

Homework #5 — Laplace Transform and Its Inverse — Chapter 4

Problem 4.7*

(a)

$$\begin{aligned}\mathcal{L}\{f_1(t)\} &= 5\mathcal{L}\{\delta(t-3)\} + 4\mathcal{L}\{u(t-2)\} + 3\mathcal{L}\left\{e^{-5(t-3)}u(t-4)\right\} \\ &= 5e^{-3s} + 4\frac{1}{s}e^{-2s} + 3\mathcal{L}\left\{e^{-5(t-3-1+1)}u(t-4)\right\} = 5e^{-3s} + 4\frac{1}{s}e^{-2s} + 3e^{-5}\frac{1}{s+5}e^{-4s}\end{aligned}$$

(b)

$$\begin{aligned}\mathcal{L}\{f_2(t)\} &= \mathcal{L}\left\{\frac{d}{dt}[t^{10}e^{-t}u(t)]\right\} + \mathcal{L}\left\{\int_{0^-}^t e^{-5\tau} \sin(5\tau)d\tau\right\} \\ &= s\mathcal{L}\{t^{10}e^{-t}u(t)\} - t^{10}e^{-t}u(t)|_{t=0^-} + \frac{1}{s}\mathcal{L}\{e^{-5t}\sin(5t)u(t)\} = s\frac{10!}{(s+1)^{11}} + \frac{1}{s}\left(\frac{5}{(s+5)^2+5^2}\right)\end{aligned}$$

(c)

$$\begin{aligned}\mathcal{L}\{f_3(t)\} &= \mathcal{L}\{(t+1)e^{-t}u(t-1)\} + \mathcal{L}\{e^{-t}\sin(2t)u(t)\} \\ \mathcal{L}\{e^{-t}\sin(2t)u(t)\} &= \frac{2}{(s+1)+2^2} \\ \mathcal{L}\{(t+1)e^{-t}u(t-1)\} &= \mathcal{L}\{(t-1+2)e^{-1}e^{-(t-1)}u(t-1)\} = \mathcal{L}\{e^{-1}(t-1)e^{-(t-1)}u(t-1)\} \\ + \mathcal{L}\{2e^{-1}e^{-(t-1)}u(t-1)\} &= e^{-1}e^{-s}\mathcal{L}\{te^{-t}u(t)\} + 2e^{-1}e^{-s}\mathcal{L}\{e^{-t}u(t)\} = \frac{e^{-(s+1)}}{(s+1)^2} + 2\frac{e^{-(s+1)}}{s+1}\end{aligned}$$

(d)

$$\begin{aligned}\mathcal{L}\{\mathbf{f}_6(t)\} &= \mathcal{L}\{\sin(t)[u(t) - u(t-1)]\} = \mathcal{L}\{\sin(t)u(t)\} - \mathcal{L}\{\sin(t)u(t-1)\} = \frac{1}{s^2+1} \\ - \mathcal{L}\{\sin(t-1+1)u(t-1)\} &= \frac{1}{s^2+1} - \mathcal{L}\{\cos(1)\sin(t-1)u(t-1) + \sin(1)\cos(t-1)u(t-1)\} \\ &= \frac{1}{s^2+1} - \cos(1)e^{-s}\frac{1}{s^2+1} - \sin(1)e^{-s}\frac{s}{s^2+1}\end{aligned}$$

Problem 4.8

(a)

$$\mathcal{L}\{f_1(t)\} = \mathcal{L}\left\{(t-3)^3u(t-3)\right\} = e^{-3s}\mathcal{L}\{t^3u(t)\} = e^{-3s}\frac{3!}{s^4}$$

(b)

$$\begin{aligned}\mathcal{L}\{f_2(t)\} &= \mathcal{L}\left\{(t-2)^2e^{-5t}u(t-1)\right\} = \mathcal{L}\left\{(t-1-1)^2e^{-5(t-1+1)}u(t-1)\right\} \\ &= \mathcal{L}\left\{\left[(t-1)^2 - 2(t-1) + 1\right]e^{-5}e^{-5(t-1)}u(t-1)\right\} = e^{-5}e^{-s}\mathcal{L}\{t^2e^{-5t}u(t)\} \\ - 2e^{-5}e^{-s}\mathcal{L}\{te^{-5t}u(t)\} + e^{-5}e^{-s}\mathcal{L}\{e^{-5t}u(t)\} &= e^{-5}e^{-s}\left[\frac{2}{(s+5)^3} - \frac{2}{(s+5)^2} + \frac{1}{s+5}\right]\end{aligned}$$

(c)*

$$\begin{aligned}
f_3(t) &= te^{-t}u(2t-3) = \begin{cases} te^{-t} & t \geq 1.5 \\ 0 & t < 1.5 \end{cases} \Rightarrow \mathcal{L}\{f_3(t)\} = \int_{1.5}^{\infty} te^{-t}e^{-st}dt \pm \int_{0^-}^{1.5} te^{-t}e^{-st}dt \\
&= \mathcal{L}\{te^{-t}u(t)\} - \int_0^{3/2} te^{-t}e^{-st}dt = \frac{1}{(s+1)^2} - \frac{1.5}{s+1}e^{-1.5(s+1)} + \frac{1}{s+1} \int_{0^-}^{1.5} e^{-(s+1)t}dt \\
&= \frac{1}{(s+1)^2} - \frac{1.5}{s+1}e^{-1.5(s+1)} - \frac{1}{(s+1)^2}(e^{-1.5(s+1)} - 1) = \frac{1}{(s+1)^2}(2 - e^{-1.5(s+1)}) - \frac{1.5}{s+1}e^{-1.5(s+1)}
\end{aligned}$$

(d)

$$\begin{aligned}
\mathcal{L}\{f_4(t)\} &= \mathcal{L}\{5tu(t-1) + 10tu(t-3)\} = 5\mathcal{L}\{(t-1+1)u(t-1)\} + 10\mathcal{L}\{(t-3+3)u(t-3)\} \\
&= 5\mathcal{L}\{(t-1)u(t-1)\} + 5\mathcal{L}\{u(t-1)\} + 10\mathcal{L}\{(t-3)u(t-3)\} + 30\mathcal{L}\{u(t-3)\} \\
5e^{-s}\mathcal{L}\{tu(t)\} + 5e^{-s}\mathcal{L}\{u(t)\} + 10e^{-3s}\mathcal{L}\{tu(t)\} + 30e^{-3s}\mathcal{L}\{u(t)\} &= 5e^{-s}\frac{1}{s^2} + 5e^{-s}\frac{1}{s} + 10e^{-3s}\frac{1}{s^2} + 30e^{-3s}\frac{1}{s}
\end{aligned}$$

Problem 4.9

(a)

$$\begin{aligned}
\mathcal{L}\{f_1(t)\} &= \mathcal{L}\{u(t-1)\sin(\pi t)\} = \mathcal{L}\{u(t-1)\sin(\pi(t-1+1))\} \\
&= \mathcal{L}\{u(t-1)\sin(\pi(t-1))\cos(\pi) + u(t-1)\cos(\pi(t-1))\sin(\pi)\} = -\mathcal{L}\{u(t-1)\sin(\pi(t-1))\} \\
&= -e^{-s}\mathcal{L}\{u(t)\sin(\pi t)\} = -e^{-s}\frac{\pi}{s^2 + \pi^2}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathcal{L}\{f_2(t)\} &= \mathcal{L}\{u(t)e^{-3t}\cos(\pi(t-2))\} = \mathcal{L}\{u(t)e^{-3t}[\cos(\pi t)\cos(-2\pi) + \sin(\pi t)\sin(2\pi)]\} \\
&= \mathcal{L}\{u(t)e^{-3t}\cos(\pi t)\} = \frac{s+3}{(s+3)^2 + \pi^2}
\end{aligned}$$

(c)

Note that $\sin(\pi(t-1+1)) = \sin(\pi(t-1))\cos(\pi) + \cos(\pi(t-1))\sin(\pi) = -\sin(\pi(t-1))$. Then

$$\begin{aligned}
\mathcal{L}\{f_3(t)\} &= \mathcal{L}\{te^{-2t}\sin(\pi t)u(t-1)\} = \mathcal{L}\{(t-1+1)e^{-2(t-1+1)}\sin(\pi(t-1+1))u(t-1)\} \\
&= \mathcal{L}\{-e^{-2}(t-1)e^{-2(t-1)}\sin(\pi(t-1))u(t-1)\} - \mathcal{L}\{e^{-2}e^{-2(t-1)}\sin(\pi(t-1))u(t-1)\} \\
&= -e^{-2}e^{-s}\mathcal{L}\{te^{-2t}\sin(\pi t)u(t)\} - e^{-2}e^{-s}\mathcal{L}\{e^{-2t}\sin(\pi t)u(t)\} \\
&= -e^{-2}e^{-s}\frac{2\pi(s+2)}{[(s+2)^2 + \pi^2]^2} - e^{-2}e^{-s}\frac{\pi}{(s+2)^2 + \pi^2}
\end{aligned}$$

(d)

$$\begin{aligned}
f_4(t) &= te^{-2t}\sin(t-2)u(t-2) = (t-2+2)e^{-2(t-2+2)}\sin(t-2)u(t-2) \\
&= e^{-4}(t-2)e^{-2(t-2)}(t-2)\sin(t-2)u(t-2) + 2e^{-4}e^{-(t-2)}\sin(t-2)u(t-2) \\
&\leftrightarrow e^{-4}\mathcal{L}\{te^{-2t}\sin(t)u(t)\}e^{-2s} + 2e^{-4}\mathcal{L}\{e^{-2t}\sin(t)u(t)\}e^{-2s} \\
&= e^{-4}\frac{2(s+2)}{[(s+2)^2 + 1]^2}e^{-2s} + 2e^{-4}\frac{1}{(s+2)^2 + 1}e^{-2s}
\end{aligned}$$

Problem 4.11

(a)

$$f_1(0^+) = \lim_{s \rightarrow \infty} \{sF_1(s)\} = \lim_{s \rightarrow \infty} \left\{ \frac{s(2s^2 + 2s + 3)}{(s+1)(s+2)(s+3)} \right\} = 2$$

$$f_1(\infty) = \lim_{s \rightarrow 0} \{sF_1(s)\} = \lim_{s \rightarrow 0} \left\{ \frac{s(2s^2 + 2s + 3)}{(s+1)(s+2)(s+3)} \right\} = 0$$

(b)

$$f_2(0^+) = \lim_{s \rightarrow \infty} \left\{ \frac{s(s+4)}{s(s+1)(s+2)} \right\} = 0, \quad f_2(\infty) = \lim_{s \rightarrow 0} \left\{ \frac{s(s+4)}{s(s+1)(s+2)} \right\} = 2$$

(c)

$$f_3(0^+) = \lim_{s \rightarrow \infty} \left\{ \frac{s^2}{(s+1)(s^2+2s+2)} \right\} = 0, \quad f_3(\infty) = \lim_{s \rightarrow 0} \left\{ \frac{s^2}{(s+1)(s^2+2s+2)} \right\} = 0$$

(d)

$$f_4(0^+) = \lim_{s \rightarrow \infty} \left\{ \frac{s^3}{s^2+5s+10} \right\} = \infty, \quad f_4(\infty) = \lim_{s \rightarrow 0} \left\{ \frac{s^3}{s^2+5s+10} \right\} = 0$$

Problem 4.13

(a) The final value theorem is not applicable to the function $F_1(s) = \frac{s^2}{(s+1)(s-2)}$ since it has a pole in the right half of the complex plane.

(b) The final value theorem is not applicable to the function $F_2(s) = \frac{1}{s^2(s+1)(s+2)}$ since it has a double pole at the origin (only a simple (distinct) pole at the origin is allowed).

(c) The final value theorem is not applicable to the function $F_3(s) = \frac{(s+2)}{(s+1)(s^2+2)}$ since it has a complex conjugate pole on the imaginary axis.

(d) The final value theorem is not applicable since the function has a pair of complex conjugate poles on the imaginary axis at $-1 \pm j\sqrt{2}$.

(e) The final value theorem can be applied since all poles are in the right half plane.

Problem 4.14

(a)

$$\begin{aligned} & \mathcal{L}^{-1} \left\{ \frac{5}{(s+1)(s+2)(s+3)(s+5)(s+10)(s+20)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{k_1}{s+1} + \frac{k_2}{s+2} + \frac{k_3}{s+3} + \frac{k_4}{s+5} + \frac{k_5}{s+10} + \frac{k_6}{s+20} \right\} \\ &= (k_1 e^{-t} + k_2 e^{-2t} + k_3 e^{-3t} + k_4 e^{-5t} + k_5 e^{-10t} + k_6 e^{-20t}) u(t) \end{aligned}$$

$$k_1 = \frac{5}{1 \times 2 \times 4 \times 9 \times 19} = \frac{5}{1368}, \quad k_2 = \frac{5}{(-1) \times 1 \times 3 \times 8 \times 18} = -\frac{5}{432}$$

$$k_3 = \frac{5}{(-2) \times (-1) \times 2 \times 7 \times 17} = \frac{5}{476}, \quad k_4 = \frac{5}{(-4) \times (-3) \times (-2) \times 5 \times 15} = -\frac{1}{360}$$

$$k_5 = \frac{5}{(-9) \times (-8) \times (-7) \times (-5) \times 10} = \frac{1}{5040}$$

$$k_6 = \frac{5}{(-19) \times (-18) \times (-17) \times (-15) \times (-10)} = -\frac{1}{174420}$$

The above results for the coefficients k_i can be verified using the MATLAB statements

```
k=5; z=[ ]; p=[-1 -2 -3 -5 -10 -20]; [num,den]=zp2tf(z,p,k); residue(num,den).
```

(b)

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^5(s+4)}\right\} &= \mathcal{L}^{-1}\left\{\frac{k_{11}}{s+2} + \frac{k_{12}}{(s+2)^2} + \frac{k_{13}}{(s+2)^3} + \frac{k_{14}}{(s+2)^4} + \frac{k_{15}}{(s+2)^5} + \frac{k_6}{s+4}\right\} \\ &= \left(k_{11}e^{-2t} + k_{12}te^{-2t} + k_{13}\frac{t^2}{2!}e^{-2t} + k_{14}\frac{t^3}{3!}e^{-2t} + k_{15}\frac{t^4}{4!}e^{-2t} + k_6e^{-4t}\right)u(t)\end{aligned}$$

where

$$\begin{aligned}k_6 &= \frac{1}{(s+2)^5}|_{s=-4} = -\frac{1}{32}, \quad k_{15} = \frac{1}{s+4}|_{s=-2} = \frac{1}{2}, \quad k_{14} = \frac{d}{ds}\left\{\frac{1}{s+4}\right\}|_{s=-2} = -\frac{1}{(s+4)^2}|_{s=-2} = -\frac{1}{4} \\ k_{13} &= \frac{1}{(5-3)!} \lim_{s \rightarrow -2} \left\{\frac{d^2}{ds^2}\left(\frac{1}{s+4}\right)\right\}|_{s=-2} = \frac{1}{(s+4)^3}|_{s=-2} = \frac{1}{8} \\ k_{12} &= \frac{1}{(5-2)!} \lim_{s \rightarrow -2} \left\{\frac{d^3}{ds^3}\left(\frac{1}{s+4}\right)\right\}|_{s=-2} = -\frac{1}{(s+4)^4}|_{s=-2} = -\frac{1}{16} \\ k_{11} &= \frac{1}{(5-1)!} \lim_{s \rightarrow -2} \left\{\frac{d^4}{ds^4}\left(\frac{1}{s+4}\right)\right\}|_{s=-2} = \frac{1}{(s+4)^5}|_{s=-2} = \frac{1}{32}\end{aligned}$$

Comment: MATLAB 6.1 failed to find the coefficients k_i in this case (`den=[1 14 80 240 400 352 128]; num=1; residue(num,den)`).

(c)

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s-1}{(s+1)^3(s+2)^4}\right\} &= \mathcal{L}^{-1}\left\{\frac{k_{11}}{s+1} + \frac{k_{12}}{(s+1)^2} + \frac{k_{13}}{(s+1)^3} + \frac{k_{41}}{s+2} + \frac{k_{42}}{(s+2)^2} + \frac{k_{43}}{(s+2)^3} + \frac{k_{44}}{(s+2)^4}\right\} \\ &= \left(k_{11}e^{-t} + k_{12}te^{-t} + k_{13}\frac{t^2}{2!}e^{-t} + k_{41}e^{-2t} + k_{42}te^{-2t} + k_{43}\frac{t^2}{2!}e^{-2t} + k_{44}\frac{t^3}{3!}e^{-2t}\right)u(t)\end{aligned}$$

$$\begin{aligned}k_{13} &= \frac{s-1}{(s+2)^4}|_{s=-1} = -2, \quad k_{12} = \frac{d}{ds}\left\{\frac{s-1}{(s+2)^4}\right\}|_{s=-1} = \frac{6-3s}{(s+2)^5}|_{s=-1} = 9 \\ k_{11} &= \frac{1}{2} \frac{d^2}{ds^2}\left\{\frac{s-1}{(s+2)^4}\right\}|_{s=-1} = -\frac{6}{(s+2)^6}|_{s=-1} = -6 \\ k_{44} &= \frac{s-1}{(s+1)^3}|_{s=-2} = 3, \quad k_{43} = \frac{d}{ds}\left\{\frac{s-1}{(s+1)^3}\right\}|_{s=-2} = \frac{2}{(s+1)^4}|_{s=-2} = 2 \\ k_{42} &= \frac{1}{2} \frac{d^2}{ds^2}\left\{\frac{s-1}{(s+1)^3}\right\}|_{s=-2} = -\frac{4}{(s+1)^5}|_{s=-2} = 4, \quad k_{41} = \frac{1}{3!} \frac{d^3}{ds^3}\left\{\frac{s-1}{(s+1)^3}\right\}|_{s=-2} = \frac{20/3}{(s+1)^6}|_{s=-2} = \frac{20}{3}\end{aligned}$$

Comment: MATLAB 6.1 failed to find the coefficients k_i in this case also.

Problem 4.16

(a) In the first example of this problem we have first to perform the long division since $n = m = 2$, hence the partial fraction expansion procedure can not be applied directly

$$\begin{aligned}s^2 : (s^2 - s - 2) &= 1 + \frac{s+2}{s^2 - s - 2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{s^2}{(s+1)(s-2)}\right\} = \mathcal{L}^{-1}\left\{1 + \frac{s+2}{s^2 - s - 2}\right\} \\ &= \mathcal{L}^{-1}\left\{1 + \frac{-1/3}{s+1} + \frac{4/3}{s-2}\right\} = \delta(t) - \left(\frac{1}{3}e^{-t} - \frac{4}{3}e^{2t}\right)u(t) = \mathcal{L}^{-1}\{F_1(s)\} = f_1(t)\end{aligned}$$

(b)

$$\begin{aligned}
f_2(t) &= \mathcal{L}^{-1}\{F_2(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{k_{11}}{s} + \frac{k_{12}}{s^2} + \frac{k_3}{s+1} + \frac{k_4}{s+2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{-0.75}{s} + \frac{0.5}{s^2} + \frac{1}{s+1} + \frac{-0.25}{s+2}\right\} = \left(-\frac{3}{4} + \frac{1}{2}t + e^{-t} - \frac{1}{4}e^{-2t}\right)u(t), \quad k_4 = \frac{1}{s^2(s+1)}|_{s=-2} = -\frac{1}{4} \\
k_3 &= \frac{1}{s^2(s+2)}|_{s=-1} = 1, \quad k_{12} = \frac{1}{(s+1)(s+2)}|_{s=0} = \frac{1}{2}, \quad k_{11} = \frac{d}{ds}\left\{\frac{1}{(s+1)(s+2)}\right\}|_{s=0} = -\frac{3}{4}
\end{aligned}$$

(c)

$$\begin{aligned}
f_3(t) &= \mathcal{L}^{-1}\{F_3(s)\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)(s^2+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{k_1}{s+j\sqrt{2}} + \frac{k_1^*}{s-j\sqrt{2}} + \frac{k_3}{s+1}\right\} \\
k_3 &= \frac{(s+2)}{(s^2+2)}|_{s=-1} = \frac{1}{3}, \quad k_1 = \frac{s+2}{(s+1)(s-j\sqrt{2})}|_{s=-j\sqrt{2}} = \frac{-j\sqrt{2}+2}{(-j\sqrt{2}+1)(-2j\sqrt{2})} \\
&= \frac{2-j\sqrt{2}}{-4-2j\sqrt{2}} \times \frac{-4+2j\sqrt{2}}{-4+2j\sqrt{2}} = \frac{-4+j8\sqrt{2}}{24} = -\frac{1}{6} + j\frac{\sqrt{2}}{3} = a + jb \Rightarrow |k_1| = \frac{1}{2}, \angle k_1 = 109.47^\circ \\
&\mathcal{L}^{-1}\left\{\frac{k_1}{s+j\sqrt{2}} + \frac{k_1^*}{s-j\sqrt{2}} + \frac{k_3}{s+1}\right\} = [2|k_1|e^{\alpha t} \cos(\beta t + \angle k_1) + k_3 e^{-t}]u(t) \\
&= \left[\cos(-\sqrt{2}t + 109.47^\circ) + \frac{1}{3}e^{-t}\right]u(t), \quad p_1 = \alpha + j\beta = -j\sqrt{2} \Rightarrow \alpha = 0, \beta = -\sqrt{2}
\end{aligned}$$

(d)

$$\begin{aligned}
\mathcal{L}^{-1}\{F_4(s)\} &= \mathcal{L}^{-1}\left\{\frac{s+3}{(s+2)(s^2+2s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{k_1}{s+1+j} + \frac{k_1^*}{s+1-j} + \frac{k_3}{s+2}\right\} \\
&= (2|k_1|e^{\alpha_1 t} \cos(\beta_1 t + \angle k_1) + k_3 e^{-2t})u(t), \quad \alpha_1 = -1, \quad \beta_1 = -1, \quad p_1 = \alpha_1 + j\beta_1 = -1 - j
\end{aligned}$$

$$\begin{aligned}
k_3 &= \frac{s+3}{s^2+2s+2}|_{s=-2} = \frac{1}{2}, \quad k_1 = \frac{s+3}{(s+2)(s+1-j)}|_{s=-1-j} = -\frac{1}{4} + \frac{3}{4}j \\
&= \frac{\sqrt{10}}{4} \angle \tan^{-1}(-3) = \frac{\sqrt{10}}{4} \angle 108.43^\circ
\end{aligned}$$

$$f_4(t) = \left(\frac{\sqrt{10}}{2}e^{-t} \cos(-t + 108.43^\circ) + \frac{1}{2}e^{-2t}\right)u(t)$$

(e)

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)(s+3)(s+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{k_1}{s+1} + \frac{k_2}{s+3} + \frac{k_3}{s+5}\right\} = (k_1 e^{-t} + k_2 e^{-3t} + k_3 e^{-5t})u(t)$$

where

$$k_1 = \frac{s+2}{(s+3)(s+5)}|_{s=-1} = \frac{1}{8}, \quad k_2 = \frac{s+2}{(s+1)(s+5)}|_{s=-3} = \frac{1}{4}, \quad k_3 = \frac{s+2}{(s+1)(s+3)}|_{s=-5} = -\frac{3}{8}$$

Problem 4.20

(a)

$$\begin{aligned}\mathcal{L}^{-1}\{F_1(s)\} &= \mathcal{L}^{-1}\left\{\frac{s+e^{-5s}}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-5s}}{(s+1)^2}\right\} \\ \frac{s}{(s+1)^2} &= \frac{k_{11}}{s+1} + \frac{k_{12}}{(s+1)^2}, \quad k_{12} = s|_{s=-1} = -1, \quad k_{11} = \frac{d}{ds}\{s\}|_{s=-1} = 1|_{s=-1} = 1 \\ \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{-1}{(s+1)^2}\right\} = (e^{-t} - te^{-t})u(t) \\ \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} &= te^{-t}u(t) \Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{-5s}}{(s+1)^2}\right\} = (t-5)e^{-(t-5)}u(t-5) = f_1(t)\end{aligned}$$

Note that the term e^{-5s} only indicates a time delay of five units.

(b)

$$\begin{aligned}\mathcal{L}^{-1}\{F_2(s)\} &= \mathcal{L}^{-1}\left\{\frac{2-e^{-5s}}{s^2(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2(s+3)}\right\} + \mathcal{L}^{-1}\left\{\frac{-e^{-5s}}{s^2(s+3)}\right\} \\ \frac{1}{s^2(s+3)} &= \frac{k_{11}}{s} + \frac{k_{12}}{s^2} + \frac{k_3}{s+3}, \quad k_{12} = \frac{1}{s+3}|_{s=0} = \frac{1}{3}, \quad k_{11} = \frac{d}{ds}\left\{\frac{1}{s+3}\right\}|_{s=0} = \frac{-1}{(s+3)^2}|_{s=0} = -\frac{1}{9} \\ k_3 &= \frac{1}{s^2}|_{s=-3} = \frac{1}{9}, \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/9}{s} + \frac{1/3}{s^2} + \frac{1/9}{s+3}\right\} = \left(-\frac{1}{9} + \frac{1}{3}t + \frac{1}{9}e^{-3t}\right)u(t) = f(t) \\ \mathcal{L}^{-1}\left\{2\frac{1}{s^2(s+3)} - e^{-5s}\frac{1}{s^2(s+3)}\right\} &= 2f(t) - f(t-5) \\ &= \left(-\frac{2}{9} + \frac{2}{3}t + \frac{2}{9}e^{-3t}\right)u(t) - \left(-\frac{1}{9} + \frac{1}{3}(t-5) + \frac{1}{9}e^{-3(t-5)}\right)u(t-5) = f_2(t)\end{aligned}$$

(c) This function is similar to the function in part (d) whose Laplace inverse is derived in detail below. here, we use MATLAB to find the Laplace inverse of the required function, which produces

$$f_3(t) = \left(\frac{1}{4} + \frac{1}{2}(t-3) - \frac{1}{4}e^{-2(t-3)}\right)u(t-3)$$

(d)

$$\begin{aligned}f_4(t) &= \mathcal{L}^{-1}\{F_4(s)\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+4)s}e^{-s}\right\} \Rightarrow \\ \mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+4)s}\right\} &= \mathcal{L}^{-1}\left\{\frac{k_1}{s+j2} + \frac{k_1^*}{s-j2} + \frac{k_3}{s}\right\} = f(t) \text{ and } \mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+4)s}e^{-s}\right\} = f(t-1) = f_4(t) \\ k_3 &= \frac{s+1}{s^2+4}|_{s=0} = \frac{1}{4}, \quad k_1 = \frac{s+1}{(s-j2)s}|_{s=-j2} = \frac{1-j2}{(-2j2)(-j2)} = -\frac{1}{8} + j\frac{1}{4} = a + jb = k_1 \\ \mathcal{L}^{-1}\left\{\frac{k_1}{s+j2} + \frac{k_1^*}{s-j2} + \frac{k_3}{s}\right\} &= [2|k_1|e^{\alpha t} \cos(\beta t + \angle k_1) + k_3]u(t), \quad |k_1| = \frac{\sqrt{5}}{8}, \quad \angle k_1 = 115.57^\circ \\ f(t) &= \left[\frac{\sqrt{5}}{4} \cos(-2t + 115.57^\circ) + \frac{1}{4}\right]u(t), \quad p_1 = -j2 = \alpha + j\beta\end{aligned}$$

Note that e^{-s} indicates a time delay of one unit. This term must not be included in the procedure for finding the coefficients k_1 and k_3 . Hence

$$f_4(t) = \mathcal{L}^{-1}\{F_4(s)\} = \left[\frac{\sqrt{5}}{4} \cos(-2(t-1) + 115.57^\circ) + \frac{1}{4}\right]u(t-1)$$

(e) Applying the MATLAB function `residue` to $F_5(s) = (s+2)/(s^2(s^2+2))$, we obtain the coefficients $k = \{-0.25 \pm j0.3536, 0.5, 1\}$ at the poles $p = \{\pm\sqrt{2}j, 0, 0\}$. The corresponding result is given by

$$f(t) = [0.5 + t + 2 \times 0.4330 \cos(\sqrt{2}t + 125.3^\circ)] u(t)$$

$$\Rightarrow f_5(t) = [0.5 + (t-2) + 2 \times 0.4330 \cos(\sqrt{2}(t-2) + 125.3^\circ)] u(t-2)$$

Problem 4.22

The first part of this problem is done in Problem 2.38, where we have shown that

$$\frac{D^n}{Dt^n}\{e^{\alpha t}u(t)\} = \alpha^n e^{\alpha t}u(t) + \alpha^{n-1}\delta(t) + \alpha^{n-2}\delta^{(1)}(t) + \cdots + \alpha\delta^{(n-2)}(t) + \delta^{(n-1)}(t)$$

In this problem we have instead of α the exponent equal to $-\alpha$ so that the corresponding derivative is

$$\frac{D^n}{Dt^n}\{e^{-\alpha t}u(t)\} = (-\alpha)^n e^{-\alpha t}u(t) + (-\alpha)^{n-1}\delta(t) + (-\alpha)^{n-2}\delta^{(1)}(t) + \cdots + (-\alpha)\delta^{(n-2)}(t) + \delta^{(n-1)}(t)$$

Taking the Laplace transform of the last expression, we obtain

$$(-\alpha)^n \frac{1}{s+\alpha} + (-\alpha)^{n-1} + (-\alpha)^{n-2}s + \cdots + (-\alpha)s^{n-2} + s^{n-1}$$

$$\frac{1}{s+\alpha} \left((-\alpha)^n + (-\alpha)^{n-1}s - (-\alpha)^n + (-\alpha)^{n-2}s^2 - (-\alpha)^{n-1}s + \cdots - \alpha s^{n-1} + s^n + \alpha s^{n-1} \right) = \frac{s^n}{s+\alpha}$$