

## Homework #7 — Chapter 5 — Z-Transform and Its Inverse

### Problem 5.4

The Euler formula gives

$$\cos(\omega kT) = \frac{1}{2}(e^{j\omega kT} + e^{-j\omega kT})$$

Using the result that  $(e^{j\omega T})^k u[k] \leftrightarrow z/(z - e^{j\omega T})$ , we have

$$\begin{aligned}\mathcal{Z}\{\sin(\omega kT)\} &= \mathcal{Z}\left\{\frac{1}{2j}(e^{j\omega kT} - e^{-j\omega kT})\right\} = \left(\frac{1}{2j}\right) \frac{z}{z - e^{j\omega T}} - \left(\frac{1}{2j}\right) \frac{z}{z - e^{-j\omega T}} \\ &= \left(\frac{1}{2j}\right) \frac{z(e^{j\omega T} - e^{-j\omega T})}{z^2 - (e^{j\omega T} + e^{-j\omega T})z + 1} = \frac{z \sin(\omega T)}{z^2 - 2 \cos(\omega T)z + 1}\end{aligned}$$

Similarly

$$\begin{aligned}\mathcal{Z}\{\cos(\omega kT)\} &= \mathcal{Z}\left\{\frac{1}{2}(e^{j\omega kT} + e^{-j\omega kT})\right\} = \left(\frac{1}{2}\right) \frac{z}{z - e^{j\omega T}} + \left(\frac{1}{2}\right) \frac{z}{z - e^{-j\omega T}} \\ &= \left(\frac{1}{2}\right) \frac{2z^2 - z(e^{j\omega T} + e^{-j\omega T})}{z^2 - (e^{j\omega T} + e^{-j\omega T})z + 1} = \frac{z^2 - z \cos(\omega T)}{z^2 - 2 \cos(\omega T)z + 1}\end{aligned}$$

### Problem 5.5

$$f[k] = 2^k u[k] \Rightarrow \mathcal{Z}\{f[k]\} = \frac{z}{z-2} = F(z)$$

$$\mathcal{Z}\{kf[k]\} = -z \frac{dF(z)}{dz} = -z \frac{d}{dz} \left\{ \frac{z}{z-2} \right\} = \frac{2z}{(z-2)^2} \quad \text{time multiplication}$$

$$\mathcal{Z}\{a^k f[k]\} = F\left(\frac{z}{a}\right) = \frac{\frac{z}{a}}{\frac{z}{a}-2} = \frac{z}{z-6} \quad \text{frequency scaling}$$

$$\mathcal{Z}\{f[k-2]u[k-2]\} = z^{-2}F(z) = \frac{z^{-2}z}{z-2} = \frac{z^{-1}}{z-2} = \frac{1}{z(z-2)} \quad \text{right time-shift}$$

$$\mathcal{Z}\{f[k+1]u[k]\} = \sum_{k=0}^{\infty} f[k+1]u[k]z^{-k} = \sum_{i=1}^{\infty} f[i]u[i-1]z^{-(i-1)}$$

$$= z \left( \sum_{i=1}^{\infty} f[i]z^{-i} + f[0] - f[0] \right) = zF(z) - zf[0] = z \frac{z}{z-2} - z2^0 = \frac{2z}{z-2}$$

### Problem 5.6

$$f[k] = ku[k] \Rightarrow \mathcal{Z}\{ku[k]\} = \frac{z}{(z-1)^2} = F(z)$$

$$\mathcal{Z}\{(k-1)f[k]\} = \mathcal{Z}\{kf[k]\} - \mathcal{Z}\{f[k]\} = -z \frac{dF(z)}{dz} - F(z) = -z \frac{d}{dz} \left\{ \frac{z}{(z-1)^2} \right\} - \frac{z}{(z-1)^2}$$

$$\begin{aligned}
&= \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2} = \frac{2z}{(z-1)^3} \\
\mathcal{Z}\{5^k f[k]\} &= F\left(\frac{z}{5}\right) = \frac{\frac{z}{5}}{\left(\frac{z}{5}-1\right)^2} = \frac{5z}{(z-5)^2} \\
\mathcal{Z}\{f[k-1]u[k-1]\} &= z^{-1}F(z) = \frac{z^{-1}z}{(z-1)^2} = \frac{1}{(z-1)^2} \\
5\mathcal{Z}\{f[k+3]u[k]\} &= 5 \sum_{k=0}^{\infty} f[k+3]u[k]z^{-k} = 5 \sum_{i=3}^{\infty} f[i]u[i-3]z^{-(i-3)} \\
&= 5z^3 \left( \sum_{i=3}^{+\infty} f[i]z^{-i} + f[0] + f[1]z^{-1} + f[2]z^{-2} - f[0] - f[1]z^{-1} - f[2]z^{-2} \right) \\
&= 5(z^3 F(z) - z^3 f[0] - z^2 f[1] - z f[2]) \\
&= 5 \left( z^3 \frac{z}{(z-1)^2} - 2z - z^2 - 0z^3 \right) = 5 \left( \frac{z^4}{(z-1)^2} - 2z - z^2 \right) = \frac{20z^2 - 10z}{(z-1)^2}
\end{aligned}$$

### Problem 5.7

$$f_1[k] = \begin{cases} 11 & k = 7 \\ -16 & k = 12 \\ 5 & k = 19 \\ 0 & \text{otherwise} \end{cases} \Rightarrow F_1(z) \triangleq \frac{11}{z^7} - \frac{16}{z^{12}} + \frac{5}{z^{19}}, \quad f_2[k] = \begin{cases} 7 & k = 8 \\ -9 & k = 12 \\ 0 & \text{otherwise} \end{cases} \Rightarrow F_2(z) \triangleq \frac{7}{z^8} - \frac{9}{z^{12}}$$

### Problem 5.8

(a)

$$\begin{aligned}
\mathcal{Z}\left\{2\delta[k-1] + 3^{k+1}ku[k] + \sin\left(k\frac{\pi}{2}\right)u[k-2]\right\} &= 2\mathcal{Z}\{\delta[k-1]\} + 3\mathcal{Z}\{3^kku[k]\} + \mathcal{Z}\left\{\sin\left((k-2+2)\frac{\pi}{2}\right)u[k-2]\right\} \\
&= 2z^{-1} + 3\frac{3z}{(z-3)^2} + \mathcal{Z}\left\{\left(\sin\left((k-2)\frac{\pi}{2}\right)\cos(\pi) + \cos\left((k-2)\frac{\pi}{2}\right)\sin(\pi)\right)u[k-2]\right\} \\
&= \frac{2}{z} + 3\frac{3z}{(z-3)^2} - \mathcal{Z}\left\{\sin\left((k-2)\frac{\pi}{2}\right)u[k-2]\right\} = \frac{2}{z} + \frac{9z}{(z-3)^2} - z^{-2} \frac{z\sin\left(\frac{\pi}{2}\right)}{z^2 - 2z\cos\left(\frac{\pi}{2}\right) + 1} = \frac{2}{z} + \frac{9z}{(z-3)^2} - \frac{z^{-1}}{z^2 + 1}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathcal{Z}\{5^k(u[k] - u[k-11])\} &= \mathcal{Z}\{5^k u[k]\} - \mathcal{Z}\{5^{k-11+11}u[k-11]\} \\
&= \frac{\frac{z}{5}}{\frac{z}{5}-1} - 5^{11}z^{-11}\mathcal{Z}\{5^k u[k]\} = \frac{z}{z-5} - 5^{11}z^{-11}\frac{z}{z-5} = \frac{z}{z-5}(1 - 5^{11}z^{-10})
\end{aligned}$$

(c)

$$\begin{aligned}
\mathcal{Z}\{ku[k-1]\} &= \mathcal{Z}\{(k-1+1)u[k-1]\} = \mathcal{Z}\{(k-1)u[k-1]\} - \mathcal{Z}\{u[k-1]\} = z^{-1}\frac{z}{(z-1)^2} - z^{-1}\frac{z}{z-1} \\
&= \frac{1}{(z-1)^2} - \frac{1}{z-1} = \frac{-z+2}{(z-1)^2}
\end{aligned}$$

$$-\frac{1}{k}(u[k-1] - u[k-3]) = \begin{cases} -1 & k = 1 \\ -0.5 & k = 2 \\ 0 & \text{otherwise} \end{cases} \Rightarrow \mathcal{Z}\left\{-\frac{1}{k}(u[k-1] - u[k-3])\right\} \triangleq -\frac{1}{z} - \frac{0.5}{z^2}$$

$$\begin{aligned}\mathcal{Z}\left\{e^{-k-2} \sin \left[k \frac{\pi}{2}\right] u[k]\right\} &= \frac{1}{e^2} \mathcal{Z}\left\{\left(\frac{1}{e}\right)^k \sin \left[k \frac{\pi}{2}\right] u[k]\right\} = \frac{1}{e^2} \frac{\frac{1}{e} z \sin \left(\frac{\pi}{2}\right)}{z^2 - 2 \frac{1}{e} \cos \left(\frac{\pi}{2}\right) + \frac{1}{e^2}} = \frac{z}{e^3 z^2 + e} \\ \mathcal{Z}\{5\delta[k-2]\} &= 5z^{-2}\end{aligned}$$

(d)

$$\begin{aligned}\mathcal{Z}\{(k-1)u[k] - ku[k-3]\} &= \mathcal{Z}\{ku[k]\} - \mathcal{Z}\{u[k]\} - \mathcal{Z}\{(k-3+3)u[k-3]\} \\ &= \frac{z}{(z-1)^2} - \frac{z}{z-1} - z^{-3} \mathcal{Z}\{ku[k]\} + 3z^{-3} \mathcal{Z}\{u[k]\} = \frac{z}{(z-1)^2} - \frac{z}{z-1} - z^{-3} \frac{z}{(z-1)^2} + 3z^{-3} \frac{z}{z-1}\end{aligned}$$

The  $\mathcal{Z}$ -transform of the last term in this Problem 5.8(d) is found in Problem 5.8(a).

### Problem 5.11

(a)

$$\begin{aligned}\frac{1}{z}F_1(z) &= \frac{z+1}{(z+2)(z-0.5)} = \frac{c_1}{z+2} + \frac{c_2}{z-0.5}, \quad c_1 = \frac{z+1}{z-0.5}|_{z=-2} = \frac{2}{5}, \quad c_2 = \frac{z+1}{z+2}|_{z=0.5} = \frac{3}{5} \\ f_1[k] &= \mathcal{Z}^{-1}\{F_1(z)\} = \mathcal{Z}^{-1}\left\{\frac{0.4z}{z+2} + \frac{0.6z}{z-0.5}\right\} = 0.4(-2)^k u[k] + 0.6(0.5)^k u[k]\end{aligned}$$

(b)

$$\begin{aligned}\frac{1}{z}F_2(z) &= \frac{6}{(z-1)^2(z+0.5)} = \frac{c_{11}}{z-1} + \frac{c_{12}}{(z-1)^2} + \frac{c_3}{z+0.5}, \quad c_3 = \frac{6}{(z-1)^2}|_{z=-0.5} = \frac{8}{3} \\ c_{12} &= \frac{6}{z+0.5}|_{z=1} = 4, \quad c_{11} = \frac{d}{dz}\left(\frac{6}{z+0.5}\right)|_{z=1} = \frac{-6}{(z+0.5)^2}|_{z=1} = -\frac{8}{3} \\ f_2[k] &= \mathcal{Z}^{-1}\{F_2(z)\} = \mathcal{Z}^{-1}\left\{-\left(\frac{8}{3}\right)\frac{z}{z-1} + \frac{4z}{(z-1)^2} + \left(\frac{8}{3}\right)\frac{z}{z+0.5}\right\} = \left(-\frac{8}{3} + 4k + \frac{8}{3}(-0.5)^k\right)u[k]\end{aligned}$$

(c)

$$\begin{aligned}\frac{1}{z}F_3(z) &= \frac{2}{z^2(z+1)(z-1)} = \frac{c_{11}}{z} + \frac{c_{12}}{z^2} + \frac{c_3}{z+1} + \frac{c_4}{z-1}, \quad c_4 = \frac{2}{z^2(z+1)}|_{z=1} = 1, \quad c_3 = \frac{2}{z^2(z-1)}|_{z=-1} = -1 \\ c_{12} &= \frac{2}{(z+1)(z-1)}|_{z=0} = -2, \quad c_{11} = \frac{d}{dz}\left(\frac{2}{(z+1)(z-1)}\right)|_{z=0} = \frac{-4z}{(z^2-1)^2}|_{z=0} = 0 \\ f_3[k] &= \mathcal{Z}^{-1}\{F_3(z)\} = \mathcal{Z}^{-1}\left\{\frac{0z}{z} + \frac{-2z}{z^2} + \frac{-z}{z+1} + \frac{z}{z-1}\right\} = -2\delta[k-1] - (-1)^k u[k] + u[k]\end{aligned}$$

(d)

$$\begin{aligned}\frac{1}{z}F_4(z) &= \frac{5z^2}{(z+0.5)^2(z-0.5)} = \frac{c_{11}}{z+0.5} + \frac{c_{12}}{(z+0.5)^2} + \frac{c_3}{z-0.5}, \quad c_3 = \frac{5z^2}{(z+0.5)^2}|_{z=0.5} = 1.25 \\ c_{12} &= \frac{5z^2}{z-0.5}|_{z=-0.5} = -1.25, \quad c_{11} = \frac{d}{dz}\left(\frac{5z^2}{z-0.5}\right)|_{z=-0.5} = \frac{10z(z-0.5)-5z^2}{(z-0.5)^2}|_{z=-0.5} = \frac{15}{4} \\ f_4[k] &= \mathcal{Z}^{-1}\{F_4(z)\} = \mathcal{Z}^{-1}\left\{\frac{\frac{15}{4}z}{z+\frac{1}{2}} + \frac{-\frac{5}{4}z}{(z+\frac{1}{2})^2} + \frac{\frac{5}{4}z}{z-\frac{1}{2}}\right\} = \frac{5}{4}\left\{3\left(-\frac{1}{2}\right)^k - 2k\left(-\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^k\right\}u[k]\end{aligned}$$

(e)

$$F_5(z) = \frac{5z}{z^2 - z + 1} = \frac{5}{\sin\left(\frac{\pi}{3}\right)} \frac{z \sin\left(\frac{\pi}{3}\right)}{z^2 - 2z \cos\left(\frac{\pi}{3}\right) + 1} = \frac{10}{\sqrt{3}} \mathcal{Z}\left\{\sin\left(k \frac{\pi}{3}\right) u[k]\right\} \leftrightarrow \frac{10}{\sqrt{3}} \sin\left(k \frac{\pi}{3}\right) u[k]$$

**Problem 5.12**

(a)

$$\begin{aligned}\frac{1}{z}F_1(z) &= \frac{1}{(z^2 + z + 1)(z + \frac{1}{3})} = \frac{c_1}{z - p_1} + \frac{c_1^*}{z - p_1^*} + \frac{9/7}{z + \frac{1}{3}} \\ p_1 &= -\frac{1}{2} - j\frac{\sqrt{3}}{2} \Rightarrow |p_1| = 1, \angle p_1 = \tan^{-1}(\sqrt{3}) = 240^\circ \\ c_1 &= -\frac{9}{14} - j\frac{\sqrt{3}}{14} \Rightarrow |c_1| = \frac{\sqrt{84}}{14}, \angle c_1 = \tan^{-1}\left(\frac{\sqrt{3}}{9}\right) = 190.9^\circ \\ f_1[k] &= \frac{9}{7} \left(-\frac{1}{3}\right)^k u[k] + \frac{2}{14} \sqrt{84} \cos(k240^\circ + 190.9^\circ) u[k]\end{aligned}$$

(b)

$$\begin{aligned}\frac{1}{z}F_2(z) &= \frac{z}{(z^2 + \frac{1}{4})(z^2 + \frac{1}{9})} = \frac{c_1}{z - j\frac{1}{2}} + \frac{c_1^*}{z + j\frac{1}{2}} + \frac{c_3}{z - j\frac{1}{3}} + \frac{c_3^*}{z + j\frac{1}{3}} \\ p_1 &= j\frac{1}{2} \Rightarrow |p_1| = \frac{1}{2}, \angle p_1 = 90^\circ, p_3 = j\frac{1}{3} \Rightarrow |p_3| = \frac{1}{3}, \angle p_3 = 90^\circ \\ c_1 &= -\frac{18}{5} \Rightarrow |c_1| = \frac{18}{5}, \angle c_1 = 180^\circ, c_3 = \frac{18}{5} \Rightarrow |c_3| = \frac{18}{5}, \angle c_3 = 0^\circ \\ f_1[k] &= \frac{36}{5} \left(\frac{1}{2}\right)^k \cos(k90^\circ + 180^\circ) u[k] + \frac{36}{5} \left(\frac{1}{3}\right)^k \cos(k90^\circ) u[k]\end{aligned}$$

(c)

$$\begin{aligned}\frac{1}{z}F_3(z) &= \frac{1}{z(z^2 + 2z + 2)} = \frac{c_1}{z - p_1} + \frac{c_1^*}{z - p_1^*} + \frac{1/2}{z} \\ p_1 &= -1 - j1 \Rightarrow |p_1| = \sqrt{2}, \angle p_1 = \tan^{-1}(1) = 225^\circ \\ c_1 &= -\frac{1}{4} - j\frac{1}{4} \Rightarrow |c_1| = \frac{1}{4}\sqrt{2}, \angle c_1 = \tan^{-1}(1) = 225^\circ \\ f_3[k] &= \frac{1}{2}\delta[k] + \frac{\sqrt{2}}{2} \left(\sqrt{2}\right)^k \cos(k225^\circ + 225^\circ) u[k]\end{aligned}$$

(d)

$$\begin{aligned}\frac{1}{z}F_4(z) &= \frac{z + \frac{1}{2}}{(z^2 - z + \frac{1}{2})(z - 1)} = \frac{c_1}{z - p_1} + \frac{c_1^*}{z - p_1^*} + \frac{3}{z - 1} \\ p_1 &= \frac{1}{2} + j\frac{1}{2} \Rightarrow |p_1| = \frac{\sqrt{2}}{2}, \angle p_1 = \tan^{-1}(1) = 45^\circ \\ c_1 &= -\frac{3}{2} + j\frac{1}{2} \Rightarrow |c_1| = \frac{1}{2}\sqrt{10}, \angle c_1 = \tan^{-1}\left(-\frac{1}{3}\right) = 161.6^\circ \\ f_4[k] &= 3u[k] + \sqrt{10} \left(\frac{\sqrt{2}}{2}\right)^k \cos(k45^\circ + 161.6^\circ) u[k]\end{aligned}$$

**Problem 5.13**

(a)

$$\begin{aligned}\frac{2z - 1}{z^2 + 1} &= \frac{2z}{z^2 + 1} - \frac{1}{z^2 + 1} = 2z^{-1} \frac{z^2}{z^2 + 1} - z^{-2} \frac{z^2}{z^2 + 1}, \text{ from the Table } \frac{z^2}{z^2 + 1} \leftrightarrow \cos\left(k\frac{\pi}{2}\right) \\ &\Rightarrow 2 \cos\left((k-1)\frac{\pi}{2}\right) u[k-1] - \cos\left((k-2)\frac{\pi}{2}\right) u[k-2]\end{aligned}$$

(b)

$$\begin{aligned}\frac{1}{z}F(z) &= \frac{22-5z}{(z-2)^2(z+1)} = \frac{c_{11}}{z-2} + \frac{c_{12}}{(z-2)^2} + \frac{c_3}{z+1}, \quad c_3 = \frac{22-5z}{(z-2)^2}|_{z=-1} = 3 \\ c_{12} &= \frac{22-5z}{z+1}|_{z=2} = 4, \quad c_{11} = \frac{d}{dz}\left(\frac{22-5z}{z+1}\right)|_{z=2} = \frac{-27}{(z+1)^2}|_{z=1} = -3 \\ f[k] &= \mathcal{Z}^{-1}\{F(z)\} = \mathcal{Z}^{-1}\left\{\frac{-3z}{z-2} + \frac{4z}{(z-2)^2} + \frac{3z}{z+1}\right\} = \left(-3(2)^k + 2k(2)^k + 3(-1)^k\right)u[k]\end{aligned}$$

(c)

$$a^k \cos\left(k\frac{\pi}{2}\right)u[k] = \frac{z^2}{z^2+a^2} \Rightarrow \frac{1}{z^2+4} = z^{-2}\frac{z^2}{z^2+2^2} \leftrightarrow 2^{k-2} \cos\left((k-2)\frac{\pi}{2}\right)u[k-2]$$

(d)

$$\begin{aligned}\frac{z}{16z^2+1} &= \frac{z^{-1}}{16}\frac{z^2}{z^2+(\frac{1}{4})^2} \leftrightarrow \frac{1}{16}\left(\frac{1}{16}\right)^{k-1}u[k-1]\cos\left((k-1)\frac{\pi}{2}\right) \text{ see Problem 5.13(c)} \\ \frac{z^2-1}{z^2+1} &= \frac{z^2}{z^2+1} - z^{-2}\frac{z^2}{z^2+1} \leftrightarrow u[k]\cos\left(k\frac{\pi}{2}\right) - u[k-2]\cos\left((k-2)\frac{\pi}{2}\right)\end{aligned}$$

(e) Using the fact that  $z^2/(z^2+4) \leftrightarrow 2^k \cos(k\pi/2)$ , we have

$$\frac{2z}{z^2+4} = 2z^{-1}\frac{z^2}{z^2+2^2} \leftrightarrow 22^{k-1} \cos\left((k-1)\frac{\pi}{2}\right)u[k]$$

(f)

$$\frac{2}{z} + \frac{5}{z^3} - \frac{3}{z^7} \leftrightarrow 2\delta[k-1] + 5\delta[k-3] - 3\delta[k-7] = f[k] = \begin{cases} 2, & k=1 \\ 5, & k=3 \\ -3, & k=7 \\ 0 & \text{otherwise} \end{cases}$$

#### Problem 5.14

(a)

$$f_1[0] = \lim_{z \rightarrow \infty} \{F_1(z)\} = \lim_{z \rightarrow \infty} \left\{ \frac{z(z+1)}{(z+2)(z-0.5)} \right\} = 1$$

(b)

$$f_2[0] = \lim_{z \rightarrow \infty} \{F_2(z)\} = \lim_{z \rightarrow \infty} \left\{ \frac{6z}{(z-1)^2(z+0.5)} \right\} = 0$$

(c)

$$f_3[0] = \lim_{z \rightarrow \infty} \{F_3(z)\} = \lim_{z \rightarrow \infty} \left\{ \frac{2}{z(z+1)(z-1)} \right\} = 0$$

(d)

$$f_4[0] = \lim_{z \rightarrow \infty} \{F_4(z)\} = \lim_{z \rightarrow \infty} \left\{ \frac{5z^3}{(z+1)^2(z-0.5)} \right\} = 5$$

(e)

$$f_5[0] = \lim_{z \rightarrow \infty} \{F_5(z)\} = \lim_{z \rightarrow \infty} \left\{ \frac{5z}{z^2-z+1} \right\} = 0$$

**Problem 5.15**

(a) Since the function  $(z - 1)F_1(z) = \frac{(z-1)z(z+1)}{(z+2)(z-0.5)}$  has a pole outside of the unit circle ( $p = 2$ ) the final value theorem is not applicable.

(b) The function  $(z - 1)F_2(z) = \frac{(z-1)6z}{(z-1)^2(z+0.5)} = \frac{6z}{(z-1)(z+0.5)}$  has a pole on the unit circle ( $p = 1$ ), hence the final value theorem is not applicable.

(c) The function  $(z - 1)F_3(z) = \frac{(z-1)^2}{z(z+1)(z-1)} = \frac{2}{z(z+1)}$  has a pole on the unit circle ( $p = -1$ ), hence the final value theorem is not applicable.

(d) The function  $(z - 1)F_4(z)$  has all poles inside the unit circle so that the final value theorem is applicable

$$f_4[\infty] = \lim_{z \rightarrow 1} \left( \frac{z-1}{z} F(z) \right) = \lim_{z \rightarrow 1} \left( \frac{z-1}{z} \frac{5z^3}{(z+0.5)^2(z-0.5)} \right) = 0$$

(e) The function  $(z - 1)F_5(z) = (z - 1) \frac{5z}{z^2 - z + 1}$  has the poles  $z_{1,2} = 0.5(1 \pm j\sqrt{3}) = \cos(\frac{\pi}{3}) \pm j \sin(\frac{\pi}{3})$ , hence the final value theorem is not applicable.