, \quad \angle p_1 = 256.4^\circ$$

so that the system impulse response,  $h[k] = \mathcal{Z}^{-1}\{H(z)\}$ , is given by

$$h[k] = \delta[k] + 2 \times 0.7859 \left( \frac{\sqrt{2}}{2} \right)^k \cos(k256.4^\circ + 129.5^\circ) u[k]$$

The system step response is obtained by finding the inverse  $\mathcal{Z}$  of the function

$$\frac{1}{z} H(z) \frac{z}{z-1} = H(z) \frac{1}{z-1} = \frac{c_1}{z + \frac{1}{6} + j\frac{\sqrt{17}}{6}} + \frac{c_1^*}{z + \frac{1}{6} - j\frac{\sqrt{17}}{6}} + \frac{c_2}{z-1}$$

The coefficients are given by

$$c_1 = -\frac{459}{1122} + j\frac{9\sqrt{17}}{1122} = -0.4091 + j0.0331 \Rightarrow |c_1| = 0.4104, \quad \angle c_1 = 175.4^\circ, \quad c_2 = \frac{9}{11}$$

The required step response is

$$y_{step}[k] = 2 \times 0.4104 \left( \frac{\sqrt{2}}{2} \right)^k \cos(k256.4^\circ + 175.4^\circ) u[k] + \frac{9}{11} u[k]$$

### Problem 5.25

(a)

$$\begin{aligned} H(z) &= \frac{z+4}{z^2+4z+4} = \frac{z+4}{(z+2)^2}, \quad h[k] = \mathcal{Z}^{-1}\{H(z)\} \\ \frac{1}{z} H(z) &= \frac{z+4}{z(z+2)^2} = \frac{c_1}{z} + \frac{c_2}{z+2} + \frac{c_3}{(z+2)^2} = \frac{1}{z} - \frac{1}{z+2} - \frac{1}{(z+2)^2} \\ h[k] &= \mathcal{Z}^{-1} \left\{ 1 - \frac{z}{z+2} - \frac{z}{(z+2)^2} \right\} = \delta[k] - (-2)^k u[k] + \frac{1}{2} k (-2)^k u[k] \end{aligned}$$

(b)

$$\begin{aligned} H(z) &= \frac{z+1}{z^2 + \frac{1}{6}z - \frac{1}{6}} = \frac{z+4}{(z+2)^2}, \quad h[k] = \mathcal{Z}^{-1}\{H(z)\} \\ \frac{1}{z} H(z) &= \frac{z+1}{z(z+\frac{1}{2})(z-\frac{1}{3})} = \frac{c_1}{z} + \frac{c_2}{z+\frac{1}{2}} + \frac{c_3}{z-\frac{1}{3}} = -\frac{6}{z} + \frac{1.2}{z+\frac{1}{2}} + \frac{4.8}{z-\frac{1}{3}} \\ h[k] &= \mathcal{Z}^{-1} \left\{ -6 + \frac{1.2z}{z+\frac{1}{2}} + \frac{4.8z}{z-\frac{1}{3}} \right\} = -6\delta[k] + 1.2 \left( -\frac{1}{2} \right)^k u[k] + 4.8k \left( \frac{1}{3} \right)^k u[k] \end{aligned}$$

### Problem 5.29

(a)

The response steady state *constant* value does not exist for the system defined by  $H_1(z)$  since it has a pole on the unit circle at  $-1$ . However, note that this system has the oscillatory steady state response (see the textbook pages 9–10 for the definition of the steady state response) given by

$$y_{1step}[k] = \left( \frac{5}{27}(-1)^k + \frac{15}{11} \right) u[k]$$

**(b)**

Since all poles of  $(z - 1)Y_{2step}(z)$  are inside the unit circle, the constant steady state response due to  $f[k] = 5u[k]$  is simply given by  $5H(1) = 5/1.5 = 10/3$ .

### Problem 5.33

$$\begin{aligned} \text{Complete system response : } & z^2Y(z) - z^2y[0] - zy[1] + 5zY(z) - 5zy[0] + 3Y(z) = zU(z) - zu[0] \\ (z^2 + 5z + 4)Y(z) &= z^2y[0] + zy[1] + 5zy[0] + z\frac{z}{z-1} - zu[0] = z^2 + 2z + 5z + \frac{z^2}{z-1} - z = \frac{z^3 + 6z^2 - 6z}{z-1} \\ \frac{1}{z}Y(z) &= \frac{z^2 + 6z - 6}{(z-1)(z+1)(z+4)} = \frac{c_1}{z-1} + \frac{c_2}{z+1} + \frac{c_3}{z+4} \Rightarrow Y(z) = \frac{1}{10}\frac{z}{(z-1)} + \frac{11}{6}\frac{z}{(z+1)} - \frac{14}{15}\frac{z}{(z+4)} \\ y[k] &= \mathcal{Z}^{-1}\{Y(z)\} = \frac{1}{10}u[k] + \frac{11}{6}(-1)^k u[k] - \frac{14}{15}(-4)^k u[k] \\ H(z) &= \frac{z}{z^2 + 5z + 4} \Rightarrow h[k] = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{3}\frac{z}{z+1} - \frac{1}{3}\frac{z}{z+4}\right\} = \frac{1}{3}(-1)^k u[k] - \frac{1}{3}(-4)^k u[k] \end{aligned}$$

### Problem 5.34

Applying the  $\mathcal{Z}$  transform to the difference equation  $y[k] + 5y[k-1] + 4y[k-2] = u[k-1]$ ,  $y[-2] = 1$ ,  $y[-1] = 2$ , we have

$$Y(z) + 5z^{-1}Y(z) + 5y[-1] + 4z^{-2}Y(z) + 4z^{-1}y[-1] + 4y[-2] = z^{-1}U(z)$$

which implies

$$Y(z) = \frac{-5y[-1] - 4z^{-1}y[-1] - 4y[-2]}{1 + 5z^{-1} + 4z^{-2}} + \frac{z^{-1}}{(1 + 5z^{-1} + 4z^{-2})(z-1)} \frac{z}{(z-1)} = \frac{-14z^2 - 8z}{z^2 + 5z + 4} + \frac{z}{(z^2 + 5z + 4)(z-1)} \frac{z}{(z-1)}$$

or

$$\frac{1}{z}Y(z) = \frac{-14z^2 + 7z + 8}{(z+1)(z+4)(z-1)} = \frac{13/6}{z+1} - \frac{244/15}{z+4} + \frac{1/10}{z-1}$$

The system response in time is given by

$$y[k] = \left( \frac{13}{6}(-1)^k - \frac{244}{15}(-4)^k + \frac{1}{10} \right) u[k]$$

Despite the fact that the system in this problem is the integral formulation of the system in Problem 5.33, represented by the derivative formulation, the responses are different since the initial conditions in Problem 5.34 are not obtained by propagating the initial conditions from Problem 5.33 through the difference equation. Instead, the initial conditions in Problem 5.34 are arbitrarily assigned.

### Problem 5.38

Taking the  $\mathcal{Z}$  transform of the both equations we have

$$\begin{aligned} Y_1(z) &= \frac{1}{z}Y_1(z) + y_1[-1] + \frac{1}{2z}Y_2(z) + \frac{1}{2}y_2[-1], \quad y_1[-1] = 2, \quad y_2[-1] = -1 \\ Y_2(z) &= -\frac{1}{3z}Y_1(z) - \frac{1}{3}y_1[-1] + \frac{1}{z}F(z), \quad F(z) = \frac{z}{z+1} \end{aligned}$$

which produces

$$\begin{aligned} \left(\frac{z-1}{z}\right)Y_1(z) &= \frac{1}{2z}Y_2(z) + \frac{3}{2} \\ Y_2(z) &= -\frac{1}{3z}Y_1(z) - \frac{2}{3} + \frac{1}{z+1} \end{aligned}$$

Replacing the expression for  $Y_2(z)$  from the second equation into the first equation leads

$$\left(\frac{z-1}{z}\right)Y_1(z) = \frac{1}{2z} \left( -\frac{1}{3z}Y_1(z) - \frac{2}{3} + \frac{1}{z+1} \right) + \frac{3}{2} \Rightarrow Y_1(z) = \frac{\frac{1}{2}z(3z^2 + \frac{7}{6}z + \frac{1}{3})}{(z^2 - z + \frac{1}{6})(z+1)}$$

Using the obtained expression for  $Y_1(z)$  in the second equation implies the expression for  $Y_2(z)$  as

$$Y_2(z) = \frac{z(-\frac{2}{3}z^2 + \frac{1}{2}z - \frac{5}{6})}{(z^2 - z + \frac{1}{6})(z+1)}$$

The corresponding partial fractions and the time domain results are given by

$$\begin{aligned} \frac{1}{z}Y_1(z) &= \frac{3/13}{z+1} + \frac{1.9559}{z-0.7887} - \frac{0.6866}{z-0.2113}, \quad 0.7887 = \frac{1}{2} + \frac{\sqrt{3}}{6}, \quad 0.2113 = \frac{1}{2} - \frac{\sqrt{3}}{6} \\ \Rightarrow y_1[k] &= \left( \frac{3}{13}(-1)^k + 1.9559(0.7887)^k - 0.6866(0.2113)^k \right) u[k] \\ \frac{1}{z}Y_2(z) &= -\frac{12/13}{z+1} - \frac{0.8266}{z-0.7887} + \frac{1.0831}{z-0.2113} \\ \Rightarrow y_2[k] &= \left( -\frac{12}{13}(-1)^k - 0.8266(0.7887)^k + 1.0831(0.2113)^k \right) u[k] \end{aligned}$$