

Solutions to HW#4 — CHAPTER 3 — Problems on the Fourier Transform

Problem 3.8

(a) By the modulation property we have

$$u_h(t) \cos(\omega_0 t) \leftrightarrow \frac{1}{2}[U_h(j(\omega + \omega_0)) + U_h(j(\omega - \omega_0))], \quad U_h(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

which implies

$$\frac{1}{2}\left(\frac{1}{j(\omega + \omega_0)} + \pi\delta(\omega + \omega_0)\right) + \frac{1}{2}\left(\frac{1}{j(\omega - \omega_0)} + \pi\delta(\omega - \omega_0)\right) = \frac{\pi}{2}(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)) - \frac{j\omega}{\omega^2 - \omega_0^2}$$

(b) From the modulation property we have

$$u_h(t)e^{-\alpha t} \leftrightarrow \frac{1}{\alpha + j\omega} \Leftrightarrow u_h(t)e^{-\alpha t} \cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[\frac{1}{\alpha + j(\omega + \omega_0)} + \frac{1}{\alpha + j(\omega - \omega_0)}\right] = \frac{\alpha + j\omega}{(\alpha + j\omega)^2 + \omega_0^2}$$

The modulation property also implies

$$u_h(t)e^{-\alpha t} \leftrightarrow \frac{1}{\alpha + j\omega} \Leftrightarrow u_h(t)e^{-\alpha t} \sin(\omega_0 t) \leftrightarrow \frac{j}{2}\left[\frac{1}{\alpha + j(\omega + \omega_0)} - \frac{1}{\alpha + j(\omega - \omega_0)}\right] = \frac{\omega_0}{(\alpha + j\omega)^2 + \omega_0^2}$$

(c) Using the frequency shift property, it follows that

$$e^{j\omega_0 t}u_h(t) \leftrightarrow U_h(j(\omega - \omega_0)) = \frac{1}{j(\omega - \omega_0)} + \pi\delta(\omega - \omega_0)$$

(d) By the time multiplication property (also the frequency differentiation property), we have

$$e^{-\alpha t}u_h(t) \leftrightarrow \frac{1}{\alpha + j\omega} \Leftrightarrow t^{n-1}e^{-\alpha t}u_h(t) \leftrightarrow j^{n-1}\frac{d^{n-1}}{d\omega^{n-1}}\left\{\frac{1}{\alpha + j\omega}\right\} = \frac{(n-1)!}{(\alpha + j\omega)^n}$$

(e) The function is Fourier transformable (in terms of regular functions), hence the result can be obtained by direct integration as follows

$$\mathcal{F}\{e^{-t}[u_h(t) - u_h(t-1)]\} \triangleq \int_0^1 e^{-t}e^{-j\omega t}dt = \frac{1}{1+j\omega}\left(1 - e^{-(1+j\omega)}\right)$$

Since

$$\cos(2\pi t)[u_h(t+1) - u_h(t-1)] = \cos(2\pi t)p_2^h(t), \quad p_2^h(t) \leftrightarrow \tau\text{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

we obtain the following result by using the modulation property and $\tau = 2$

$$\frac{1}{2}\tau\text{sinc}\left(\frac{(\omega + 2\pi)\tau}{2\pi}\right) + \frac{1}{2}\tau\text{sinc}\left(\frac{(\omega - 2\pi)\tau}{2\pi}\right) = \text{sinc}\left(\frac{\omega + 2\pi}{\pi}\right) + \text{sinc}\left(\frac{\omega - 2\pi}{\pi}\right)$$

Problem 3.10

We know that $p_\tau^h(t) \leftrightarrow \tau\text{sinc}(\omega\tau/2\pi)$. By duality we have

$$X(jt) = \tau\text{sinc}\left(\frac{t\tau}{2\pi}\right) \leftrightarrow 2\pi p_\tau^h(-\omega) \Leftrightarrow \text{sinc}\left(\frac{t\tau}{2\pi}\right) \leftrightarrow \frac{2\pi}{\tau}p_\tau^h(\omega)$$

Hence, we have the following Fourier pair for the time scaled sinc signal

$$\text{sinc}(at) \leftrightarrow \frac{1}{a}p_\tau^h(\omega), \quad a = \frac{\tau}{2\pi} \Leftrightarrow \text{sinc}(t) \leftrightarrow p_\tau^h(a\omega) = p_\tau^h\left(\frac{\tau\omega}{2\pi}\right)$$

Using the definition of the rectangular pulse, we have

$$\begin{aligned} p_\tau^h\left(\frac{\tau\omega}{2\pi}\right) &= \begin{cases} 1, & -\frac{\tau}{2} < \frac{\tau\omega}{2\pi} < \frac{\tau}{2} \\ 0, & \frac{\tau\omega}{2\pi} < -\frac{\tau}{2}, \frac{\tau\omega}{2\pi} > \frac{\tau}{2} \\ 0.5, & \frac{\tau\omega}{2\pi} = \pm\frac{\tau}{2} \end{cases} = \begin{cases} 1, & -\pi < \omega < \pi \\ 0, & \omega < -\pi, \omega > \pi \\ 0.5, & \omega = \pm\pi \end{cases} = p_{2\pi}^h(\omega) \\ &= \begin{cases} 1, & -0.5 < f < 0.5 \\ 0, & f < -0.5, f > 0.5 \\ 0.5, & f = \pm 0.5 \end{cases} = p_1^h(f), \quad \omega = 2\pi f \end{aligned}$$

so that the final result can be written as

$$\text{sinc}(t) \leftrightarrow p_{2\pi}^h(\omega) = p_1^h(f)$$

Problem 3.11

(a) This is a periodic signal represented in its exponential Fourier series form. According to (3.48), the corresponding Fourier transform pair is given by

$$x_1(t) = \sum_{n=-\infty}^{n=\infty} \frac{1}{n\pi} e^{jn\omega_0 t} \leftrightarrow 2\pi \sum_{n=-\infty}^{n=\infty} \frac{1}{n\pi} \delta(\omega - n\omega_0)$$

(b) Note that

$$e^{-7t} u_h(t) \text{sgn}(2t - 5) = \begin{cases} e^{-7t}, & t > 2.5 \\ -e^{-7t}, & 0 < t < 2.5 \\ 0, & t < 0 \end{cases}$$

Hence this function is Fourier transformable in terms of regular function and we can use the definition integral of the Fourier transform

$$\begin{aligned} \mathcal{F}\{e^{-7t} u_h(t) \text{sgn}(2t - 5)\} &= \int_0^{5/2} (-e^{-7t}) e^{-j\omega t} dt + \int_{5/2}^{\infty} e^{-7t} e^{-j\omega t} dt \\ &= \frac{1}{7+j\omega} \left(e^{-(7+j\omega)\frac{5}{2}} - 1 \right) + \frac{1}{7+j\omega} e^{-(7+j\omega)\frac{5}{2}} = \frac{1}{7+j\omega} \left(2e^{-(7+j\omega)\frac{5}{2}} - 1 \right) \end{aligned}$$

(c) Since $\text{sgn}(t) \leftrightarrow 2/j\omega$ then by the duality property

$$\frac{2}{jt} \leftrightarrow 2\pi \text{sgn}(-\omega) \Leftrightarrow \frac{1}{t} \leftrightarrow j\pi \text{sgn}(-\omega) = -j\pi \text{sgn}(\omega)$$

(d) The result established in Problem 3.10 states

$$\text{sinc}(t) \leftrightarrow p_{2\pi}^h(\omega)$$

The combined time scaling and shifting property derived in Problem 3.7(a) implies

$$\text{sinc}(2t - 5) \leftrightarrow \frac{1}{2} e^{-j(\frac{\omega}{2})5} p_{2\pi}\left(\frac{\omega}{2}\right)$$

Problem 3.12

Using the definition of the Fourier transform we obtain

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{e^{-|t|}\right\} \triangleq \int_{-\infty}^0 e^t e^{-j\omega t} dt + \int_0^{\infty} e^{-t} e^{-j\omega t} dt = \frac{1}{1-j\omega} + \frac{1}{1+j\omega} = \frac{2}{\omega^2 + 1} = X(j\omega)$$

(a) It follows from the direct integration as above can that $\mathcal{F}\{x_1(t)\} = \mathcal{F}\{e^{-\alpha|t|}\} = 2\alpha/(\omega^2 + \alpha^2)$. By the modulation property, we have

$$\mathcal{F}\{x_2(t)\} = \mathcal{F}\left\{e^{-2|t|} \sin(t)\right\} = \frac{j}{2} \left[\frac{4}{(\omega+1)^2 + 4} - \frac{4}{(\omega-1)^2 + 4} \right]$$

Similarly, $\mathcal{F}\{x_3(t)\} = \mathcal{F}\{te^{-2|t|}\}$ follows from the time multiplication property

$$\mathcal{F}\left\{te^{-2|t|}\right\} = j \frac{d}{d\omega} \left\{ \frac{4}{\omega^2 + 4} \right\}$$

(b) Using the duality property of the Fourier transform, we obtain

$$X(jt) = \frac{2}{t^2 + 1} \leftrightarrow 2\pi x(-\omega) = 2\pi e^{-|-\omega|} = 2\pi e^{-|\omega|} \Rightarrow \frac{1}{t^2 + 1} \leftrightarrow \pi e^{-|\omega|}$$

Problem 3.13

The results established in Problem 3.7 can be used for the first three terms of the signal $x_1(t)$. The fourth term in $x_1(t)$ requires a new combined property: time shifting and modulation. This combined property can be derived as follows

$$\begin{aligned} \int_{-\infty}^{\infty} x(t - t_0) \cos(\omega_0 t) e^{-j\omega t} dt &= \frac{1}{2} \int_{-\infty}^{\infty} x(t - t_0) [e^{j\omega_0 t} + e^{-j\omega_0 t}] e^{-j\omega t} dt = \frac{1}{2} \int_{-\infty}^{\infty} x(t - t_0) e^{-j(\omega - \omega_0)t} dt \\ &+ \frac{1}{2} \int_{-\infty}^{\infty} x(t - t_0) e^{-j(\omega + \omega_0)t} dt = \frac{1}{2} \int_{-\infty}^{\infty} x(\sigma) e^{-j(\omega - \omega_0)(t_0 + \sigma)} d\sigma + \frac{1}{2} \int_{-\infty}^{\infty} x(\sigma) e^{-j(\omega + \omega_0)(t_0 + \sigma)} d\sigma \\ &\frac{1}{2} e^{-j(\omega - \omega_0)t_0} \int_{-\infty}^{\infty} x(\sigma) e^{-j(\omega - \omega_0)\sigma} d\sigma + \frac{1}{2} e^{-j(\omega + \omega_0)t_0} \int_{-\infty}^{\infty} x(\sigma) e^{-j(\omega + \omega_0)\sigma} d\sigma \\ &\frac{1}{2} e^{-j(\omega - \omega_0)t_0} X(j(\omega - \omega_0)) + \frac{1}{2} e^{-j(\omega + \omega_0)t_0} X(j(\omega + \omega_0)) \end{aligned}$$

Note that in the derivations a change of variables $t - t_0 = \sigma$ has been used. The Fourier transform of the signal $x_1(t)$ is given by

$$\begin{aligned} \mathcal{F}\left\{x(5t - 4) + tx(t - 1) + \frac{dx(3t)}{dt} + x(t - 1) \cos(4t)\right\} &= \frac{1}{5} e^{-j(\frac{\omega}{5})^4} X\left(\frac{j\omega}{5}\right) \\ &+ e^{-j\omega} \left[X(j\omega) + j \frac{dX(j\omega)}{d\omega} \right] + j \frac{\omega}{3} X\left(\frac{j\omega}{3}\right) + \frac{1}{2} \left[e^{-j(\omega-4)} X(j(\omega-4)) + e^{-j(\omega+4)} X(j(\omega+4)) \right] \end{aligned}$$

where

$$\begin{aligned} X\left(\frac{j\omega}{5}\right) &= \frac{2a}{(\frac{\omega}{5})^2 + a^2}, \quad X\left(\frac{j\omega}{3}\right) = \frac{2a}{(\frac{\omega}{3})^2 + a^2}, \quad \frac{dX(j\omega)}{d\omega} = \frac{-4\omega}{(\omega^2 + a^2)} \\ X(j(\omega + 4)) &= \frac{2a}{(\omega + 4)^2 + a^2}, \quad X(j(\omega - 4)) = \frac{2a}{(\omega - 4)^2 + a^2} \end{aligned}$$

Problem 3.18

(a)

$$\begin{aligned} \frac{dx_1(t)}{dt} &= p_1(t + 0.5) - 0.5p_2(t - 1) \leftrightarrow e^{j\frac{\omega}{2}} \mathcal{F}\{p_1(t)\} - 0.5e^{-j\omega} \mathcal{F}\{p_2(t)\} \\ &= e^{j\frac{\omega}{2}} \text{sinc}\left(\frac{\omega}{2\pi}\right) - e^{-j\omega} \text{sinc}\left(\frac{\omega}{\pi}\right) = X(j\omega) \end{aligned}$$

Using the integral property, we have

$$\begin{aligned}\mathcal{F}\{x_1(t)\} &= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega) = \frac{1}{j\omega}\left[e^{j\frac{\omega}{2}}\text{sinc}\left(\frac{\omega}{2\pi}\right) - e^{-j\omega}\text{sinc}\left(\frac{\omega}{\pi}\right)\right] + \pi 0\delta(\omega) \\ &= \frac{1}{j\omega}\left[e^{j\frac{\omega}{2}}\text{sinc}\left(\frac{\omega}{2\pi}\right) - e^{-j\omega}\text{sinc}\left(\frac{\omega}{\pi}\right)\right]\end{aligned}$$

Note that this result could have been obtained by using the result from Example 3.13 that states $\Delta_\tau(t) \leftrightarrow \frac{\tau}{2}\text{sinc}^2\left(\frac{\omega\tau}{4\pi}\right)$, which for $2\Delta_2(t)$ implies $2\Delta_2(t) \leftrightarrow 2\text{sinc}^2\left(\frac{\omega}{2\pi}\right)$.

(b)

$$\begin{aligned}X_2(j\omega) &= \int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt = \int_{-1}^0 e^{-j\omega t}dt + \int_0^2 (-1)e^{-j\omega t}dt \\ &= -\frac{1}{j\omega}(1 - e^{j\omega}) + \frac{1}{j\omega}(e^{-2j\omega} - 1) = \frac{1}{j\omega}(e^{j\omega} + e^{-2j\omega} - 2)\end{aligned}$$

Problem 3.20

(a)

$$\begin{aligned}\frac{dx_1(t)}{dt} &= 2p_1(t+0.5) - 2p_1(t-0.5) \leftrightarrow 2[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}]\mathcal{F}\{p_1(t)\} \\ &= 2\left[\cos\left(\frac{\omega}{2}\right) + j\sin\left(\frac{\omega}{2}\right) - \cos\left(\frac{\omega}{2}\right) + j\sin\left(\frac{\omega}{2}\right)\right]\mathcal{F}\{p_1(t)\} = 4j\sin\left(\frac{\omega}{2}\right)\text{sinc}\left(\frac{\omega}{2\pi}\right) = X(j\omega)\end{aligned}$$

Using the integral property, we have

$$\begin{aligned}\mathcal{F}\{x_1(t)\} &= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega) = \frac{1}{j\omega}4j\sin\left(\frac{\omega}{2}\right)\text{sinc}\left(\frac{\omega}{2}\right) + \pi 0\delta(\omega) = \frac{4}{\omega}\sin\left(\frac{\omega}{2}\right)\text{sinc}\left(\frac{\omega}{2\pi}\right) \\ &= \frac{4}{\omega}\sin\left(\frac{\omega}{2}\right)\text{sinc}\left(\frac{\omega}{2\pi}\right) = 2\text{sinc}^2\left(\frac{\omega}{2\pi}\right)\end{aligned}$$

Note that this result could have been obtained by using the result from Example 3.13 which states $\Delta_\tau(t) \leftrightarrow \frac{\tau}{2}\text{sinc}^2\left(\frac{\omega\tau}{4\pi}\right)$, which for $2\Delta_2(t)$ implies $2\Delta_2(t) \leftrightarrow 2\text{sinc}^2\left(\frac{\omega}{4\pi}\right)$.

(b)

$$\begin{aligned}\mathcal{F}\{x_2(t)\} &= \mathcal{F}\{p_2(t+1) - p_2(t-1)\} = e^{j\omega}\mathcal{F}\{p_2(t)\} - e^{-j\omega}\mathcal{F}\{p_2(t)\} \\ &= 2(e^{j\omega} - e^{-j\omega})\text{sinc}\left(\frac{\omega}{\pi}\right) = 4j\sin(\omega)\text{sinc}\left(\frac{\omega}{\pi}\right) = 4j\omega\text{sinc}^2\left(\frac{\omega}{\pi}\right)\end{aligned}$$

Problem 3.23

(a) Using the Euler formula, we have

$$\frac{\cos(\omega)}{j\omega+2}e^{-j3\omega} = \frac{e^{j\omega} + e^{-j\omega}}{2(j\omega+2)}e^{-j3\omega} = \frac{e^{-j2\omega} + e^{-j4\omega}}{2(j\omega+2)}$$

Since $1/(j\omega+2) \leftrightarrow u_h(t)e^{-2t}$, then by the time shift property we obtain the following function in the time domain

$$\frac{1}{2}\left(e^{-2(t-2)}u_h(t-2) + e^{-2(t-4)}u_h(t-4)\right)$$

Similarly, we have

$$\begin{aligned} \frac{\cos(\omega + 2)}{j\omega + 2} e^{-j3\omega} &= \frac{e^{j(\omega+2)} + e^{-j(\omega+2)}}{2(j\omega + 2)} e^{-j3\omega} = \frac{e^{j2}e^{-j2\omega} + e^{-j2}e^{-j4\omega}}{2(j\omega + 2)} \\ &\leftrightarrow \frac{e^{j2}}{2} e^{-2(t-2)} u_h(t-2) + \frac{e^{-j2}}{2} e^{-2(t-4)} u_h(t-4) \end{aligned}$$

(b) We know from Problem 3.10 that $p_\tau^h(\omega) \leftrightarrow (\tau/2\pi)\text{sinc}(t\tau/2\pi)$, which for $\tau = 2$ produces the following pair $p_2^h(\omega) \leftrightarrow (1/\pi)\text{sinc}(t/\pi)$. Using the Euler formula and applying the time shifting property, we obtain

$$p_2^h(\omega) \cos(\omega) = \frac{1}{2} p_2^h(\omega) (e^{j\omega} + e^{-j\omega}) \leftrightarrow \frac{1}{2\pi} \left[\text{sinc}\left(\frac{t+1}{\pi}\right) + \text{sinc}\left(\frac{t-1}{\pi}\right) \right]$$

Using the time scaling property, it follows from the relationship $p_\tau^h(t) \leftrightarrow \tau\text{sinc}(\omega\tau/2\pi)$ that the following pair exists $x(t) = p_\tau^h(2t) \leftrightarrow (1/2)\tau\text{sinc}((\omega/2)\tau/2\pi) = X(j\omega)$. By using the duality property, we have

$$X(jt) = \frac{1}{2} \tau \text{sinc}\left(\frac{t\tau}{4\pi}\right) \leftrightarrow 2\pi p_\tau^h(2(-\omega)) = 2\pi p_\tau^h(2\omega)$$

Applying the Euler formula and using the preceding result, we obtain

$$\begin{aligned} p_2^h(2\omega) \sin(2\omega) &= \frac{1}{2j} p_2^h(2\omega) (e^{j2\omega} - e^{-j2\omega}) \leftrightarrow 2jX(t+2) - 2jX(t-2) \\ &= \frac{j\tau}{2\pi} \left[\text{sinc}\left(\frac{(t+2)\tau}{4\pi}\right) - \text{sinc}\left(\frac{(t-2)\tau}{4\pi}\right) \right], \quad \tau = 2 \end{aligned}$$

(c) We know that $1/(j\omega + 5) \leftrightarrow e^{-5t} u_h(t)$. In the frequency domain, the multiplication by $j\omega$ indicates the time derivative and the multiplication by $e^{-j3\omega}$ indicates the time shift by three time units. Hence, the result is given by

$$j\omega \frac{1}{j\omega + 5} e^{-j3\omega} \leftrightarrow \frac{d}{dt} \left\{ e^{-5(t-3)} u_h(t-3) \right\}$$

(d) This problem demonstrates the time domain convolution. Namely

$$\frac{1}{j(\omega - 2)} = \frac{1}{2} \frac{2}{j(\omega - 2)} \leftrightarrow \frac{1}{2} \text{sgn}(t) e^{j2t} = x_1(t)$$

and

$$\tau \text{sinc}\left(\frac{\omega\tau}{2}\right) \leftrightarrow \frac{1}{\pi} p_\tau\left(\frac{t}{\pi}\right) = x_2(t)$$

Since the product of two functions in the frequency domain corresponds to the time domain convolution, we have

$$\frac{1}{j(\omega - 2)} \tau \text{sinc}\left(\frac{\omega\tau}{2}\right) \leftrightarrow x_1(t) * x_2(t)$$

The second part of this problem can be solved similarly. From the table of common pairs, we have the following two pairs

$$e^{-at} u_h(t), \quad a > 0 \quad \leftrightarrow \quad \frac{1}{1+j\omega}, \quad \text{sgn}(t) \leftrightarrow \frac{2}{j\omega}$$

By the time multiplication property, it follows

$$t e^{-at} u_h(t) \leftrightarrow j \frac{d}{d\omega} \left\{ \frac{1}{1+j\omega} \right\} = j \frac{-j}{(1+j\omega)^2} = \frac{1}{(1+j\omega)^2}$$

By the time convolution property, we have

$$\frac{1}{j\omega} \frac{1}{(1+j\omega)^2} \leftrightarrow \frac{1}{2} \text{sinc}(t) * \{te^{-at}u_h(t)\}, \quad a > 0$$

Problem 3.24

From FIGURE 3.25 we have $X(\omega) = 2p_4(\omega) - p_2(\omega)$. We know from Problem 3.10 that $p_\tau(\omega) \leftrightarrow (\tau/2\pi)\text{sinc}(t\tau/2\pi)$. Hence

$$2p_4(\omega) - p_2(\omega) \leftrightarrow 2 \frac{4}{2\pi} \text{sinc}\left(t \frac{4}{2\pi}\right) - \frac{2}{2\pi} \text{sinc}\left(t \frac{2}{2\pi}\right) = \frac{4}{\pi} \text{sinc}\left(\frac{2t}{\pi}\right) - \frac{1}{\pi} \text{sinc}\left(\frac{t}{\pi}\right)$$

Problem 3.25

(a) Using the Euler formula, we have

$$\frac{1}{1+j\omega} \cos(2\omega)e^{-j5\omega} = \frac{1}{1+j\omega} \frac{(e^{j2\omega} + e^{-j2\omega})}{2} e^{-j5\omega} = \frac{1}{2} \frac{1}{(1+j\omega)} (e^{-j3\omega} + e^{-j7\omega})$$

Since $e^{-t}u_h(t) \leftrightarrow 1/(1+j\omega)$, then by the time shifting property, the following signal is obtained in the time domain

$$\frac{1}{2} e^{-(t-3)} u_h(t-3) + \frac{1}{2} e^{-(t-7)} u_h(t-7)$$

We know from Problem 3.10 that $p_\tau^h(\omega) \leftrightarrow (\tau/2\pi)\text{sinc}(t\tau/2\pi)$. For $\tau = 6$ we obtain $p_6^h(\omega) \leftrightarrow (3/\pi)\text{sinc}(3t/\pi)$. The sine function can be expressed in terms of exponential functions by using the Euler formula. The inverse Fourier transform produces the corresponding time shifted functions, that is

$$p_6^h(\omega) \sin(\omega) = p_6^h(\omega) \frac{e^{j\omega} - e^{-j\omega}}{2j} \leftrightarrow \frac{3}{2j\pi} \left[\text{sinc}\left(\frac{3(t+1)}{\pi}\right) - \text{sinc}\left(\frac{3(t-1)}{\pi}\right) \right]$$

(b) From the table of common pairs we know that $2/(1+\omega^2) \leftrightarrow e^{-|t|}$ (see also Problem 3.12). Using the time shifting property we obtain

$$\frac{1}{1+\omega^2} \cos(5\omega) = \frac{1}{2} \frac{1}{(1+\omega^2)} (e^{j5\omega} + e^{-j5\omega}) \leftrightarrow \frac{1}{4} (e^{-|t+5|} + e^{-|t-5|})$$

Knowing that $p_\tau^h(\omega) \leftrightarrow (\tau/2\pi)\text{sinc}(t\tau/2\pi)$, for $\tau = 3$ we have the following pair $p_3^h(\omega) \leftrightarrow (3/2\pi)\text{sinc}(3t/2\pi)$. The cosine function should be first expressed in terms of exponential functions by using the Euler formula. The application of the inverse Fourier transform produces the time shifted functions, that is

$$p_3^h(\omega) \cos(\omega) = p_3^h(\omega) \frac{e^{j\omega} + e^{-j\omega}}{2} \leftrightarrow \frac{3}{4\pi} \left[\text{sinc}\left(\frac{3(t+1)}{2\pi}\right) + \text{sinc}\left(\frac{3(t-1)}{2\pi}\right) \right]$$

Problem 3.34

The system transfer function and the system impulse response are obtained from the system differential equation as

$$\frac{dy(t)}{dt} + y(t) = x(t) \Rightarrow H(j\omega) = \frac{1}{1+j\omega} \Rightarrow h(t) = \mathcal{F}^{-1}\{H(j\omega)\} = e^{-t}u_h(t)$$

The system zero-state response due to the input $x(t) = \sin(t) \leftrightarrow j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$ is obtained from

$$Y(j\omega) = X(j\omega)H(j\omega) = j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)] \frac{1}{1+j\omega} = j\pi \frac{1}{1-j\omega_0} \delta(\omega + \omega_0) - j\pi \frac{1}{1+j\omega_0} \delta(\omega - \omega_0)$$

For $\omega_0 = 1$, we have

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\{Y(j\omega)\} = \mathcal{F}^{-1}\left\{j\pi\frac{1}{1-j1}\delta(\omega+1) - j\pi\frac{1}{1+j1}\delta(\omega-1)\right\} \\ &= \mathcal{F}^{-1}\{0.25(-1+j)2\pi\delta(\omega+1) - 0.25(1+j)2\pi\delta(\omega-1)\} \\ &= 0.25(-1+j)e^{-jt} - 0.25(1+j)e^{jt} = -0.5\cos(t) + 0.5\sin(t) \end{aligned}$$

In the above derivations we have used the result $e^{\mp j\omega_0 t} \leftrightarrow 2\pi\delta(\omega \pm \omega_0)$.

Problem 3.39

The linear dynamic system in Example 3.19 is defined by $y^{(2)}(t) + 2y^{(1)}(t) + 3y(t) = x(t)$. Its transfer function is given by

$$H(j\omega) = \frac{1}{(j\omega)^2 + 2(j\omega) + 3} = \frac{1}{3 - \omega^2 + j2\omega} = \frac{1}{\sqrt{(3 - \omega)^2 + 4\omega^2}} \angle \left\{ -\tan^{-1} \left(\frac{2\omega}{3 - \omega^2} \right) \right\} = |H(j\omega)| \angle H(j\omega)$$

Note that $\sin(\alpha) = \cos(\alpha - \pi/2)$, hence the input signal is given by $x(t) = 5\cos(2t + \pi/6 - \pi/2) = 5\cos(2t - \pi/3)$. Using formula (3.83) the system steady state response is given by

$$\begin{aligned} y_{ss}(t) &= |H(2)|5\cos\left(2t - \frac{\pi}{3} + \angle H(2)\right) = 5\frac{1}{\sqrt{(3 - 2)^2 + 4 \times 2^2}} \cos\left(2t - \frac{\pi}{3} - \tan^{-1}\left(\frac{4}{3 - 4}\right)\right) \\ &= \frac{5}{\sqrt{17}} \cos\left(2t - \frac{\pi}{3} - \tan^{-1}(-4)\right) = 1.2127 \times \cos(2t + 15.96^\circ) \end{aligned}$$

Problem 3.41

Using formula (3.83) the system steady state response is obtained as

$$\begin{aligned} y_{ss}(t) &= |H(j1)|2\cos\left(t + \frac{\pi}{3} + \angle H(j1)\right) = 2\frac{|j\omega|}{|(1 + j\omega)^2|} \cos\left(t + \frac{\pi}{3} + \angle(j\omega) - 2\angle(1 + j\omega)\right) \\ &= \frac{2\omega}{\sqrt{(1 - \omega^2)^2 + 4\omega^2}} \cos\left(t + \frac{\pi}{3} + \frac{\pi}{2} - 2\tan^{-1}(\omega)\right) = \frac{2\omega}{1 + \omega^2} \cos\left(t + \frac{\pi}{3} + \frac{\pi}{2} - 2\tan^{-1}(1)\right) = \cos\left(t + \frac{\pi}{3}\right) \end{aligned}$$