

where

$$Q \triangleq \begin{bmatrix} FPh(t_1) & FPh(t_2) & \cdots & FPh(t_\gamma) \\ GPh(t_1) & GPh(t_2) & \cdots & GPh(t_\gamma) \\ FG_1Ph(t_1) & FG_1Ph(t_2) & \cdots & FG_1Ph(t_\gamma) \\ FG_2Ph(t_1) & FG_2Ph(t_2) & \cdots & FG_2Ph(t_\gamma) \\ \vdots & \vdots & \vdots & \vdots \\ FG_qPh(t_1) & FG_qPh(t_2) & \cdots & FG_qPh(t_\gamma) \end{bmatrix}.$$

With the proper choice of the input  $\mathbf{u}(t)$ ,  $Q$  can be guaranteed to be nonsingular. Then

$$\begin{aligned} & [A \ B \ N_1 \ N_2 \ \cdots \ N_q] \\ & = \{F - [\mathbf{x}_0 \ \mathbf{0} \ \cdots \ \mathbf{0}]\} \\ & \quad \times [\mathbf{h}(t_1) \ \mathbf{h}(t_2) \ \cdots \ \mathbf{h}(t_\gamma)]Q^{-1}. \end{aligned} \quad (37)$$

Now, the unknown parameters  $A, B, N_1, \dots, N_q$  are determined.

## V. NUMERICAL EXAMPLES

*Example 1:* Consider the bilinear system of the form (1), where

$$\begin{aligned} A &= \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad B = 0, \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \mathbf{u}(t) &= \exp(-t), \quad \text{and} \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned} \quad (38)$$

If we solve (38) for  $\mathbf{x}(t)$  directly, the analytic solution for  $\mathbf{x}(t)$  can be shown to be

$$\begin{aligned} \mathbf{x}(t) &= \frac{1}{2} \cdot \begin{bmatrix} \exp(-t - \exp(-t) + 1) + \exp(-3t - \exp(-t) + 1) \\ \exp(-t - \exp(-t) + 1) - \exp(-3t - \exp(-t) + 1) \end{bmatrix}. \end{aligned} \quad (39)$$

The comparison between the Haar solution and the analytic solution for  $t \in [0, 8)$  is shown in Fig. 1, which confirms that high accuracy has been obtained from the Haar approach with the integration time step 0.0625.

*Example 2:* Consider a second-order bilinear system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) + N_1\mathbf{x}(t)u_1(t) + N_2\mathbf{x}(t)u_2(t), \quad (40)$$

The response data due to the input  $u_1(t) = \exp(-t)$  and  $u_2(t) = \cos(t)$  are shown in Table II.

Now, assume the parameters  $A, B, N_1$ , and  $N_2$  are unknown. By (37) we have 42. By comparing (41) to (42), shown at the bottom of the previous page, we see that the estimation process is very effective and the result is satisfactory. In this example,  $m = 16$  is used. If a larger value of  $m$  is chosen, a better estimation is expected.

## VI. CONCLUSION

Some fundamental properties on Haar wavelets such as (18), (23), and (26) have been derived and some effective algorithms have been applied to solve the rather difficult bilinear problems successfully. The main contributions should be ascribed to the nice local orthogonal Haar wavelets. The application region beyond bilinear systems can be widely enlarged to include time-varying, nonlinear, stochastic optimum controls, etc. We are fully confident of the future development for the HT method, since the sound base has been established.

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## Reduced-Order $H_\infty$ Optimal Filtering for Systems with Slow and Fast Modes

Myo-Taeg Lim and Zoran Gajic

**Abstract**—In this paper we present a method that allows complete time-scale separation and parallelism of the  $H_\infty$  optimal filtering problem for linear systems with slow and fast modes (singularly perturbed linear systems). The algebraic Riccati equation of singularly perturbed  $H_\infty$  filtering problem is decoupled into two completely independent reduced-order pure-slow and pure-fast  $H_\infty$  algebraic Riccati equations. The corresponding  $H_\infty$  filter is decoupled into independent reduced-order, well-defined pure-slow and pure-fast filters driven by system measurements. The proposed exact closed-loop decomposition technique produces many savings in both on-line and off-line computations and allows parallel processing of information with different sampling rates for slow and fast signals.

**Index Terms**—Filters,  $H_\infty$  optimization, reduced-order systems, singularly perturbed systems.

## I. INTRODUCTION

During the last 15 years the  $H_\infty$  optimization became one of the most interesting and challenging areas of optimal control and filtering. The  $H_\infty$  filter has recently become popular in signal processing (see [1] and [2] and references therein). The main advantage of the  $H_\infty$  optimization is that such controllers and filters are robust with respect to

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internal and external disturbances. The additional advantage of the  $H_\infty$  filter over the standard Kalman filter is that the former does not require knowledge of the system and measurement noise intensity matrices: data hardly exactly known.

In this paper we study the  $H_\infty$  filter for linear systems with slow and fast modes. The standard Kalman filter of singularly perturbed linear systems has been studied in [3] and [4]. The difficulties encountered with the  $H_\infty$  filter of singularly perturbed linear systems are that the corresponding algebraic filter Riccati equation is ill conditioned and contains an indefinite coefficient matrix multiplying the quadratic term, which makes this equation much more difficult for studying than the corresponding one of the standard singularly perturbed optimal Kalman filtering problem.

In [5], the algebraic regulator Riccati equation of the  $H_\infty$  optimal linear regulator problem is decoupled into the reduced-order pure-slow and pure-fast algebraic regulator Riccati equations. In this paper, we first extend the results of [5] to the decomposition of the  $H_\infty$  algebraic filter Riccati equation so that the coefficients of the  $H_\infty$  pure-slow and pure-fast filters can be obtained with very high accuracy (theoretically with perfect accuracy), which is very important due to the fact that  $H_\infty$  filters can be fragile in the sense that very small perturbations in the filter coefficients can destroy the filter's stability, [6]. In that direction, we use duality between optimal linear filters and regulators, which in this case requires some modifications as indicated throughout the paper. In the second part of the paper, we show how to decompose the  $H_\infty$  singularly perturbed filter into independent well-conditioned  $H_\infty$  reduced-order filters. In addition, we explain why the transformation used in the decomposition of the algebraic  $H_\infty$  filter Riccati equation does not decouple the  $H_\infty$  filters (which is the case in the standard Kalman filtering of singularly perturbed linear systems, [4]). The filters obtained are completely independent and can work in parallel. Each of them can process information with a different sampling rate. The fast filter requires a small sampling period and the slow one can process information with a relatively large sampling period.

## II. PROBLEM FORMULATION

Consider the linear singularly perturbed system

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) + D_1 w(t) \\ \epsilon \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) + D_2 w(t) \end{aligned} \quad (1)$$

with the corresponding measurements

$$y(t) = C_1 x_1(t) + C_2 x_2(t) + v(t) \quad (2)$$

where  $x_1(t) \in R^{n_1}$  and  $x_2(t) \in R^{n_2}$  are slow and fast state variables, respectively,  $y(t) \in R^p$  are system measurements,  $w(t) \in R^r$ , and  $v(t) \in R^p$  are system and measurement disturbances.  $A_i$ ,  $i = 1, 2, 3, 4$  and  $C_j$ ,  $D_j$ ,  $j = 1, 2$  are constant matrices of appropriate dimensions.  $\epsilon$  is a small positive singular perturbation parameter which indicates system separation into slow and fast time scales.

In this paper we design a filter to estimate system states  $x_1(t)$  and  $x_2(t)$ . The states to be estimated are given by a linear combination

$$z(t) = G_1 x_1(t) + G_2 x_2(t). \quad (3)$$

The estimation problem is to obtain an estimate  $\hat{z}(t)$  of  $z(t) \in R^q$  using the measurements  $y(t)$  [2], [7], [8]. The measure of the infinite horizon estimation problem is defined as a disturbance attenuation function

$$J = \frac{\int_0^\infty \|z(t) - \hat{z}(t)\|_R^2 dt}{\int_0^\infty (\|w(t)\|_{W^{-1}}^2 + \|v(t)\|^2) dt} \quad (4)$$

where  $R \geq 0$  and  $W > 0$  are weighting matrices to be chosen by designers. The  $H_\infty$  filter is to ensure that the energy gain from the disturbances to the estimation errors  $z(t) - \hat{z}(t)$  is less than a prespecified level  $\gamma^2$ . That is,

$$\sup_{w,v} J < \gamma^2 \quad (5)$$

where  $\sup$  stands for supremum and  $\gamma^2$  is a prescribed level of noise attenuation. The  $H_\infty$  filter of (1)–(2) is given by [2]

$$\begin{aligned} \dot{\hat{x}}_1(t) &= A_1 \hat{x}_1(t) + A_2 \hat{x}_2(t) + K_1 \nu(t) \\ \epsilon \dot{\hat{x}}_2(t) &= A_3 \hat{x}_1(t) + A_4 \hat{x}_2(t) + K_2 \nu(t) \\ \nu(t) &= y(t) - C_1 \hat{x}_1(t) - C_2 \hat{x}_2(t) \end{aligned} \quad (6)$$

where the filter gains  $K_1$  and  $K_2$  are obtained from

$$K_1 = P_1 C_1^T + P_2 C_2^T, \quad K_2 = \epsilon P_2^T C_1^T + P_3 C_2^T \quad (7)$$

with matrices  $P_1$ ,  $P_2$ , and  $P_3$  representing the positive definite stabilizing solution of the following algebraic Riccati equation [2], [7]:

$$AP + PA^T - P \left( C^T C - \frac{1}{\gamma^2} G^T R G \right) P + D W D^T = 0 \quad (8)$$

where

$$\begin{aligned} A &= \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon} A_3 & \frac{1}{\epsilon} A_4 \end{bmatrix} & D &= \begin{bmatrix} D_1 \\ \frac{1}{\epsilon} D_2 \end{bmatrix} \\ P &= \begin{bmatrix} P_1 & P_2 \\ P_2^T & \frac{1}{\epsilon} P_3 \end{bmatrix} & C &= [C_1 \quad C_2] & G &= [G_1 \quad G_2]. \end{aligned} \quad (9)$$

In the following, we will achieve the slow-fast  $H_\infty$  filter decomposition in which both filters will be independent and directly driven by the system measurements and thus we will eliminate the need for communication of estimates.

## III. DECOMPOSITION OF THE $H_\infty$ FILTER ALGEBRAIC RICCATI EQUATION

Consider the optimal closed-loop filter (6) driven by the system measurements

$$\begin{aligned} \dot{\hat{x}}_1(t) &= (A_1 - K_1 C_1) \hat{x}_1(t) + (A_2 - K_1 C_2) \hat{x}_2(t) + K_1 y(t) \\ \epsilon \dot{\hat{x}}_2(t) &= (A_3 - K_2 C_1) \hat{x}_1(t) + (A_4 - K_2 C_2) \hat{x}_2(t) + K_2 y(t) \end{aligned} \quad (10)$$

with the optimal filter gains  $K_1$  and  $K_2$  defined in (7)–(9). By duality between the optimal filter and regulator, the  $H_\infty$  filter algebraic Riccati equation (8) can be solved by using the same decomposition method (with some small modifications) as the one used for solving the corresponding  $H_\infty$  regulator algebraic Riccati equation [5]. By invoking the results from [4] and [5] and using duality between the optimal  $H_\infty$  linear-quadratic controllers and optimal  $H_\infty$  filters the following matrices have to be formed:

$$\begin{aligned} T_1 &= \begin{bmatrix} A_1^T & - \left( C_1^T C_1 - \frac{1}{\gamma^2} G_1^T R G_1 \right) \\ -D_1 W D_1^T & -A_1 \end{bmatrix} \\ T_2 &= \begin{bmatrix} A_3^T & - \left( C_1^T C_2 - \frac{1}{\gamma^2} G_1^T R G_2 \right) \\ -D_1 W D_2^T & -A_2 \end{bmatrix} \\ T_3 &= \begin{bmatrix} A_2^T & - \left( C_2^T C_1 - \frac{1}{\gamma^2} G_2^T R G_1 \right) \\ -D_2 W D_1^T & -A_3 \end{bmatrix} \\ T_4 &= \begin{bmatrix} A_4^T & - \left( C_2^T C_2 - \frac{1}{\gamma^2} G_2^T R G_2 \right) \\ -D_2 W D_2^T & -A_4 \end{bmatrix}. \end{aligned} \quad (11)$$

It can be shown after some algebra that matrices  $(T_1, T_2, T_3, T_4)$  comprise the system matrix of a standard singularly perturbed system, namely

$$\begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ \frac{1}{\epsilon}T_3 & \frac{1}{\epsilon}T_4 \end{bmatrix} \begin{bmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix}. \quad (12)$$

Note that in contrast to the results of [5], where the state–costate variables have to be partitioned as  $x^T = [x_1^T \ x_2^T]$  and  $p^T = [p_1^T \ \epsilon p_2^T]$ , in the case of the dual filter variables we must use the following partitions  $x^T = [x_1^T \ \epsilon x_2^T]$  and  $p^T = [p_1^T \ p_2^T]$  (duality modification). Since matrices  $T_1, T_2, T_3$ , and  $T_4$  correspond to the system matrices of a singularly perturbed linear system, the slow–fast decomposition of (12) can be achieved by using the Chang decoupling equations [9] of the form

$$\begin{aligned} T_4 M - T_3 - \epsilon M(T_1 - T_2 M) &= 0 \\ -N(T_4 + \epsilon M T_2) + T_2 + \epsilon(T_1 - T_2 M)N &= 0. \end{aligned} \quad (13)$$

The unique solutions of algebraic equations (13) exist by the implicit function theorem, for  $\epsilon$  sufficiently small, under the assumption that the matrix  $T_4$  is nonsingular. The solutions of the above equations can be easily obtained in terms of linear algebraic equations by using either the Newton method or the fixed point iterations with the initial conditions given by  $M^{(0)} = M + O(\epsilon) = T_4^{-1}T_3$  and  $N^{(0)} = N + O(\epsilon) = T_2 T_4^{-1}$ . Using the results of [10] and duality between  $H_\infty$  optimal linear-quadratic regulators and  $H_\infty$  optimal filters, it follows that the matrix  $T_4$  is nonsingular under the following assumption.

*Assumption 1:* The triple  $(A_4, C_2, D_2)$  is controllable observable.

The Chang decoupling transformation corresponding to (12) and (13) is given by [9]

$$\mathbf{T} = \begin{bmatrix} I - \epsilon N M & -\epsilon N \\ M & I \end{bmatrix}. \quad (14)$$

From the results of [5], we have

$$P = \left( \Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \right) \left( \Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \right)^{-1} \quad (15)$$

where the pure-slow and pure-fast well-conditioned reduced-order algebraic  $H_\infty$  filter Riccati equations are given by

$$\begin{aligned} P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s &= 0 \\ P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f &= 0 \end{aligned} \quad (16)$$

with

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = T_1 - T_2 M, \quad \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = T_4 + \epsilon M T_2. \quad (17)$$

The  $\Omega_i, i = 1, 2, 3, 4$ , matrices in (15) are

$$\begin{aligned} \Omega &= \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_1^{-1} \mathbf{T}^{-1} E_2 \\ &= E_1^{-1} \begin{bmatrix} I & \epsilon N \\ -M & I - \epsilon M N \end{bmatrix} E_2. \end{aligned} \quad (18)$$

The permutation matrices dual to those from [5] (note that  $E_1$  is different than the corresponding one from [5], which is another duality modification) are given by

$$E_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & \frac{1}{\epsilon} I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix}$$

$$E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix}. \quad (19)$$

Since

$$\begin{aligned} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} &= T_1 - T_2 M = T_1 - T_2 (M^{(0)} + O(\epsilon)) \\ &= T_1 - T_2 T_4^{-1} T_3 + O(\epsilon) \\ T_s + O(\epsilon) &= \begin{bmatrix} A_s^T & -\left(C_s^T C_s - \frac{1}{\gamma^2} G_s^T R_s G_s\right) \\ -D_s W_s D_s^T & -A_s \end{bmatrix} \\ &+ O(\epsilon) \end{aligned} \quad (20)$$

and

$$\begin{aligned} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} &= T_4 + \epsilon M T_2 = T_4 + O(\epsilon) \\ &= \begin{bmatrix} A_4^T & -\left(C_2^T C_2 - \frac{1}{\gamma^2} G_2^T R G_2\right) \\ -D_2 W D_2^T & -A_4 \end{bmatrix} \\ &+ O(\epsilon). \end{aligned} \quad (21)$$

it follows that by perturbing the coefficients of the algebraic Riccati equations (16) we get the following  $H_\infty$  symmetric algebraic filter Riccati equations:

$$\begin{aligned} P_s^{(0)} A_s^T + A_s P_s^{(0)} + D_s W_s D_s^T \\ - P_s^{(0)} \left( C_s^T C_s - \frac{1}{\gamma^2} G_s^T R_s G_s \right) P_s^{(0)} &= 0 \end{aligned} \quad (22)$$

$$\begin{aligned} P_f^{(0)} A_4^T + A_4 P_f^{(0)} + D_2 W D_2^T \\ - P_f^{(0)} \left( C_2^T C_2 - \frac{1}{\gamma^2} G_2^T R G_2 \right) P_f^{(0)} &= 0. \end{aligned} \quad (23)$$

An important feature of (22) and (23), which distinguishes these equations from the standard algebraic filter Riccati equation, is that the quadratic terms have indefinite coefficient matrices. The algorithm of [11], developed for solving the  $H_\infty$  algebraic Riccati equations in terms of Lyapunov iterations, converges globally to the unique positive definite stabilizing solution of (23) under Assumption 1. The same algorithm finds the unique positive definite stabilizing solution of (22) under the following assumption.

*Assumption 2:* The triple  $(A_s, C_s, \sqrt{D_s W_s D_s^T})$  is controllable–observable.

The matrix  $D_s W_s D_s^T$  can be obtained from (20) by compatibly partitioning the matrix  $T_s$ . The matrix  $C_s$  can be found analytically by using the procedure of [12] as follows:

$$C_s^T = C_0^T \left( I + C_2 A_4^{-1} D_2 W D_2^T A_4^{-T} C_2^T \right)^{-1/2}. \quad (24)$$

Note that this expression requires invertibility of matrix  $A_4$ , which is the standard assumption in theory of singularly perturbed linear systems, [14].

Equations (22) and (23) can be also solved with the MATLAB package by using the Schur method.

The existence of the unique solutions of (16) is guaranteed by the implicit function theorem, [15], since their  $O(\epsilon)$  perturbations are uniquely obtained from (22) and (23).

IV. DECOMPOSITION OF THE  $H_\infty$  FILTER

It is interesting to point out that for the singularly perturbed Kalman filtering problem, the transformation that relates the old and new coordinates given by [4]

$$\Gamma = (\Pi_1 + \Pi_2 P) \quad (25)$$

where

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = E_2^T \begin{bmatrix} I - \epsilon N M & -\epsilon N \\ M & I \end{bmatrix} E_1 \quad (26)$$

is used to decouple both the algebraic filter Riccati equation and the Kalman filter into independent pure-slow and pure-fast components [4]. However, in the case of the  $H_\infty$  filtering the similarity transformation

$$\begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} = \Gamma^{-1} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \quad (27)$$

does not produce in the new coordinates the optimal pure-slow and optimal pure-fast filters, that is

$$\begin{bmatrix} \dot{\hat{\eta}}_s(t) \\ \dot{\hat{\eta}}_f(t) \end{bmatrix} = \Gamma^{-1} \begin{bmatrix} A_1 - K_1 C_1 & A_2 - K_1 C_2 \\ \frac{1}{\epsilon}(A_3 - K_2 C_1) & \frac{1}{\epsilon}(A_4 - K_2 C_2) \end{bmatrix} \times \Gamma \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} + \Gamma^{-1} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} y(t) \quad (28)$$

does not lead to a block diagonal filter matrix in the new coordinates. The reason for this inconsistency lies in the fact that the closed-loop  $H_\infty$  filtering problem matrix is

$$A - P \left( C^T C - \frac{1}{\gamma^2} G^T R G \right) = A - K C - \frac{1}{\gamma^2} P G^T R G. \quad (29)$$

This matrix is indeed block diagonalized by the similarity transformation  $\Gamma$ . However, the  $H_\infty$  optimal filter defined in (10) has the feedback matrix given by

$$\begin{aligned} A - P C^T C &= A - K C \\ &= \begin{bmatrix} A_1 - K_1 C_1 & A_2 - K_1 C_2 \\ \frac{1}{\epsilon}(A_3 - K_2 C_1) & \frac{1}{\epsilon}(A_4 - K_2 C_2) \end{bmatrix} \end{aligned} \quad (30)$$

This singularly perturbed matrix can be diagonalized by using another Chang transformation of the form

$$\mathbf{T}_F = \begin{bmatrix} I - \epsilon H L & -\epsilon H \\ L & I \end{bmatrix}, \quad \mathbf{T}_F^{-1} = \begin{bmatrix} I & \epsilon H \\ -L & I - \epsilon L H \end{bmatrix} \quad (31)$$

where  $L$  and  $H$  matrices satisfy the Chang decoupling equations

$$\begin{aligned} (A_4 - K_2 C_2)L - (A_3 - K_2 C_1) - \epsilon[(A_1 - K_1 C_1) \\ - (A_2 - K_1 C_2)L] &= 0 \\ -H(A_4 - K_2 C_2) + (A_2 - K_1 C_2) - \epsilon H L(A_2 - K_1 C_2) \\ + \epsilon[(A_1 - K_1 C_1) - (A_2 - K_1 C_2)L]H &= 0. \end{aligned} \quad (32)$$

The unique solutions of these equations exist under the assumption that the matrix  $A_4 - K_2 C_2$  is nonsingular. Note that based on theory of singular perturbations [14], the matrix  $A_4 - P_3 C_2^T C_2 - (1/\gamma^2) P_3 G_2^T R G_2$  is nonsingular since it represents the fast feedback matrix. By the result from [8], the stability of the matrix  $A_4 - P_3 C_2^T C_2 - (1/\gamma^2) P_3 G_2^T R G_2$  implies that the matrix  $A_4 - P_3 C_2^T C_2$  is nonsingular also. Using (7) we see that  $A_4 - K_2 C_2 + O(\epsilon)$  is a stable matrix. Thus, the matrix  $A_4 - K_2 C_2$  is stable for sufficiently small values of the small singular perturbation parameter  $\epsilon$ . The unique solutions of (32) can be easily obtained either

by using the Newton method or the fixed point iterations starting with the following initial conditions:

$$\begin{aligned} L^{(0)} &= (A_4 - K_2 C_2)^{-1} (A_3 - K_2 C_1) \\ M^{(0)} &= (A_2 - K_1 C_2)(A_4 - K_2 C_2)^{-1}. \end{aligned} \quad (33)$$

Hence, the application of the following similarity transformation:

$$\begin{bmatrix} \hat{\zeta}_s(t) \\ \hat{\zeta}_f(t) \end{bmatrix} = \mathbf{T}_F^{-1} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \quad (34)$$

produces in the new coordinates the  $H_\infty$  optimal pure-slow and optimal pure-fast reduced-order filters, that is,

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\zeta}}_s(t) \\ \dot{\hat{\zeta}}_f(t) \end{bmatrix} &= \mathbf{T}_F^{-1} \begin{bmatrix} A_1 - K_1 C_1 & A_2 - K_1 C_2 \\ \frac{1}{\epsilon}(A_3 - K_2 C_1) & \frac{1}{\epsilon}(A_4 - K_2 C_2) \end{bmatrix} \\ &\quad \times \mathbf{T}_F \begin{bmatrix} \hat{\zeta}_s(t) \\ \hat{\zeta}_f(t) \end{bmatrix} + \mathbf{T}_F^{-1} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} y(t) \\ \begin{bmatrix} \dot{\hat{\zeta}}_s(t) \\ \dot{\hat{\zeta}}_f(t) \end{bmatrix} &= \begin{bmatrix} a_s & 0 \\ 0 & \frac{1}{\epsilon} a_f \end{bmatrix} \begin{bmatrix} \hat{\zeta}_s(t) \\ \hat{\zeta}_f(t) \end{bmatrix} + \begin{bmatrix} K_s \\ \frac{1}{\epsilon} K_f \end{bmatrix} y(t) \end{aligned} \quad (35)$$

with the pure-slow and pure-fast filter gains given by

$$\begin{bmatrix} K_s \\ \frac{1}{\epsilon} K_f \end{bmatrix} = \mathbf{T}_F^{-1} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix}. \quad (36)$$

Using the expression for the similarity transformation defined in (31) we can obtain analytical expressions for  $a_s$ ,  $a_f$ ,  $K_s$ ,  $K_f$  as follows:

$$\begin{aligned} a_s &= (A_1 - K_1 C_1) - (A_2 - K_1 C_2)L \\ a_f &= (A_4 - K_2 C_2) + \epsilon L(A_2 - K_1 C_2) \\ K_s &= K_1 - H K_2 - \epsilon H L K_1 \\ K_f &= K_2 + \epsilon L K_1. \end{aligned} \quad (37)$$

The reduced-order independent pure-slow and pure-fast filtering equations (35) represent the main result of this paper. Due to complete independence of the slow and fast filters, the slow and fast signals can be now processed with different sampling rates. In contrast, the original full-order filter (10) requires the fast sampling rate for processing of both the slow and fast signals.

## V. CONCLUSION

An approach to solve the  $H_\infty$  filtering problem for linear systems with slow and fast modes is proposed. The completely independent reduced-order pure-slow and pure-fast  $H_\infty$  filters driven by the system measurements are obtained. The proposed method allows independent and parallel processing of information in slow and fast time scales.

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## Bifurcations and Chaos in the Tolerance Band PWM Technique

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**Abstract**—In this paper we study the dynamical behavior of the tolerance band PWM technique, which is used in controlling power electronic ac inverters and dc–dc converters. We demonstrate numerically as well as experimentally the existence of three basic modes: quasiperiodic, chaotic, and square wave mode. We also observe saddle node bifurcation as the cause of boundary crises with transient chaos, merging crisis following symmetry breaking bifurcations and interior crisis. The critical value of the amplitude of external reference for saddle node bifurcation is evaluated analytically by the Tsytkin method.

**Index Terms**—Chaos, hysteresis band method, PWM, tolerance band method.

### I. INTRODUCTION

Most power electronic circuits are controlled by pulse width modulation (PWM) schemes. Various PWM control schemes are now available, each suitable for particular applications. A systematic categorization can be found in [1].

It has been reported that dc-dc converters with the PWM principles of current mode control [2], [3] and duty cycle control [4]–[9] exhibit

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chaotic behavior over a wide range of parameter values. Various nonlinear phenomena and pathways to chaos have been investigated.

In this paper we investigate the dynamics of another PWM technique, the tolerance band method [1]. This PWM technique is generally applied to current controlled ac inverters for variable frequency drives and uninterruptible power supply (UPS) systems. In addition, modern concepts for voltage and current control in three-phase ac inverters with decoupling of state space variables include this method [10]. Therefore, the nonlinear dynamics of the tolerance band PWM technique is of importance to the engineering community.

For the tolerance band method, no time base exists. The switching action is controlled by the upper and lower threshold voltage of an on–off controller with hysteresis. One of the system variables is compared with the tolerance band around the reference waveform. If the state variable tends to go above or below the tolerance band, appropriate switching action takes place and the variable is forced to follow the reference waveform within the tolerance band.

Depending on the control strategy the feedback can be generated by the load-voltage, the load-current or the inverter output current. We study the load-voltage feedback with resistive load  $R$ , as shown in Fig. 1. The variable frequency is produced by variation of the frequency of the sinusoidal reference signal  $W(\tau)$ . To eliminate the higher harmonics of the load current the LC filter is used.

Dc-dc converters can also use this principle, where the sinusoidal reference is changed to a constant value.

### II. MODEL OF THE SYSTEM, DIFFERENT EQUATION, AND THE POINCARÉ MAP

From the point of view of control engineering the system in Fig. 1 shows a nonautonomous relay connected to a linear plant of second order. Because the switching characteristic of the controller and the linearity of the plant, the nonlinearity can be reduced to piecewise linearity. This allows analytic integration of the differential equation in each piece. The dependence of the signals on time is described by the normalized time variable  $\tau$ , given by  $\tau = \omega_0 t$ , where  $\omega_0 = (\sqrt{LC})^{-1}$  is the resonant frequency of the linear system of second order. Also, the frequency of the sinusoidal drive signal  $\omega_w$  is normalized to  $\Omega$  when the time  $t$  is substituted, i.e.,  $\omega_w t = (\omega_w/\omega_0)\tau = \Omega\tau$ . Furthermore, the correcting variable of the controller  $u(\tau)$ , the output voltage  $x(\tau)$ , and the drive signal  $w(\tau)$  and the corresponding amplitude  $a$ , the error signal  $e(\tau)$ , and the hysteresis or the width of the tolerance band of the controller  $h$  are normalized to the maximum value of the on–off controller output voltage  $u_m$

$$\begin{aligned} U(\tau) &= \frac{u(\tau)}{2u_m} = \pm \frac{1}{2}, & X(\tau) &= \frac{x(\tau)}{2u_m} \\ W(\tau) &= \frac{w(\tau)}{2u_m}, & E(\tau) &= \frac{e(\tau)}{2u_m} \\ A &= \frac{a}{2u_m}, & H &= \frac{h}{2} \frac{1}{2u_m}. \end{aligned} \quad (1)$$

In normalized notation, the expression  $2H$  is the width of the tolerance band.  $H$  represents the threshold value where the switching action takes place. The piecewise linear differential equation in normalized notation is as follows:

$$X''(\tau) + 2DX'(\tau) + X(\tau) = U(E(\tau)) \quad (2)$$

where

$$\begin{aligned} E(\tau) &= W(\tau) - X(\tau) \\ W(\tau) &= A \cos[\Omega\tau + \Phi] \end{aligned}$$