Homework #11 — Introduction to Feedback Systems — Section 12.1

Problem 12.1

$$M(s) = \frac{Y(s)}{F(s)} = \frac{\frac{K}{s(s+p)}}{1 + \frac{K}{s(s+p)}} = \frac{K}{s^2 + ps + K}$$

The sensitivity of the closed-loop system transfer function with respect to parameter p is given by

$$S^p(s) = \frac{\frac{\Delta M(s)}{M(s)}}{\frac{\Delta p}{p}} = \frac{p}{\Delta p}\frac{\Delta M(s)}{M(s)} = \frac{p}{\Delta p}\frac{1}{M(s)}\left(\frac{K}{s^2 + (p+\Delta p)s + K} - \frac{K}{s^2 + ps + K}\right)$$
$$= \frac{p}{\Delta p}\frac{s^2 + ps + K}{K}\left(\frac{K}{s^2 + (p+\Delta p)s + K} - \frac{K}{s^2 + ps + K}\right) = \frac{p}{\Delta p}\left(\frac{s^2 + ps + K}{s^2 + (p+\Delta p)s + K} - 1\right)$$
$$= \frac{p}{\Delta p}\frac{-(\Delta p)s}{s^2 + (p+\Delta p)s + K} = \frac{-ps}{s^2 + (p+\Delta p)s + K} \approx \frac{-ps}{s^2 + ps + K}$$

In the last step, the approximation is done assuming that Δp is small.

Problem 12.2

The original transfer function $H(s) = 1000/(s+10)$ has the static gain equal to $H(0) = 1000/10 = 100$. The closed-loop system transfer function is

$$M(s) = \frac{Y(s) + \Delta Y(s)}{F(s)} = \frac{100(H(s) \pm \Delta H(s))}{1 + 100(H(s) \pm \Delta H(s))} = \frac{100(10 \pm 5)}{s + 1010 \pm 500}$$

When the disturbance sign is positive, the static gain is

$$M^+(0) = \frac{100(15)}{1510} = 0.9934$$

When the disturbance sign is negative, the static gain is

$$M^-(0) = \frac{100(5)}{510} = 0.9803$$

In both cases the tolerance is less than 1% of the original DC gain (100). Hence, the combined tolerance is less than 2%.

Problem 12.3

The sensitivity function for the unity feedback system defined in Problem 12.2, with the transfer function $H(s) = 1000/(s+10)$, is given by

$$S(s) = \frac{1}{1 + G(s)H(s)} = \frac{1}{1 + \frac{1000}{s+10}} = \frac{s+10}{s+1010}$$

COMMENT: STUDENTS WILL NOT BE ASKED TO PLOT THE SENSITIVITY FUNCTION

Problem 12.4

$$Y(s) = H(s)D(s) + H(s)G_c(s)(F(s) - G(s)Y(s))$$
$$\Rightarrow (1 + H(s)G_c(s)G(s))Y(s) = H(s)D(s) + H(s)G_c(s)F(s)$$
$$\Rightarrow Y(s) = \frac{H(s)G_c(s)}{1 + H(s)G_c(s)G(s)}F(s) + \frac{H(s)}{1 + H(s)G_c(s)G(s)}D(s)$$

For large values of the controller transfer function $G_c(s)$ and/or large values of the feedback element $G(s)$, the disturbance is attenuated on the system output.