

Homework #6 — Laplace Transform in Systems Analysis — Chapter 4

Problem 4.25

(a)

$$h_1(t) = \mathcal{L}^{-1}\{H_1(s)\} = \mathcal{L}^{-1}\left\{\frac{s+3}{s(s+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}\right\} = (k_1 + k_2 e^{-t} + k_3 e^{-2t}) u(t)$$

$$k_1 = \frac{s+3}{(s+1)(s+2)}|_{s=0} = \frac{3}{2}, \quad k_2 = \frac{s+3}{s(s+2)}|_{s=-1} = -2, \quad k_3 = \frac{s+3}{s(s+1)}|_{s=-2} = \frac{1}{2}$$

$$h_1(t) = \left(\frac{3}{2} - 2e^{-t} + \frac{1}{2}e^{-2t}\right) u(t)$$

(b)

$$h_2(t) = \mathcal{L}^{-1}\{H_2(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{k_1}{s+j} + \frac{k_1^*}{s-j} + \frac{k_3}{s+2}\right\}$$

$$= [2|k_1|e^{\alpha t} \cos(\beta t + \angle k_1) + k_3 e^{-2t}] u(t), \quad k_3 = \frac{s}{s^2+1}|_{s=-2} = -\frac{2}{5}$$

$$k_1 = \frac{s}{(s-j)(s+2)}|_{s=-j} = \frac{-j}{(-2j)(2-j)} = \frac{1}{4-j2} \times \frac{4+j2}{4+j2} = \frac{4+j2}{16+4} = \frac{1}{5} + j\frac{1}{10}$$

$$|k_1| = \sqrt{\frac{1}{25} + \frac{1}{100}} = \frac{\sqrt{5}}{10}, \quad \angle k_1 = \tan^{-1}\left(\frac{1}{2}\right) = 26.57^\circ, \quad p_1 = -j = \alpha + j\beta \Rightarrow \alpha = 0, \beta = -1$$

$$h_2(t) = \left[\frac{\sqrt{5}}{5} \cos(-t + 26.57^\circ) - \frac{2}{5}e^{-2t}\right] u(t)$$

(c)

$$h_3(t) = \mathcal{L}^{-1}\{H_3(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{k_{11}}{s} + \frac{k_{12}}{s^2} + \frac{k_3}{s+5}\right\} = (k_{11} + k_{12}t + k_3 e^{-5t}) u(t)$$

$$k_{12} = \frac{1}{s+5}|_{s=0} = \frac{1}{5}, \quad k_{11} = \frac{d}{ds}\left\{\frac{1}{s+5}\right\}|_{s=0} = \frac{-1}{(s+5)^2}|_{s=0} = -\frac{1}{25}, \quad k_3 = \frac{1}{s^2}|_{s=-5} = \frac{1}{25}$$

$$h_3(t) = \left(-\frac{1}{25} + \frac{1}{5}t + \frac{1}{25}e^{-5t}\right) u(t)$$

(d)

$$h_4(t) = \mathcal{L}^{-1}\{H_4(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{(s+1)(s^2+s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{k_1}{s+\frac{1}{2}+j\frac{\sqrt{3}}{2}} + \frac{k_1^*}{s+\frac{1}{2}-j\frac{\sqrt{3}}{2}} + \frac{k_3}{s+1}\right\}$$

$$h_4(t) = [2|k_1|e^{\alpha t} \cos(\beta t + \angle k_1) + k_3 e^{-t}] u(t), \quad k_3 = \frac{2}{s^2+s+1}|_{s=-1} = 2$$

$$k_1 = \frac{2}{(s+1)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}|_{s=-\frac{1}{2}-j\frac{\sqrt{3}}{2}} = \frac{2}{\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)(-j\sqrt{3})} = -\frac{4}{3+j\sqrt{3}} \times \frac{3-j\sqrt{3}}{3-j\sqrt{3}} = \frac{-12+j4\sqrt{3}}{12}$$

$$= -1 + j\frac{\sqrt{3}}{3} = k_1 \Rightarrow |k_1| = \frac{2}{\sqrt{3}}, \quad \angle k_1 = 150^\circ, \quad p_1 = -\frac{1}{2} - j\frac{\sqrt{3}}{2} = \alpha + j\beta \Rightarrow \alpha = -\frac{1}{2}, \beta = -\frac{\sqrt{3}}{2}$$

$$h_4(t) = \left[\frac{4}{\sqrt{3}}e^{-0.5t} \cos\left(-\frac{\sqrt{3}}{2}t + 150^\circ\right) + 2e^{-t}\right] u(t)$$

Problem 4.26

(a)

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 3s + 1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s + \frac{1}{2})^2 + \frac{3}{4}}\right\} = \frac{2}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{(s + \frac{1}{2})^2 + \frac{3}{4}}\right\} = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)u(t)$$

(b)

$$h(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2 + 3s + 2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1} - \frac{2}{s+2}\right\} = (3e^{-t} - 2e^{-2t})u(t)$$

(c)

$$h(t) = \mathcal{L}^{-1}\left\{\frac{2s^2 + s + 2}{s^3 + 3s^2 + 3s + 1}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{3}{(s+1)^2} + \frac{3}{(s+1)^3}\right\} = \left(4e^{-t} - 3te^{-t} + \frac{3}{2}t^2e^{-t}\right)u(t)$$

Problem 4.33

(a)

$$\begin{aligned} \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y &= e^{-3t}u(t), \quad y(0^-) = 1, \quad y^{(1)}(0^-) = 0 \\ s^2Y(s) - sy(0^-) - y^{(1)}(0^-) + 2sY(s) - 2y(0^-) + Y(s) &= \frac{1}{s+3} \\ Y(s) &= \frac{1}{s^2 + 2s + 1} \left(s + 0 + 2 + \frac{1}{s+3} \right) = \frac{s^2 + 5s + 7}{(s+1)^2(s+3)} = \frac{k_{11}}{s+1} + \frac{k_{12}}{(s+1)^2} + \frac{k_3}{s+3} \\ y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{0.75}{s+1} + \frac{1.5}{(s+1)^2} + \frac{0.25}{s+3}\right\} = (0.75e^{-t} + 1.5te^{-t} + 0.25e^{-3t})u(t) \end{aligned}$$

(b)

$$\begin{aligned} \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y &= \sin(3t)u(t), \quad y(0^-) = 1, \quad y^{(1)}(0^-) = 2 \\ s^2Y(s) - sy(0^-) - y^{(1)}(0^-) + 2sY(s) - 2y(0^-) + Y(s) &= \frac{3}{s^2 + 9} \\ Y(s) &= \frac{1}{s^2 + 2s + 1} \left(s + 2 + 2 + \frac{3}{s^2 + 9} \right) = \frac{s^3 + 4s^2 + 9s + 39}{(s+1)^2(s^2 + 9)} = \frac{k_{11}}{s+1} + \frac{k_{12}}{(s+1)^2} + \frac{k_3}{(s+j3)} + \frac{k_3^*}{(s-j3)} \\ y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1.06}{s+1} + \frac{3.3}{(s+1)^2} + \frac{-0.03 - j0.04}{s+j3} + \frac{-0.03 + j0.04}{s-j3}\right\} \\ &= [1.06e^{-t} + 3.3te^{-t} + 2|k_1| \cos(-3t + \angle k_1)]u(t), \quad |k_1| = 0.05, \quad \angle k_1 = 233.13^\circ \\ k_1 &= \frac{s^3 + 4s^2 + 9s + 39}{(s+1)^2(s-j3)}|_{s=-j3} = \frac{-j^3 27 + j^2 36 - j 27 + 39}{(-j3+1)^2(-2j3)} = \frac{1}{-12 + 16j} = -0.03 - j0.04 \end{aligned}$$

Problem 4.39

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = e^{-t}u(t-1), \quad y(0^-) = 1, \quad y^{(1)}(0^-) = 1$$

Applying the Laplace transform, we have

$$s^2Y(s) - sy(0^-) - y^{(1)}(0^-) + 2sY(s) - 2y(0^-) + Y(s) = \mathcal{L}\left\{e^{-1}e^{-(t-1)}u(t-1)\right\} = \frac{e^{-1}}{s+1}e^{-s}$$

Replacing the values for the initial conditions and solving with respect to $Y(s)$, we obtain

$$Y(s) = \frac{s+3}{(s+1)^2} + \frac{e^{-1}}{(s+1)^3}e^{-s}$$

Using the Laplace inverse, we obtain the complete system response

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s+3}{(s+1)^2} + \frac{e^{-1}}{(s+1)^3}e^{-s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{2}{(s+1)^2}\right\} + \frac{e^{-1}}{2}(t-1)^2e^{-(t-1)}u(t-1) \\ y(t) &= y_{zi}(t) + y_{zs}(t) = (e^{-t} + 2te^{-t})u(t) + \frac{e^{-1}}{2}(t-1)^2e^{-(t-1)}u(t-1) \end{aligned}$$

Problem 4.40

Using the linearity principle we have

$$\begin{aligned} f(t) &= u(t) - r(t) + r(t-2) + u(t-3) \Rightarrow y(t) = y_{step}(t) - y_{ramp}(t) + y_{ramp}(t-2) + y_{step}(t-3) \\ h(t) &= \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = te^{-2t} \Rightarrow y_{step}(t) = \int_0^t h(\tau)d\tau = \int_0^t \tau e^{-2\tau}d\tau = \frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t} \\ y_{ramp}(t) &= \int_0^t y_{step}(\tau)d\tau = \int_0^t \left(\frac{1}{4} - \frac{1}{4}e^{-2\tau} - \frac{1}{2}\tau e^{-2\tau}\right)d\tau = \frac{1}{4}(-1 + t + e^{-2t} + te^{-2t}), \quad t \geq 0 \end{aligned}$$