332: 345 EXAM I Solutions – Fall 2004

#1) 5 pts. (A simpler version of HW Problem 1.10)

We use the time invariance principle and first find the solution to

$$\frac{dy(t)}{dt} + y(t) = \sin(t), \quad y(0) = 0, \quad t \ge 0$$

At the end we will shift the solution obtained by 3 time units. The homogeneous solution is obtained from

$$\frac{dy_h(t)}{dt} + y_h(t) = 0 \quad \Rightarrow \quad y_h(t) = Ce^{-t}$$

The particular solution satisfies

$$\frac{dy_p(t)}{dt} + y_p(t) = \sin(t) \quad \Rightarrow \quad y_p(t) = \alpha \sin(t) + \beta \cos(t)$$

Plugging this solution into the differential equation implies $\alpha = 0.5$ and $\beta = -0.5$ so that the particular solution is given by

$$y_p(t) = 0.5\sin(t) - 0.5\cos(t)$$

The system response is given by

$$y(t) = y_h(t) + y_p(t) = Ce^{-t} + 0.5\sin(t) - 0.5\cos(t)$$

Its initial condition produces

$$y(0) = 0 = y_h(0) + y_p(0) = C - 0.5 \quad \Rightarrow \quad C = 0.5 \quad \Rightarrow \quad y_h(t) = 0.5e^{-t}$$

Hence, the system response due to $\sin(t)$ is

$$y(t) = y_h(t) + y_p(t) = 0.5e^{-t} + 0.5\sin(t) - 0.5\cos(t)$$

The system response due to $\sin(t-3)$, by the time invariance, is given by

$$y(t) = y_h(t) + y_p(t) = (0.5e^{-(t-3)} + 0.5\sin(t-3) - 0.5\cos(t-3))u(t-3)$$ **2.5 pts**

The system zero-state response satisfies

$$\frac{dy_{zs}(t)}{dt} + y_{zs}(t) = \sin(t-3), \quad y_{zs}(0) = 0, \quad t \ge 0$$

Using the previously obtained result, the zero-state response is given by

$$y_{zs}(t) = (0.5e^{-(t-3)} + 0.5\sin(t-3) - 0.5\cos(t-3))u(t-3)$$ **1pt**

The zero-input response satisfies

$$\frac{dy_{zi}(t)}{dt} + y_{zi}(t) = 0, \quad y_{zi}(0) = 0, \quad t \ge 0$$

The zero-input response is given by

$$y_{zi}(t) = Ce^{-t} = 0 \quad t \ge 0$$ **0.5pt**

The system response in terms of its zero-state and zero-input components, is given by

$$y(t) = y_{zs}(t) + y_{zi}(t) = (0.5e^{-(t-3)} + 0.5\sin(t-3) - 0.5\cos(t-3))u(t-3)$$

The steady state response is practically obtained for large values of time, that is

$$y_{ss}(t) = y(t), \quad for \quad t \quad large \quad \Rightarrow \quad y_{ss}(t) \approx 0.5\sin(t-3) - 0.5\cos(t-3)$$ **0.5pt**

According to the textbook definition of the transient response, we have

$$y_{tr}(t) = y(t), \quad for \quad t \quad small$$

Using the electrical circuit definition of the transient response we have

$$\bar{y}_{ss}(t) = 0.5(\sin(t-3) - 0.5\cos(t-3))u(t-3), \quad \Rightarrow \quad \bar{y}_{tr}(t) = y(t) - \bar{y}_{ss}(t) = 0.5e^{-(t-3)}u(t-3)$$ **0.5pt**

#2a) 5 pts (Similar to WH Problems 2.20 to 2.22)

(i) 1pt $$(t-1)^2\delta(t-1)=(1-1)\delta(t-1)=0\delta(t-1)=0$$

(ii) 1pt $$\int_{-\infty}^{+\infty}\cos(\pi t)\delta^{(3)}(t-1)dt=(-1)^3\frac{d^3}{dt^3}\{\cos(\pi t)\}_{|t=1}=(-1)^3\pi^3\sin(\pi)=0$$

(iii) 1pt The delta impulse signal is located at $t=2/3$ (within integration limits), hence

$$\int_{-\infty}^{2}\sin(\pi t)\delta(3t-2)dt=\frac{1}{3}\{\sin(\pi t)\}_{|t=2/3}=\frac{1}{3}\sin(\frac{2\pi}{3})=\frac{\sqrt{3}}{6}$$

(iv) 1pt The delta impulse signal is located exactly at the upper integration bound (need a factor of 0.5)

$$\int_{-\infty}^{3}e^{-5t}\sin(3t)\delta(t-3)dt=\frac{1}{2}e^{-5t}\sin(3t)_{|t=3}=\frac{1}{2}e^{-15}\sin(9)$$

(v) 1pt The delta impulse signal is outside of integration limits (at $t=-5$), hence

$$\int_{-3}^{4}f(t)\delta^{(1)}(t+5)dt=\int_{-3}^{4}0dt=0$$

#2b) 5 pts (The rectangular and unit step signals are similar to those from HW Problem 2.26. Plotting graphs of combined signals is similar to HW Problems 2.6, 2.8, 2.30)

$$p_4(2t-5)=\begin{cases}1, & -2\le 2t-5\le 2\\ 0, & \text{otherwise}\end{cases}=\begin{cases}1, & 1.5\le t\le 3.5\\ 0, & \text{otherwise}\end{cases}=p_2(t-2.5)$$ **1pt**

$$u(-4t+2)=\begin{cases}1, & -4t+2\ge 0\\ 0, & \text{otherwise}\end{cases}=\begin{cases}1, & t<0.5\\ 0, & \text{otherwise}\end{cases}=u(-t+0.5)$$ **1pt**

1 pt is awarded for each of the above two signals assuming that their graphs are also correctly plotted.
The correct graph of the signal $r(t-3)$ is worth **0.5 pt**. The combined graph of all three signals is worth **0.5pt**.

The generalized derivative can be found either directly (HW Problem 2.29) or using the graph (Problem 2.30). The result is identical for both approaches. The direct method

$$\frac{Dx(t)}{Dt}=\frac{D}{Dt}(2p_2(t-2.5)-u(-t+0.5)+r(t-3))=2\delta(t-1.5)-2\delta(t-3.5)-(-\delta(t-0.5)+u(t-3))$$ **2 pts**

#3a) 5pts (The Fourier Series for the sawtooth signal are derived in Example 3.2 of the textbook)
This is an odd signal so that $a_n=0, n=0,1,2,....$ The coefficient b_n is obtained from (using the formula for integration given on the exam sheet)

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2}\frac{2E}{T}t\sin(n\omega_0 t)dt=2\int_{-1/2}^{1/2}2t\sin(2\pi nt)dt=8\int_{0}^{1/2}t\sin(2\pi nt)dt=\frac{2(-1)^{n+1}}{n\pi}$$ **4 pts**

The Fourier series are given by

$$x(t)=\sum_{n=1}^{\infty}b_n\sin(n\omega_0 t)=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin(2\pi nt)$$ **1pt**

#3b) 5 pts (Similar to HW problems 3.37, 3.38, 3.42)

$$H(j\omega)=\frac{j\omega}{1+j\omega}=|H(j\omega)|\arg\{H(j\omega)\},\quad |H(j\omega)|\frac{\omega}{\sqrt{1+\omega^2}},\quad \arg\{H(j\omega)\}=\frac{\pi}{2}-\tan^{-1}(\omega)$$

$$X_n(j\omega)=0.5(a_n-jb_n)=-j0.5b_n=j\frac{(-1)^n}{n\pi}=|X_n(j\omega)|\arg\{X_n(j\omega)\}$$

$$\arg\{X_n(j\omega)\}=(-1)^n\frac{\pi}{2},\quad |X_n(j\omega)|=\frac{1}{n\pi}$$ **2 pts**

The system output is periodic with the same period as the input signal and represented by the Fourier series with $Y(jn\omega_0)=H(jn\omega_0)X_n(jn\omega_0)$

$$|Y_n(jn\omega_0)|=|H(jn\omega_0)||X_n(jn\omega_0)|=\frac{n\omega_0}{\sqrt{1+n^2\omega_0^2}}\frac{1}{n\pi},\quad n=1,2,...,\quad \omega_0=\frac{2\pi}{T}=2\pi$$

$$\arg\{Y(jn\omega_0\}=\arg\{H(jn\omega_0\}+\arg\{X_n(jn\omega_0\}=\frac{\pi}{2}-\tan^{-1}(\omega)+(-1)^n\frac{\pi}{2}$$ **2 pts**

The output signal is given by

$$y(t)=\sum_{n=1}^{\infty}|Y_n(jn\omega_0)|\cos(n\omega_0 t+\arg\{Y_n(jn\omega_0)\})$$ **1pt**

#3c) 5 pts. ((i) is taken from HW 3.8e, (ii) from 3.8d, and (iii) from HW 3.10.)

$$(i)\quad F\{\sin(2\pi t)[u_h(t-2)-u_h(t-1)]\}=-F\{\sin(2\pi t)p_1(t-1.5)\}=-\frac{j}{2}\{F_p(j\omega+2\pi)-F_p(j\omega-2\pi)\}$$

$$F_p(j\omega)=F\{p_1(t-1.5)\}=e^{-j1.5\omega}F\{p_1(t)\}=e^{-j1.5\omega}\sin c(\frac{\omega}{2\pi})$$

2pts

$$(ii)\quad F\{t^2e^{-3t}u_h(t)\}=j^2\frac{d^2}{d\omega^2}\{F\{e^{-3t}u_h(t)\}=-\frac{d^2}{d\omega^2}\left(\frac{1}{3+j\omega}\right)=\frac{2}{(3+j\omega)^3}$$

1pt

$$(iii)\quad p_\tau(t)\leftrightarrow\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)\ \Rightarrow\ \tau\,\mathrm{sinc}\left(\frac{t\tau}{2\pi}\right)\leftrightarrow 2\pi p_\tau(-\omega)\ \Rightarrow\ \mathrm{sinc}\left(\frac{t\tau}{2\pi}\right)\leftrightarrow\frac{2\pi}{\tau}p_\tau(\omega)$$

$$\Rightarrow\ \mathrm{sinc}(t)\leftrightarrow p_\tau\left(\frac{\tau\omega}{2\pi}\right)=p_{2\pi}(\omega)=p_1(f)\ \Rightarrow\ \mathrm{sinc}(3t-4)\leftrightarrow\frac{e^{-j\frac{4}{3}\omega}}{3}p_{2\pi}\left(\frac{\tau\omega}{3}\right)$$

2pts

#3d) 3 pts. (Similar to HW Problem 3.24)

$$\mathrm{sinc}\left(\frac{t\tau}{2\pi}\right)\leftrightarrow\frac{2\pi}{\tau}p_\tau(\omega)\ \Rightarrow\ p_\tau(\omega)\leftrightarrow\frac{\tau}{2\pi}\mathrm{sinc}\left(\frac{t\tau}{2\pi}\right)\ \Rightarrow\ p_2(\omega)\leftrightarrow\frac{1}{\pi}\mathrm{sinc}\left(\frac{t}{\pi}\right)$$

$$p_4(\omega)\leftrightarrow\frac{2}{\pi}\mathrm{sinc}\left(\frac{2t}{\pi}\right)$$

$$X(j\omega)=2p_4(\omega)-p_2(\omega-1)\leftrightarrow\frac{4}{\pi}\mathrm{sinc}\left(\frac{2t}{\pi}\right)-e^{jt}\frac{1}{\pi}\mathrm{sinc}\left(\frac{t}{\pi}\right)$$

#3e) 2 pts. (Similar to HW Problems 3.39, 341)

$$y(t) = 5\left|H(j10)\right| \cos(10t + \frac{\pi}{3} + \arg\{H(j10)\}$$

$$H(j\omega) = \frac{j\omega}{1 + j\omega} \quad \Rightarrow \quad \left|H(j10)\right| = \frac{10}{\sqrt{1+10^2}}, \qquad \arg\{H(j10)\} = \frac{\pi}{2} - \tan^{-1}(10)$$

Exams Statistics

Average 13.1 pts (35pts); Max = 32 pts; Min =0.5 pts.
13 students scored 25 pts and above
28 students scored 20 points and above
48 students scored below 10 points
13 students scored less than 6 points (ADVISED TO DROP THE CLASS)

New grading scale $A \ge 87, \quad B^+ \ge 79, \quad B \ge 72, \quad C^+ \ge 64, \quad C \ge 57, \quad D \ge 47$

OFFICE HOURS WHEN YOU CAN SEE AND PICKUP YOUR EXAM I

Wednesday, Nov. 3, 2:30-4:40pm, Nov. 18, Th3, Nov. 22, M3, Nov. 23, 2004.

Please check Exam I Solutions before coming to ELE 222 to get the exam. Once you take the exam from my office you will not be allowed to come back and complain about grading.