# Solutions to HW#3, Chapter 3, Problems on Fourier Series

**Problem 3.1**

For the periodic signal in FIGURE 3.19[1], the trigonometric Fourier series coefficients are obtained as follows

$$x(t) = x(t+T), \quad x(t) = \frac{t}{T}, \quad 0 < t < T, \quad \omega_0 = \frac{2\pi}{T}$$

$$a_n = \frac{2}{T}\int_0^T x(\tau)\cos(n\omega_0\tau)d\tau = \frac{2}{T}\int_0^T \frac{\tau}{T}\cos(n\omega_0\tau)d\tau = \frac{2}{T^2}\left[\frac{\cos(n\omega_0\tau)}{n^2\omega_0^2} + \frac{\tau\sin(n\omega_0\tau)}{n\omega_0}\right]\Big|_{\tau=0}^{\tau=T}$$

$$= \frac{2}{(Tn\omega_0)^2}\cos\left(n\frac{2\pi}{T}T\right) - \frac{2}{(Tn\omega_0)^2}\cos(0) = 0, \quad n = 1, 2, \ldots, \quad a_0 = \frac{2}{T}\int_0^T \frac{\tau}{T}d\tau = \frac{2}{T^2}\frac{T^2}{2} = 1$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0\tau)d\tau = \frac{2}{T}\int_0^T \frac{\tau}{T}\sin(n\omega_0\tau)d\tau = \frac{2}{T^2}\left[\frac{\sin(n\omega_0\tau)}{n^2\omega_0^2} - \frac{\tau\cos(n\omega_0\tau)}{n\omega_0}\right]\Big|_{\tau=0}^{\tau=T}$$

$$= \frac{2}{(Tn\omega_0)^2}\sin\left(n\frac{2\pi}{T}T\right) - \frac{2}{Tn\omega_0}\cos\left(n\frac{2\pi}{T}T\right) = -\frac{2}{Tn\omega_0} = -\frac{1}{n\pi}, \quad n = 1, 2, \ldots$$

Note that in the preceding derivations, we have used the standard results taken from a table of integrals

$$\int \tau\sin(\alpha\tau)d\tau = \frac{1}{\alpha^2}\sin(\alpha\tau) - \frac{\tau}{\alpha}\cos(\alpha\tau), \qquad \int \tau\cos(\alpha\tau)d\tau = \frac{1}{\alpha^2}\cos(\alpha\tau) + \frac{\tau}{\alpha}\sin(\alpha\tau)$$

The real coefficient trigonometric Fourier series representation of this signal is given by

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) = \frac{1}{2} - \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin(n\omega_0 t)$$

**Answer 3.2:**

$$X_0 = \frac{1}{2}a_0 = 0.5 \quad \Rightarrow \quad |X_0| = 0.5, \ \measuredangle X_0 = 0,$$
$$X_n = \frac{1}{2}(a_n - jb_n) = -\frac{jb_n}{2} = \frac{j}{2n\pi} \quad \Rightarrow \quad |X_n| = \frac{1}{2n\pi}, \ \measuredangle X_n = \frac{\pi}{2}, \quad n = 1, 2, \ldots$$

$$x(t) = X_0 + 2\sum_{n=1}^{\infty}|X_n|\cos(n\omega_0 t + \measuredangle X_n) = 0.5 + \sum_{n=1}^{\infty}\frac{1}{n\pi}\cos\left(n\omega_0 t + \frac{\pi}{2}\right)$$

[1] Note that "FIGURE x.y" denotes a figure from the textbook and "Figure x.y denotes a figure from the solution manual.

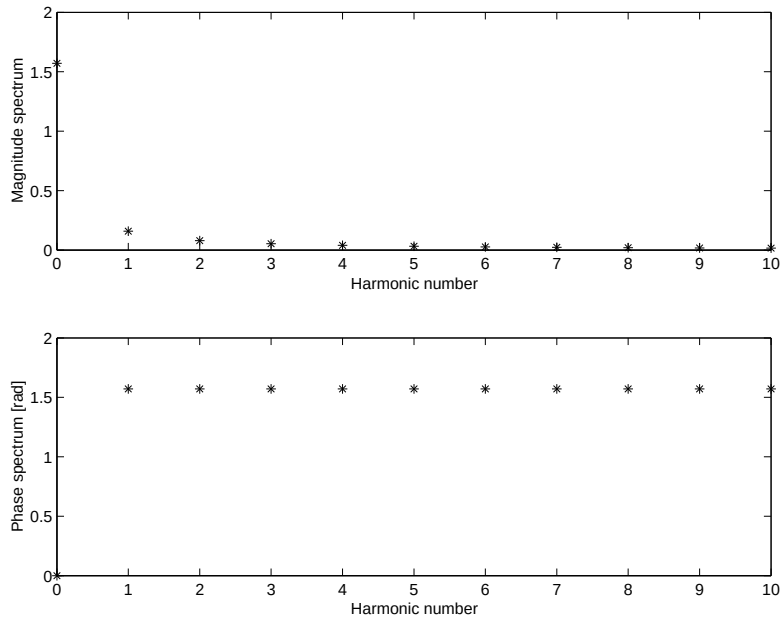


Figure 3.1

**Problem 3.3**
The required Fourier series coefficients for the signal given in FIGURE 3.20 are obtained from

$$X_0 = \frac{1}{T}\int_{t_1}^{t_2} E dt = \frac{E}{T}(t_2 - t_1), \quad X_n = \frac{1}{T}\int_0^T x(\tau)e^{-jn\omega_0\tau}d\tau = \frac{1}{T}\int_{t_1}^{t_2} Ee^{-jn\omega_0\tau}d\tau$$
$$= \frac{E}{T(-jn\omega_0)}e^{-jn\omega_0\tau}|_{\tau=t_1}^{\tau=t_2} = \frac{jE}{Tn\omega_0}\left(e^{-jn\omega_0 t_2} - e^{-jn\omega_0 t_1}\right), \quad n = \pm 1, \pm 2, \ldots$$

For $t_1 = 0$ and $t_2 = T/2$, we have

$$X_n = \frac{jE}{2n\pi}\left(e^{-j\frac{2\pi}{T}\frac{T}{2}} - 1\right) = \frac{jE}{2n\pi}\left(e^{-j\pi} - 1\right) = \frac{jE}{2n\pi}((-1)^n - 1), \quad n = \pm 1, \pm 2, \ldots, \quad X_0 = \frac{E}{T}\frac{T}{2} = \frac{E}{2}$$

$$\Rightarrow \quad x(t) = \sum_{n=-\infty}^{n=\infty} X_n e^{jn\omega_0 t} = \frac{jE}{2\pi}\sum_{n=-\infty}^{n=\infty}\frac{1}{n}((-1)^n - 1)e^{jn\omega_0 t}$$

**Answer 3.4:**

$$|X_n| = \frac{E}{2n\pi}((-1)^n - 1), \quad n = \pm 1, \pm 2, \ldots, \quad \measuredangle X_n = \begin{cases} -\frac{\pi}{2}, & n = \pm 1, \pm 3 \ldots \\ 0, & n = \pm 2, \pm 4, \ldots \end{cases}, \quad |X_0| = \frac{E}{2}, \quad \measuredangle X_0 = 0$$

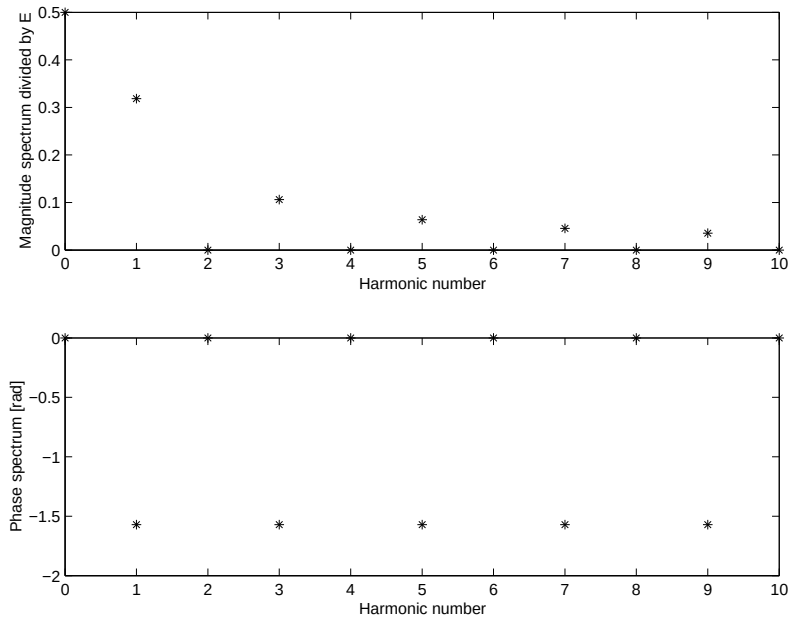


Figure 3.2

**Problem 3.6**

Since $x(t)$ is a even function then $b_n = 0$ and $X_n = 0.5(a_n - jb_n) = 0.5a_n$. Note that $\omega_0 = 2\pi/2\pi = 1$.

$$a_n = \frac{2}{T}\int_0^{\pi} \tau\cos(n\omega_0\tau)d\tau + \frac{2}{T}\int_{\pi}^{2\pi}(2\pi-\tau)\cos(n\omega_0\tau)d\tau = \frac{2}{T}\left[\frac{\cos(n\tau)}{n^2} + \frac{\tau\sin(n\tau)}{n}\right]\Big|_{\tau=0}^{\tau=\pi}$$

$$+\frac{4\pi}{Tn}\sin(n\tau)\Big|_{\tau=\pi}^{\tau=2\pi} - \frac{2}{T}\left[\frac{\cos(n\tau)}{n^2} + \frac{\tau\sin(n\tau)}{n}\right]\Big|_{\tau=\pi}^{\tau=2\pi} = \frac{1}{\pi n^2}\cos(n\pi) - \frac{1}{\pi n^2} - \frac{1}{\pi n^2}\cos(2n\pi) + \frac{1}{\pi n^2}\cos(n\pi)$$

$$= \frac{1}{\pi n^2}[2\cos(n\pi) - 1 - \cos(2n\pi)] = -\frac{2}{\pi n^2}[1-(-1)^n], \qquad n = 1, 2, 3, \ldots$$

$$a_0 = \frac{2}{T}\int_0^{\pi}\tau d\tau + \frac{2}{T}\int_{\pi}^{2\pi}(2\pi-\tau)d\tau = \pi$$

The coefficient $X_n$ is given by

$$X_n = 0.5a_n = \begin{cases} \frac{\pi}{2}, & n = 0 \\ -\frac{2}{\pi n^2}, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases}$$

The frequency magnitude and phase spectra are given by

$$|X_n| = \begin{cases} \frac{\pi}{2}, & n = 0 \\ \frac{2}{\pi n^2}, & n = 1, 3, 5, \ldots, \\ 0, & n = 2, 4, 6, \ldots \end{cases} \qquad \measuredangle X_n = \begin{cases} 0, & n = 0, 2, 4, \ldots \\ -180^\circ, & n = 1, 3, 5, \ldots \end{cases}$$

# Fourier Series in Linear System Analysis

**Problem 3.37**

The system transfer function is given by

$$H(j\omega) = \frac{1}{(j\omega)^2 + 2(j\omega) + 3} = \frac{1}{\sqrt{(3-\omega^2)^2 + 4\omega^2}} \measuredangle -\tan^{-1}\left(\frac{2\omega}{3-\omega}\right), \qquad H(0) = \frac{1}{3}$$

The periodic input in FIGURE 3.19 has the Fourier series coefficients determined in Problem 3.1 as

$$a_0 = 1, a_n = 0, n = 1, 2, \ldots, \quad b_n = -\frac{2}{Tn\omega_0} = -\frac{1}{n\pi} \quad \Rightarrow \quad X_n(jn\omega_0) = 0.5(a_n - jb_n) = \begin{cases} 1, & n = 0 \\ \frac{j}{2n\pi}, & n = 1, 2, \ldots \end{cases}$$

The complex Fourier series coefficients of the system output are

$$Y_n(jn\omega_0) = H(jn\omega_0)X_n(jn\omega_0) = \begin{cases} H(0) = \frac{1}{3}, & n = 0 \\ H(jn\omega_0)\frac{j}{2n\pi} = \frac{1}{\left(3-(n\omega_0)^2+j2n\omega_0\right)}\frac{j}{2n\pi}, & n = 1, 2, \ldots \end{cases}$$

whose magnitude and phase are obtained as

$$|Y_n(jn\omega_0)| = \begin{cases} \frac{1}{3}, & n = 0 \\ \frac{1}{2n\pi}\frac{1}{\sqrt{\left(3-(n\omega_0)^2\right)^2+(2n\omega_0)^2}}, & n = 1, 2, \ldots \end{cases}$$

$$\measuredangle Y_n(jn\omega_0) = \begin{cases} 0, & n = 0 \\ \frac{\pi}{2} - \tan^{-1}\left(\frac{2n\omega_0}{3-(n\omega_0)^2}\right), & n = 1, 2, .. \end{cases}$$

The steady state system response is

$$y_{ss}(t) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{n\pi} \frac{1}{\sqrt{\left(3-(n\omega_0)^2\right)^2 + (2n\omega_0)^2}} \cos\left(n\omega_0 t + \frac{\pi}{2} - \tan^{-1}\left(\frac{2n\omega_0}{3-(n\omega_0)^2}\right)\right)$$

**Answer 3.38:**

$$X_n = \frac{jE}{2n\pi}((-1)^n - 1), \quad n = 1, 2, \ldots, \quad X_0 = \frac{E}{2}$$

$$Y_n(jn\omega_0) = H(jn\omega_0)X_n(jn\omega_0) = \begin{cases} H(0)\frac{E}{2} = \frac{E}{6}, & n = 0 \\ H(jn\omega_0)\frac{jE}{2n\pi}((-1)^n - 1) = \frac{((-1)^n-1)}{\left(3-(n\omega_0)^2+j2n\omega_0\right)}\frac{jE}{2n\pi}, & n = 1, 2, \ldots \end{cases}$$

$$|Y_n(jn\omega_0)| = \begin{cases} \frac{E}{6}, & n = 0 \\ \frac{E((-1)^n-1)}{2n\pi}\frac{1}{\sqrt{\left(3-(n\omega_0)^2\right)^2+(2n\omega_0)^2}}, & n = 1, 2, \ldots \end{cases}$$

$$\measuredangle Y_n(jn\omega_0) = \begin{cases} 0, & n = 0 \\ \frac{\pi}{2} - \tan^{-1}\left(\frac{2n\omega_0}{3-(n\omega_0)^2}\right), & n = 1, 2, .. \end{cases}$$

$$y_{ss}(t) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{E((-1)^n - 1)}{n\pi} \frac{1}{\sqrt{\left(3-(n\omega_0)^2\right)^2 + (2n\omega_0)^2}} \cos\left(n\omega_0 t + \frac{\pi}{2} - \tan^{-1}\left(\frac{2n\omega_0}{3-(n\omega_0)^2}\right)\right)$$

**Answer 3.42:**

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)dt = \frac{2}{2}\int_0^1 \sin(n\omega_0 t)dt - \frac{2}{2}\int_1^2 \sin(n\omega_0 t)dt = -\frac{1}{n\omega_0}cos(n\omega_0 t)|_{t=0}^{t=1} + \frac{1}{n\omega_0}cos(n\omega_0 t)|_{t=1}^{t=2}$$

$$= \frac{1}{n\pi}[-\cos(n\pi) + \cos(0) + \cos(2n\pi) - \cos(n\pi)] = \frac{1}{n\pi}[2 - 2\cos(n\pi)] = \frac{2}{n\pi}[1-(-1)^n]$$

$$= \begin{cases} \frac{4}{n\pi}, & n = 1,3,... \\ 0 & n = 2,4,... \end{cases}$$

$X_n(jn\omega_0) = 0.5(a_n - jb_n) = -\frac{2j}{n\pi},\ \ n = 1,3,5,...$ and $X_n(jn\omega_0) = 0,\ n = 0,2,4,...$

$$Y_n(jn\omega_0) = |Y_n(jn\omega_0)|\arg\{Y_n(jn\omega_0)\} = H(jn\omega_0)X_n(jn\omega_0)$$
$$= |H(jn\omega_0)|\measuredangle\{H(jn\omega_0)\}|X_n(jn\omega_0)|\measuredangle\{X_n(jn\omega_0)\},\quad n = 1,3,5,...$$

$$|Y_n(jn\omega_0)| = |H(jn\omega_0)||X_n(jn\omega_0)| = \frac{1}{\sqrt{1+n^2\pi^2}}\frac{2}{n\pi}$$
$$\measuredangle\{Y_n(jn\omega_0)\} = \measuredangle\{H(jn\omega_0)\} + \measuredangle\{X_n(jn\omega_0)\} = -\tan^{-1}(n\pi) - \frac{\pi}{2},\quad n = 1,3,5,...$$

$$y_{ss}(t) = 2\sum_{n=1}^{\infty}|Y_n(jn\omega_0)|\cos(n\omega_0 t + \measuredangle\{Y_n(jn\omega_0)\}) = 2\sum_{n=1}^{\infty}\frac{1}{\sqrt{1+n^2\pi^2}}\frac{2}{n\pi}\cos\left(n\pi t - \tan^{-1}(n\pi) - \frac{\pi}{2}\right)$$

**Answer 3.43:**

$$X_n(jn\omega_0) = -\left(-j\frac{2}{n\pi}\right) = j\frac{2}{n\pi},\quad n = 1,3,5,...,\quad T = 2,\ \omega_0 = \pi$$

$$Y_n(jn\omega_0) = H(jn\omega_0)X_n(jn\omega_0),\qquad H(j\omega) = \frac{1+j\omega}{j\omega(2+j\omega)}$$

$$|H(j\omega)| = \frac{|1+j\omega|}{|j\omega||2+j\omega|} = \frac{\sqrt{1+\omega^2}}{\omega\sqrt{4+\omega^2}},\quad \measuredangle H(j\omega) = \measuredangle(1+j\omega) - \measuredangle j\omega - \measuredangle(2+j\omega) = \tan^{-1}(\omega) - \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{2}\right)$$

$$|Y_n(jn\omega_0)| = \frac{2}{n\pi}\frac{\sqrt{1+n^2\pi^2}}{(n\pi)\sqrt{4+n^2\pi^2}},\quad \measuredangle Y_n(jn\omega_0) = \measuredangle H(jn\omega_0) + \measuredangle X_n(jn\omega_0) = \tan^{-1}(n\pi) - \tan^{-1}\left(n\frac{\pi}{2}\right) - \frac{\pi}{2} + \frac{\pi}{2}$$

$$y_{ss}(t) = 2\sum_{n=1,3,...}^{\infty}\frac{2}{(n\pi)^2}\frac{\sqrt{1+n^2\pi^2}}{\sqrt{4+n^2\pi^2}}\cos\left(n\pi t + \tan^{-1}(n\pi) - \tan^{-1}\left(n\frac{\pi}{2}\right)\right)$$