

HW#10 — Solutions to Problems from Chapter 8

Problem 8.3

The system transfer function can be written in the form

$$H(s) = \frac{Ks + K}{s^6 + 26s^5 + 465s^4 + 5032s^3 + 17664s^2}$$

Using formulas (8.20) and (8.26), we obtain the state space matrices directly from the system transfer function as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -17674 & -5032 & -465 & -26 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C}^T = \begin{bmatrix} K \\ K \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad D = 0$$

Problem 8.5

(a)

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)^{-1} = \frac{1}{s^2 + 1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}$$

Using the Laplace inverse we obtain

$$\Phi(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \right\} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

(b) The system state response is obtained as follows

$$\begin{aligned} \mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} F(s) + (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}(0) = \frac{1}{s^2 + 1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} F(s) + \frac{1}{s^2 + 1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \left[\frac{\frac{1}{s^2+1}}{s} \right] \frac{1}{s} + \left[\frac{\frac{1}{s^2+1}}{s} \right] = \left[\frac{\frac{1}{(s^2+1)} \frac{1}{s} + \frac{1}{s^2+1}}{\frac{1}{s^2+1} + \frac{s}{s^2+1}} \right] = \left[\frac{\frac{1}{s} - \frac{s}{s^2+1} + \frac{1}{s^2+1}}{\frac{1}{s^2+1} + \frac{s}{s^2+1}} \right] \leftrightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 - \cos(t) + \sin(t) \\ \sin(t) + \cos(t) \end{bmatrix} u(t) \end{aligned}$$

The system output response is equal to $y(t) = \mathbf{C}\mathbf{x}(t) = x_2(t) = \cos(t) + \sin(t)$. In the case that only the output response is required and when $\mathbf{x}(t)$ is not available, the system output response is evaluated as follows

$$\begin{aligned} Y(s) &= H(s)F(s) + \mathbf{C}\Phi(s)\mathbf{x}(0) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} F(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}(0) \\ &= [0 \ 1] \frac{1}{s^2 + 1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} F(s) + [0 \ 1] \frac{1}{s^2 + 1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{s}{(s^2 + 1)} \frac{1}{s} + \frac{s}{s^2 + 1} = \frac{1}{s^2 + 1} + \frac{s}{s^2 + 1} \leftrightarrow y(t) = \cos(t) + \sin(t) \end{aligned}$$

Problem 8.6

Using the Laplace transform we obtain

$$\begin{aligned} e^{\mathbf{A}t} &= \mathcal{L}^{-1} \left\{ (s\mathbf{I} - \mathbf{A})^{-1} \right\} = \mathcal{L}^{-1} \left\{ \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix} \right\} \\ &= \mathcal{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{(s+1)^2} + \frac{1}{s+1} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{-1}{(s+1)^2} + \frac{1}{s+1} \end{bmatrix} \right\} = \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix} \end{aligned}$$

This result can be obtained using the Laplace transform as follows

$$e^{\mathbf{A}t} = \mathcal{L}^{-1} \left\{ (s\mathbf{I} - \mathbf{A})^{-1} \right\} = \mathcal{L}^{-1} \left\{ \begin{bmatrix} s+3 & 0 \\ -5 & s+1 \end{bmatrix}^{-1} \right\} = \mathcal{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{s+3} & 0 \\ \frac{5}{(s+1)(s+3)} & \frac{1}{s+1} \end{bmatrix} \right\} = \begin{bmatrix} \frac{1}{2} (e^{-t} - e^{-3t}) & 0 \\ e^{-t} & e^{-t} \end{bmatrix}$$

Problem 8.7

Using the \mathcal{Z} transform we have

$$\begin{aligned}\mathbf{A}^k &= \mathcal{Z}^{-1}\left\{z(z\mathbf{I} - \mathbf{A})^{-1}\right\} = \mathcal{Z}^{-1}\left\{z\begin{bmatrix} z & -1 \\ 0 & z-0.5 \end{bmatrix}^{-1}\right\} = \mathcal{Z}^{-1}\left\{\frac{z}{z(z-0.5)}\begin{bmatrix} z-0.5 & 1 \\ 0 & z \end{bmatrix}\right\} \\ &= \mathcal{Z}^{-1}\left\{\begin{bmatrix} 1 & \frac{1}{z-0.5} \\ 0 & \frac{z}{z-0.5} \end{bmatrix}\right\} = \begin{bmatrix} \delta[k] & (0.5)^{k-1}u[k] \\ 0 & (0.5)^k u[k] \end{bmatrix}\end{aligned}$$

(b) Using the \mathcal{Z} transform we have

$$\begin{aligned}\mathbf{A}^k &= \mathcal{Z}^{-1}\left\{z(z\mathbf{I} - \mathbf{A})^{-1}\right\} = \mathcal{Z}^{-1}\left\{z\begin{bmatrix} z & 0 \\ -2 & z \end{bmatrix}^{-1}\right\} = \mathcal{Z}^{-1}\left\{\frac{z}{z^2}\begin{bmatrix} z & 0 \\ 2 & z \end{bmatrix}\right\} \\ &= \mathcal{Z}^{-1}\left\{\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}\right\} = \begin{bmatrix} \delta[k] & 0 \\ 2\delta[k-1] & \delta[k] \end{bmatrix}, \quad k \geq 0\end{aligned}$$

Problem 8.11

The system transition matrix is given by

$$\mathbf{A}^k = \mathcal{Z}^{-1}\left\{z(z\mathbf{I} - \mathbf{A})^{-1}\right\} = \begin{bmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{bmatrix} u[k]$$

The system response due to initial conditions is

$$\mathbf{x}[k] = \mathbf{A}^k \mathbf{x}[0] = \begin{bmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3^k \\ 3^k \end{bmatrix}$$

Problem 8.15

The state space form of this system, obtained using the change of variables $x_1[k] = y[k]$, $x_2[k] = y[k+1]$, is given by

$$\begin{aligned}\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (-1)^k u[k], \quad \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} = \begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ y[k] &= [1 \quad 0] \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} = x_1[k]\end{aligned}$$

The system response in the frequency domain is

$$\begin{aligned}Y(z) &= \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1} z \mathbf{x}[0] + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} F(z) = [1 \quad 0] \begin{bmatrix} z & -1 \\ 1 & z \end{bmatrix}^{-1} z \begin{bmatrix} 1 \\ 0 \end{bmatrix} + [1 \quad 0] \begin{bmatrix} z & -1 \\ 1 & z \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{z}{z+1} \\ [1 & 0] \frac{z}{z^2+1} \begin{bmatrix} z & 1 \\ -1 & z \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + [1 & 0] \frac{1}{z^2+1} \begin{bmatrix} z & 1 \\ -1 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{z}{z+1} &= \frac{z^2}{z^2+1} + \frac{z}{(z^2+1)(z+1)} = \frac{z(z^2+z+1)}{(z^2+1)(z+1)}\end{aligned}$$

The time domain response is obtained by taking the inverse \mathcal{Z} transform

$$\begin{aligned}Y(z) &= z \left(\frac{k_1}{z+1} + \frac{k_2}{z+j} + \frac{k_2^*}{z-j} \right), \quad k_1 = \frac{1}{2}, \quad k_2 = \frac{1}{4} + j \frac{1}{4} = \frac{\sqrt{2}}{4} e^{j(\frac{\pi}{4})} \\ Y(z) \leftrightarrow y[k] &= \left(\frac{1}{2}(-1)^k + \frac{\sqrt{2}}{2} \cos \left[k \frac{\pi}{2} - \frac{\pi}{4} \right] \right) u[k]\end{aligned}$$

Plugging $k = 0, 1, 2, \dots$ we find that this output response takes the values $1, 0, 0, -1$ that repeat periodically.

Problem 8.17

(a)

$$\begin{aligned}
 e^{\mathbf{A}t} &= \mathcal{L}^{-1}\{\Phi(s)\} = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\} = \mathcal{L}^{-1}\left\{\left(s\mathbf{I} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right)^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}\right\} \\
 &= \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}\right\} \\
 &= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ -\frac{1}{s+1} + \frac{2}{s+2} & -\frac{1}{s+1} + \frac{2}{s+2} \end{bmatrix}\right\} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} = \Phi(t) = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix}
 \end{aligned}$$

(b)

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{(s+1)(s+2)} \\ -\frac{1}{(s+1)(s+2)} \end{bmatrix}$$

Note that $(s\mathbf{I} - \mathbf{A})^{-1}$ is calculated in Part (a).

(c)

$$\begin{aligned}
 \mathbf{x}(t) &= \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\mathbf{B}u(\tau)d\tau = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} \phi_{11}(t-\tau) & \phi_{12}(t-\tau) \\ \phi_{21}(t-\tau) & \phi_{22}(t-\tau) \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} d\tau \\
 &= \begin{bmatrix} -\phi_{11}(t) \\ -\phi_{21}(t) \end{bmatrix} + \int_0^t \begin{bmatrix} -\phi_{12}(t-\tau) \\ -\phi_{22}(t-\tau) \end{bmatrix} d\tau = \begin{bmatrix} -2e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} \end{bmatrix} + \int_0^t \begin{bmatrix} -e^{-(t-\tau)} + e^{-2(t-\tau)} \\ e^{-(t-\tau)} - 2e^{-2(t-\tau)} \end{bmatrix} d\tau \\
 &= \begin{bmatrix} -2e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t} \\ -e^{-t} + e^{-2t} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix} = \mathbf{Ix}(t) = \mathbf{Cx}(t) = \mathbf{y}(t)
 \end{aligned}$$

Problem 8.21

(a) The discrete-time system transition matrix in the complex domain is obtained as follows

$$\Phi(z) = (z\mathbf{I} - \mathbf{A})^{-1}z = \frac{z}{(z+1)(z+2)} \begin{bmatrix} z+3 & 1 \\ -2 & z \end{bmatrix} = \begin{bmatrix} \frac{2z}{z+1} - \frac{z}{z+2} & \frac{z}{z+1} - \frac{z}{z+2} \\ -\frac{2z}{z+1} + \frac{2z}{z+2} & -\frac{z}{z+1} + \frac{2z}{z+2} \end{bmatrix}$$

Using the inverse \mathcal{Z} -transform we have

$$\Phi[k] = \begin{bmatrix} 2(-1)^k - (-2)^k & (-1)^k - (-2)^k \\ -2(-1)^k + 2(-2)^k & (-1)^k - (-2)^k \end{bmatrix}$$

(b) The system transfer function is found using the formula

$$H(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = [0 \ 1] \begin{bmatrix} z & -1 \\ 2 & z+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{z-2}{(z+1)(z+2)}$$

(c) The system output response in the complex domain is given by

$$\begin{aligned}
 Y(z) &= \mathbf{C}\Phi(z)\mathbf{x}(0) + H(z)F(z) = [0 \ 1] \begin{bmatrix} \frac{2z}{z+1} - \frac{z}{z+2} & \frac{z}{z+1} - \frac{z}{z+2} \\ -\frac{2z}{z+1} + \frac{2z}{z+2} & -\frac{z}{z+1} + \frac{2z}{z+2} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \frac{z-2}{(z+1)(z+2)} \frac{z}{(z-1)^2} \\
 &= \frac{2z}{z+1} - \frac{2z}{z+2} + \frac{(z-2)z}{(z+1)(z+2)(z-1)^2} = \frac{z(3z^3 - 8z^2 + 8z - 4)}{(z+1)(z+2)(z-1)^2} \\
 &= \left(\frac{23}{12}\right) \frac{z}{z+1} + \left(\frac{76}{9}\right) \frac{z}{z+2} - \left(\frac{7}{6}\right) \frac{z}{z-1} - \left(\frac{1}{6}\right) \frac{z}{(z-1)^2}
 \end{aligned}$$

Applying the inverse \mathcal{Z} -transform we obtain

$$y[k] = \mathcal{Z}^{-1}\{Y(z)\} = \left\{ \frac{23}{12}(-1)^k + \frac{76}{9}(-2)^k - \frac{7}{6}(1)^k - \frac{1}{6}k \right\} u[k]$$