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General Transformation for Block Diagonalization of Weakly Coupled Linear Systems Composed of N-Subsystems

Z. Gajic and I. Borno

***Abstract*—A transformation is introduced for exact decomposition (block-diagonalization) of linear weakly coupled systems composed of N subsystems. This transformation can also be used for block diagonalization of block-diagonally dominant matrices and, under certain assumptions, it can be applied for block diagonalization of nearly completely decomposable Markov chains. A twelfth-order real-world power system example is included to demonstrate the efficiency of the proposed method.**

***Index Terms*—Block diagonalization, decoupling, large scale systems, linear systems, weak coupling.**

I. Introduction

The linear weakly coupled systems were introduced to the control audience in [1] and since then have been studied by many control researchers (see [2] and [3] and references therein). In addition, the weakly coupled systems have been studied in mathematics [5]–[7], economics [8], [9], and power system engineering [10]–[12] under the name of block diagonally dominant matrices and block diagonally dominant systems. In addition, weak coupling linear structures also appear in nearly completely decomposable continuous- and discrete-time Markov chains [13]–[15].

A decoupling transformation that exactly decomposes weakly coupled linear systems composed of two subsystems into independent subsystems was introduced in [16]. In this paper we extend the results of [16] and [17] to the general case of linear weakly coupled systems composed of N subsystems and establish conditions under which such a transformation is feasible. The estimate of the rate of convergence of the corresponding algorithm used for decomposition of N weakly coupled linear subsystems is given and compared to the case of two weakly coupled linear subsystems.

Consider a continuous-time linear system consisting of n states clustered into N groups of strongly interacting states. Weak interactions among different groups are expressed in terms of a small perturbation parameter ϵ. The dynamics of such systems are represented by the differential equation

$$\frac{dx(t)}{dt} = Ax(t) \tag{1}$$

where x is the n-dimensional state vector partitioned consistently with N subsystems as

$$x(t) = [x_1^T(t) \quad x_2^T(t) \quad \cdots \quad x_N^T(t)]^T$$

with $dim\{x_i\} = n_i$. The constant matrix A is partitioned as

$$A = \begin{bmatrix} A_{11} & \epsilon A_{12} & \cdots & \epsilon A_{1N} \\ \epsilon A_{21} & A_{22} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \epsilon A_{N1} & \epsilon A_{N2} & \cdots & A_{NN} \end{bmatrix} \tag{2}$$

where ϵ is a small parameter. Each block A_{ii} is of dimensions $n_i \times n_i$, hence, $\Sigma_{i=1}^N \ n_i = n$. All elements in matrix A are assumed to be bounded by $O(1)$. Note that (1) and (2) can also be used to model block-diagonally dominant nonlinear systems with bounded weak interconnections. In addition, it is assumed that magnitudes of all the system eigenvalues are bounded by $O(1)$, that is, $|\lambda_j| = O(1), j = 1, 2, \cdots, n$. This implies that the matrices A_{ii} are nonsingular with $det(A_{ii} = O(1), i = 1, 2, \cdots, N$, which is the standard assumption for weakly coupled linear control systems and also corresponds to the block diagonal dominance of the system matrix A. Thus, the main results presented in this paper are valid under the following assumption.

Assumption 1: The magnitudes of the system eigenvalues are bounded by $O(1)$, that is, $|\lambda_j| \leq O(1), j = 1, 2, \cdots, n$, which implies that the matrices $A_{ii}, i = 1, 2, \cdots, N$ are nonsingular with $det A_{ii} = O(1)$.

Note that when this assumption is not satisfied, the system (1), in addition to weak coupling, also displays multiple time scale phenomena (singular perturbations), [14], [18].

In the cases when a linear weakly coupled system is not in its explicit form defined by (1) and (2), one can use the methodology of [19] and [20] in order to achieve the desired weakly coupled structure.

II. Decoupling Transformation

Our goal is to find a transformation that makes the matrix A block diagonal. Consider the following change of the state variables, which represents a generalization of a transformation derived in [17] for two subsystems (see also [3, p. 72] where the results of [17] are reviewed)

$$\eta_i(t) = x_i(t) + \epsilon \sum_{j=1, j\neq i}^{N} L_{ij} x_j(t), \qquad i = 1, \cdots, N. \tag{3}$$

This leads to

$$\dot{\eta}_i(t) = \dot{x}_i(t) + \epsilon \sum_{j=1, j\neq i}^{N} L_{ij} \dot{x}_j(t), \qquad i = 1, 2, \cdots, N. \tag{4}$$

By eliminating $\dot{x}_i(t), \qquad i = 1, 2, \cdots, N$ from (1), (2), that is

$$\dot{x}_i(t) = A_{ii} x_i(t) + \epsilon \sum_{j=1, j\neq i}^{N} A_{ij} x_j(t), \qquad i = 1, 2, \cdots, N \tag{5}$$

Manuscript received October 2, 1998; revised February 15, 1999. This paper was recommended by Associate Editor L. Fortuna.

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Publisher Item Identifier S 1057-7122(00)05051-0.

and using (3), we get

$$\dot{\eta}_i(t) = \left(A_{ii} + \epsilon^2 \sum_{j=1, j\neq i}^{N} L_{ij}A_{ji}\right)\eta_i(t) + \epsilon \sum_{j=1, j\neq i}^{N} \mathcal{F}_{ij}(L_{ij}, \epsilon)x_j(t), \qquad i = 1, 2, \cdots, N \tag{6}$$

where

$$\begin{aligned}\mathcal{F}_{ij}(L_{ij}, \epsilon) = L_{ij}A_{jj} - A_{ii}L_{ij} + A_{ij} &+ \epsilon\left(\sum_{k=1, k\neq i,j}^{N} L_{ik}A_{kj}\right) \\ &- \epsilon^2\left(\sum_{k=1, k\neq i}^{N} L_{ik}A_{ki}\right)L_{ij} \\ &i, j = 1, 2, \cdots, N, \quad i \neq j.\end{aligned} \tag{7}$$

In order to achieve complete decoupling, the matrices L_{ij} must be chosen such that

$$\mathcal{F}_{ij}(L_{ij}, \epsilon) = 0, \quad \forall i, j = 1, 2, \cdots, N. \tag{8}$$

Assuming that (8) is satisfied, we get in the new coordinates a set of completely decomposed N subsystems, that is

$$\dot{\eta}_i(t) = \Omega_i\eta_i(t), \qquad i = 1, 2, \cdots, N \tag{9}$$

with

$$\Omega_i = A_{ii} + \epsilon^2 \sum_{j=1, j\neq i}^{N} L_{ij}A_{ji}, \qquad j = 1, 2, \cdots, N. \tag{10}$$

Let

$$\eta = [\eta_1^T \quad \eta_2^T \quad \cdots \quad \eta_N^T]^T$$

then

$$\dot{\eta}(t) = \Omega\eta(t) \tag{11}$$

where $\Omega = \mathrm{diag}\{\Omega_1, \Omega_2, \cdots, \Omega_N\}$.

The transformation matrix that relates the original weakly coupled linear system and the set of completely decoupled subsystems in the new coordinates is given by

$$\eta(t) = \Gamma x(t) \tag{12}$$

where

$$\Gamma(\epsilon) = \begin{bmatrix} I & \epsilon L_{12} & \cdots & \epsilon L_{1N} \\ \epsilon L_{21} & I & \cdots & \epsilon L_{2N} \\ \vdots & \cdots & \ddots & \vdots \\ \epsilon L_{N1} & \cdots & \epsilon L_{N(N-1)} & I \end{bmatrix} = I + \epsilon\Psi \tag{13}$$

with the obvious definition of Ψ.

Note that $\Gamma(\epsilon)$ is invertible for sufficiently small values of ϵ. This transformation offers the advantage that it exactly decomposes a high-order linear system into N completely decoupled reduced-order subsystems that can be solved independently. The state of the system in the original coordinates can be determined by the inverse transformation as

$$x(t) = \Gamma^{-1}(\epsilon)\eta(t). \tag{14}$$

The main problem that we are faced with is the solution of the system of algebraic equations (8). This system has the form

$$\begin{aligned}L_{ij}A_{jj} - A_{ii}L_{ij} + A_{ij} + \epsilon\left(\sum_{k=1, k\neq i,j}^{N} L_{ik}A_{kj}\right) \\ - \epsilon^2\left(\sum_{k=1, k\neq i}^{N} L_{ik}A_{ki}\right)L_{ij} = 0, \\ i, j = 1, 2, \cdots, N, \quad i \neq j.\end{aligned} \tag{15}$$

It represents a system of nonlinear algebraic equations. However, the nonlinear (quadratic) terms are nicely multiplied by the squares of the small perturbation parameter ϵ. Solving the system of algebraic equations (15) will be the focus of the next section.

III. Iterative Algorithms

In this section we present iterative algorithms for computing matrices L_{ij} by performing iterations on a set of linear algebraic equations.

Algorithm 1: The first algorithm that can be used to efficiently solve the set of algebraic equations (15) is based on the fixed-point iterations. The algorithm is given in two steps.

Step 1: Set $\epsilon = 0$ in (15) and solve the $O(\epsilon)$ perturbed set of completely decoupled reduced-order algebraic Sylvester equations

$$L_{ij}^{(0)}A_{jj} - A_{ii}L_{ij}^{(0)} + A_{ij} = 0, \qquad i, j = 1, 2, \cdots, N, \quad i \neq j. \tag{16}$$

Equation (16) has a unique solution under the assumption that matrices A_{jj} and A_{ii} have no eigenvalues in common [21], thus, we have to impose the following assumption.

Assumption 2: The matrices A_{jj} and A_{ii} have no eigenvalues in common for every $i, j, i \neq j$.

This step produces an $O(\epsilon)$ approximation for the desired solution, that is, $\|L_{ij}\| = \|L_{ij}^{(0)}\| + O(\epsilon)$. Note that under Assumptions 1 and 2 we have $\|L_{ij}^{(0)}\| = O(1)$ and $\|L_{ij}\| = O(1)$.

Step 2: In order to improve the required solution accuracy up to any arbitrary order, we propose the following fixed-point iteration scheme with $L_{ij}^{(0)}$, obtained in Step 1, playing the role of the initial conditions

$$\begin{aligned}L_{ij}^{(m+1)}A_{jj} - A_{ii}L_{ij}^{(m+1)} + A_{ij} + \epsilon\left(\sum_{k=1, k\neq i,j}^{N} L_{ik}^{(m)}A_{kj}\right) \\ - \epsilon^2\left(\sum_{k=1, k\neq i}^{N} L_{ik}^{(m)}A_{ki}\right)L_{ij}^{(m)} = 0 \\ i, j = 1, 2, \cdots, N, \quad i \neq j; \quad m = 0, 1, 2, \cdots.\end{aligned} \tag{17}$$

Algorithm 1 has the advantage that it operates on the linear decoupled Sylvester's equations to solve the set of nonlinear coupled algebraic equations (15).

The convergence proof of the fixed point algorithm (17) can be obtained under Assumptions 1 and 2 by generalizing the corresponding proofs of [3] and [16] to N subsystems. Note that under Assumptions 1 and 2, the system of nonlinear algebraic equations (15) has unique solutions for sufficiently small values of ϵ since the corresponding Jacobian is nonsingular at $\epsilon = 0$. This also implies that $\|L_{ij}^{(0)}\| = O(1)$. By generalizing the results of [3], [16], it can be established that the rate of convergence of the algorithm (17) is $O(\epsilon)$, hence, $\|L_{ij} - L_{ij}^{(m)}\| = O(\epsilon^m)$, where m is the number of iterations. It is interesting to point out that in the case of algorithms considered in [3] and [16], that is, for $N = 2$ the convergence rate of the corresponding algorithms is much faster, that is, it is equal to $O(\epsilon^{2m})$.

Algorithm 2: Since (16) produces initial guesses that are only $O(\epsilon)$ apart from the exact solutions, it seems that the Newton method is an excellent candidate for solving nonlinear algebraic equations (15). In the following, the Newton algorithm for solving the set of nonlinear coupled algebraic equations (15) is derived. The Newton method is known for its quadratic rate of convergence, hence, this algorithm will converge to the solutions of the algebraic equations (17) faster than the fixed-point Algorithm 1, which has linear convergence. The Newton algorithm can be basically derived by replacing L_{ij} with $L_{ij}^{(m+1)}, \forall i, j$, substituting $L_{ij}^{(m+1)} = L_{ij}^{(m)} + \Delta_{ij}, \forall i, j, i \neq j$ into quadratic terms, and neglecting quadratic terms with respect to Δ_{ij}. This yields the following algorithm:

$$
\begin{aligned}
&L_{ij}^{(m+1)} A_{jj} - \left\{ A_{ii} + \epsilon^2 \left(\sum_{k=1, k\neq i}^{N} L_{ik}^{(m)} A_{ki} \right) \right\} L_{ij}^{(m+1)} \\
&\quad + \epsilon \left(\sum_{k=1, k\neq i,j}^{N} L_{ik}^{(m+1)} A_{kj} \right) \\
&\quad - \epsilon^2 \left(\sum_{k=1, k\neq i}^{N} L_{ik}^{(m+1)} A_{ki} \right) L_{ij}^{(m)} \\
&\quad + A_{ij} + \epsilon^2 \left(\sum_{k=1, k\neq i}^{N} L_{ik}^{(m)} A_{ki} \right) L_{ij}^{(m)} = 0 \\
&\qquad i, j = 1, 2, \cdots, N, \quad i \neq j; \quad m = 0, 1, 2, \cdots.
\end{aligned} \tag{18}
$$

It can be seen that the Newton method leads to a set of linear equations coupled by terms which are $O(\epsilon)$ and $O(\epsilon^2)$. These equations can be solved in terms of the decoupled linear equations by using the fixed-point iterations as in Algorithm 1. Since $\|L_{ij}^{(m+1)} - L_{ij}^{(m)}\| = O(\epsilon)$, we can replace $L_{ij}^{(m+1)}$ in the third and the fourth terms of (19) by $L_{ij}^{(m)}$ without affecting the corresponding fixed point type algorithm which now has the form

$$
\begin{aligned}
&L_{ij}^{(m+1)} A_{jj} - \left\{ A_{ii} + \epsilon^2 \left(\sum_{k=1, k\neq i}^{N} L_{ik}^{(m)} A_{ki} \right) \right\} L_{ij}^{(m+1)} \\
&\quad + A_{ij} + \epsilon \left(\sum_{k=1, k\neq i,j}^{N} L_{ik}^{(m)} A_{kj} \right) = 0 \\
&\qquad i, j = 1, 2, \cdots, N, \quad i \neq j, \quad m = 0, 1, 2, \cdots.
\end{aligned} \tag{19}
$$

This algorithm can be called the hybrid Newton-fixed-point iterations algorithm.

The Newton method requires an initial guess that has to be quite close to the exact solution, otherwise the Newton method does not converge. In such cases when the initial guess is not good (small parameter ϵ is not very small), one has to use the fixed-point iterations Algorithm 1.

Note that the Sylvester equations (16) have unique solutions if the square matrices A_{ii} and A_{jj}, $i, j = 1, 2, \cdots, N, i \neq j$ have one or more common eigenvalues. However, any solution of (16) and, subsequently, any solution of (17) will produce the desired transformation. Namely, the fact that there are several solutions of (15) implies that there are several transformations having the form of (13) that block diagonalize the considered weakly coupled system composed of N subsystems. This is particularly important for nearly decomposable Markov chains for which Assumptions 1 and 2 are not satisfied. For block iterative methods for Markov chains, the reader is referred to [15]. Here we just give the known result for solvability of (16) needed when Assumptions 1 and 2 are not satisfies. The solvability condition of the Sylvester equations is given by the following lemma [22].

Lemma 1: Equation (16) has a solution if and only if matrices

$$
\begin{bmatrix} A_{jj} & 0 \\ 0 & A_{ii} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_{jj} & A_{ji} \\ 0 & A_{ii} \end{bmatrix} \tag{20}
$$

are similar.

IV. Numerical Example

In order to demonstrate the efficiency of the presented method we have run a twelfth-order power system example composed of three machines playing the role of three subsystems. Each machine is modeled as a fourth-order subsystem consisting of a third-order synchronous machine and a first-order exciter regulator system (see [23]).

Matrices $A_{ij}, i, j = 1, 2, 3$, can be found in Delacour *et al.* Their eigenvalues and determinants satisfy both Assumptions 1 and 2 since

$$
\begin{aligned}
\det\{A_{11}\} &= 142.2, \\
\lambda(A_{11}) &= \{-0.0362 \pm j7.4534, -1.3733 \pm j0.8211\}; \\
\det\{A_{22}\} &= 147.5, \\
\lambda(A_{22}) &= \{-0.0241 \pm j7.444\,61, -0.9649 \pm j1.3148\}; \\
\det\{A_{33}\} &= 1381.9, \\
\lambda(A_{33}) &= \{-16.6128, -3.9337, -0.1660 \pm j4.5956\}.
\end{aligned}
$$

Using Algorithm 1 with $\epsilon = 0.01$, we have obtained the results presented in Table I. The accuracy of the solutions obtained is measured by using the MATLAB function `norm` in the following sense error$(m) = \max_{i,j} \quad \|\mathcal{F}_{ij}(L_{ij}^{(m)}, \epsilon)\|$. Note that this error estimate is a conservative measure so that the results presented in Table I are slightly worse

$$
\Omega = \begin{bmatrix}
0.0000 & 1 & -0.2660 & -0.0090 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2.7498 & -2.78 & -1.3601 & -0.0370 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.0011 & 0 & 0.0006 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4.9447 & 0 & -55.5028 & -0.0389 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.21 & 1 & -1.5998 & -0.005 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1.90 & -1.8 & 9.2999 & -0.120 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.00 & 0 & -0.0007 & 1.000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3.10 & 0 & -55.9992 & 0.032 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.197 & 1 & -1.2001 & -0.003 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -54.400 & -20 & 70.1000 & -2.370 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.000 & 0 & 0.0000 & 1.000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.400 & 0 & -20.9994 & -0.017
\end{bmatrix}
$$

TABLE I
ERROR PROPAGATION PER ITERATION

iteration(m)	error(m)
0	6.2458×10^{0}
1	2.5816×10^{-1}
2	7.0832×10^{-3}
3	6.7502×10^{-4}
4	2.6511×10^{-5}
5	7.7792×10^{-7}
6	6.4017×10^{-8}
7	2.1459×10^{-9}
8	1.4050×10^{-10}
9	5.2846×10^{-12}
10	2.5311×10^{-13}

than predicted by the rate of convergence of the presented algorithm. In addition, the matrices A_{11} and A_{22} have a pair of complex conjugate eigenvalues close to each other causing numerical bad-conditioning. In achieving higher order of accuracy we have experienced some problems with MATLAB function `lyap`. However, by using MATLAB function `lyap2` those problems have been eliminated.

The new decoupled system matrix is obtained as shown at the bottom of the previous page.

All elements in this matrix denoted by 0 are zeros with the accuracy of at least 10^{-14}.

V. CONCLUSION

The transformation is introduced for decomposition of weakly coupled linear dynamic controllers, observers, and Kalman filters, that is, for the block-diagonal control and filtering of weakly coupled linear deterministic and stochastic systems. It can be also used to simplify computations of large systems of linear and nonlinear algebraic equations displaying block diagonal dominance. The transformation is very useful for parallel processing of information and computations on parallel computers. In addition, this matrix block diagonalization transformation might simplify many problems of linear algebra, such as the problem of finding matrix eigenvalues.

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