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Reduced-Order Solution to the Finite-Time Optimal-Control Problems of Linear Weakly Coupled Systems

Wu Chung Su and Zoran Gajic

Abstract—**The optimal solution to the finite-time optimal-control problem of weakly coupled linear systems is found in terms of completely decoupled reduced-order differential equations for both the closed-loop and open-loop control. This has been achieved via the use of the decoupling transformation that block diagonalizes the Hamiltonian matrix of the weakly coupled linear-quadratic control problem. The convergence to the optimal solution is pretty rapid. The proposed technique is very well suited for parallel computations.**

I. Introduction

The study of linear weakly coupled control systems originated in [1]. The recursive approach to linear weakly coupled systems, based on the fixed point iterations, has been developed recently. It has been shown that the recursive methods are particularly useful when a coupling parameter ϵ is not extremely small and/or when any desired order of accuracy is required, namely, $O(\epsilon^k)$, where $k = 2, 3, 4, \cdots$, [2]–[7].

The recursive methods of [2]–[7] are based on the fixed point theory applied to the corresponding algebraic equations, so that the results reported in [2]–[7] are applicable to the steady-state control problems only.

In this note, we will study the finite-time optimal-control problem of weakly coupled systems. The solution of this problem is given in terms of differential equations, which makes it more challenging for research. Both the open-loop (linear two point boundary value problem) and the closed-loop (nonlinear differential Riccati equation) optimal control problems will be studied.

The recursive reduced-order solution will be obtained by exploiting the transformation introduced in [5] which will block diagonalize the Hamiltonian form of the solution for the optimal linear-quadratic control problem. Completely decoupled sets of reduced-order differential equations are obtained in both cases: the closed-loop and open-loop control. The convergence to the optimal solution is pretty rapid, due to the fact that the algorithms derived in [5] have the rate of convergence of at least of $O(\epsilon^2)$. This produces a lot of savings in the size of computations required. In addition, the proposed method is very suitable for the parallel computations.

It is interesting to point out that the better results are expected (and obtained) for the open-loop problem since it is less computationally involved than the closed-loop problem (exactly the same sets of differential equations have to be solved, but they differ in the dimensionality).

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II. The Recursive Reduced-Order Solution of the Matrix Differential Riccati Equation

Consider the linear weakly coupled system

$$\dot{x}_1 = A_1 x_1 + \epsilon A_2 x_2 + B_1 u_1 + \epsilon B_2 u_2, \qquad x_1(t_0) = x_{10}$$
$$\dot{x}_2 = \epsilon A_3 x_1 + A_4 x_2 + \epsilon B_3 u_1 + B_4 u_2, \qquad x_2(t_0) = x_{20} \quad (1)$$

with

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = D \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} D_1 & \epsilon D_2 \\ \epsilon D_3 & D_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (2)$$

where $x_i \in R^{n_i}$, $u_i \in R^{m_i}$, $z_i \in R^{r_i}$, $i = 1, 2$ are state, control, and output variables, respectively. The system matrices are of appropriate dimensions and, in general, they are bounded functions of a small coupling parameter ϵ, [2]–[4]. In this note, we will assume that all given matrices are constant.

With (1)–(2), consider the performance criterion

$$J = \frac{1}{2} \int_{t_0}^{T} \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T D^T D \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T R \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\} dt + \frac{1}{2} \begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix}^T F \begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix} \quad (3)$$

with positive-definite R and positive-semidefinite F, which has to be minimized. It is assumed that matrices F and R have the weakly coupled structure, that is

$$F = \begin{bmatrix} F_1 & \epsilon F_2 \\ \epsilon F_2^T & F_3 \end{bmatrix}, \quad R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}. \quad (4)$$

The optimal closed-loop control law has the very well-known form [8]

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = R^{-1} \begin{bmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{bmatrix}^T P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = R^{-1} B^T P x \quad (5)$$

where P satisfies the differential Riccati equation given by

$$-\dot{P} = PA + A^T P + D^T D - PSP, \quad P(T) = F \quad (6)$$

with

$$A = \begin{bmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{bmatrix}, \quad S = BR^{-1}B^T = \begin{bmatrix} S_1 & \epsilon S_2 \\ \epsilon S_2^T & S_3 \end{bmatrix}. \quad (7)$$

Due to weakly coupled structure of all coefficients in (6), the solution of that equation has the form

$$P = \begin{bmatrix} P_1 & \epsilon P_2 \\ \epsilon P_2^T & P_3 \end{bmatrix}. \quad (8)$$

In this section, we will exploit the Hamiltonian form of the solution of the Riccati differential equation and a nonsingular transformation introduced in [5] in order to obtain an efficient recursive method for solving (6).

The solution of (1) can be sought in the form

$$P(t) = M(t) N^{-1}(t) \quad (9)$$

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where matrices $M(t)$ and $N(t)$ satisfy a system of linear equations [8]

$$\dot{M} = -A^T M(t) - D^T D N(t), \quad M(T) = F \quad (10)$$

$$\dot{N}(t) = -SM(t) + AN(t), \quad N(T) = I. \quad (11)$$

The following lemma, proved in [9], guarantees the existence of the invertible solution for $N(t)$.

Lemma: If the triple (A, B, D) is stablizable observable, then the matrix $N(t)$ is invertible for any $t \in (t_0, T)$.

The Hamiltonian approach is considered as the most efficient numerical method for the solution of the differential Riccati equation [10].

Knowing the nature of the solution of (6), we introduce compatible partitions of $M(t)$ and $N(t)$ matrices as

$$M(t) = \begin{bmatrix} M_1(t) & \epsilon M_2(t) \\ \epsilon M_3(t) & M_4(t) \end{bmatrix}, \quad N(t) = \begin{bmatrix} N_1(t) & \epsilon N_2(t) \\ \epsilon N_3(t) & N_4(t) \end{bmatrix}. \quad (12)$$

Partitioning (10) and (11), according to (6), will reveal a decoupled structure, that is, M_1, M_3, N_1, and N_3 are independent of equations for M_2, M_4, M_2, and N_4 and vice versa. Introducing the notation

$$U = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}, \quad V = \begin{bmatrix} \epsilon M_3 \\ \epsilon N_3 \end{bmatrix}, \quad X = \begin{bmatrix} \epsilon M_2 \\ \epsilon N_2 \end{bmatrix}, \quad Y = \begin{bmatrix} M_4 \\ N_4 \end{bmatrix} \quad (13)$$

and

$$T_1 = \begin{bmatrix} -A_1^T & -Q_1 \\ -S_1 & A_1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -A_3^T & -Q_2 \\ -S_2 & A_2 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} -A_2^T & -Q_2^T \\ -S_2^T & A_3 \end{bmatrix}, \quad T_4 = \begin{bmatrix} -A_4^T & -Q_3 \\ -S_3 & A_4 \end{bmatrix} \quad (14)$$

where

$$Q_1 = D_1^T D_1 + \epsilon^2 D_3^T D_3,$$
$$Q_2 = D_1^T D_2 + D_3^T D_4,$$
$$Q_3 = D_4^T D_4 + \epsilon^2 D_2^T D_2. \quad (15)$$

After doing some algebra, we get two independent systems of weakly coupled matrix differential equations

$$\dot{U} = T_1 U + \epsilon T_2 V$$
$$\dot{V} = \epsilon T_3 U + T_4 V \quad (16)$$

with terminal conditions

$$U(T) = \begin{bmatrix} F_1 \\ I \end{bmatrix}, \quad V(T) = \begin{bmatrix} \epsilon F_2^T \\ 0 \end{bmatrix} \quad (17)$$

and

$$\dot{X} = T_1 X + \epsilon T_2 Y$$
$$\dot{Y} = \epsilon T_3 X + T_4 Y \quad (18)$$

with terminal conditions

$$X(T) = \begin{bmatrix} \epsilon F_2 \\ 0 \end{bmatrix}, \quad Y(T) = \begin{bmatrix} F_3 \\ I \end{bmatrix}. \quad (19)$$

Note that these two systems have exactly the same form and they differ in terminal conditions only. From this point, we will proceed by applying the decoupling transformation introduced in [5]. This transformation is defined by

$$K = \begin{bmatrix} I & -\epsilon L \\ \epsilon H & I - \epsilon^2 HL \end{bmatrix}, \quad K^{-1} = \begin{bmatrix} I - \epsilon^2 LH & \epsilon L \\ -\epsilon H & I \end{bmatrix} \quad (20)$$

where L and H satisfy

$$T_1 L + T_2 - LT_4 - \epsilon^2 LT_3 L = 0 \quad (21)$$

$$H(T_1 - \epsilon^2 LT_3) - (T_4 + \epsilon^2 T_3 L)H + T_3 = 0. \quad (22)$$

Applied to (16)–(19), it will produce

$$\dot{\hat{U}} = (T_1 - \epsilon^2 LT_3)\hat{U}, \quad \hat{U}(T) = U(T) - \epsilon LV(T) \quad (23)$$

$$\dot{\hat{V}} = (T_4 + \epsilon^2 T_3 L)\hat{V}, \quad \hat{V}(T) = \epsilon HU(T) + (I - \epsilon^2 HL)V(T) \quad (24)$$

and

$$\dot{\hat{X}} = (T_1 - \epsilon^2 LT_3)\hat{X}, \quad \hat{X}(T) = X(T) - \epsilon LY(T) \quad (25)$$

$$\dot{\hat{Y}} = (T_4 + \epsilon^2 T_3 L)\hat{Y}, \quad \hat{Y}(T) = \epsilon HX(T) + (I - \epsilon^2 HL)Y(T). \quad (26)$$

Solutions of (23)–(26) are given by

$$\hat{U}(t) = e^{(T_1 - \epsilon^2 LT_3)(t-T)}\hat{U}(T) \quad (27)$$

$$\hat{V}(t) = e^{(T_4 + \epsilon^2 T_3 L)(t-T)}\hat{V}(T) \quad (28)$$

$$\hat{X}(t) = e^{(T_1 - \epsilon^2 LT_3)(t-T)}\hat{X}(T) \quad (29)$$

$$\hat{Y}(t) = e^{(T_4 + \epsilon^2 T_3 L)(t-T)}\hat{Y}(T) \quad (30)$$

so that in the original coordinates we have

$$U(t) = (I - \epsilon^2 LH)e^{(T_1 - \epsilon^2 LT_3)(t-T)}\hat{U}(T) + \epsilon Le^{(T_4 + \epsilon^2 T_3 L)(t-T)}\hat{V}(T) \quad (31)$$

$$V(t) = -\epsilon He^{(T_1 - \epsilon^2 LT_3)(t-T)}\hat{U}(T) + e^{(T_4 + \epsilon^2 T_3 L)(t-T)}\hat{V}(T) \quad (32)$$

$$X(t) = (I - \epsilon^2 LH)e^{(T_1 - \epsilon^2 LT_3)(t-T)}\hat{X}(T) + \epsilon Le^{(T_4 + \epsilon^2 T_3 L)(t-T)}\hat{Y}(T) \quad (33)$$

$$Y(t) = -\epsilon He^{(T_1 - \epsilon^2 LT_3)(t-T)}\hat{X}(T) + e^{(T_4 + \epsilon^2 T_3 L)(t-T)}\hat{Y}(T). \quad (34)$$

Partitioning $U(t)$, $V(t)$, $X(t)$, and $Y(t)$ according to (13) will produce all components of the matrices $M(t)$ and $N(t)$; that is

$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} = \begin{bmatrix} M_1(t) \\ N_1(t) \end{bmatrix}, \quad V(t) = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} \epsilon M_3(t) \\ \epsilon N_3(t) \end{bmatrix}$$

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} \epsilon M_2(t) \\ \epsilon N_2(t) \end{bmatrix}, \quad Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} M_4(t) \\ N_4(t) \end{bmatrix} \quad (35)$$

so that the required solution of (6) is given by

$$P(t) = \begin{bmatrix} U_1(t) & X_1(t) \\ V_1(t) & Y_1(t) \end{bmatrix} \begin{bmatrix} U_2(t) & X_2(t) \\ V_2(t) & Y_2(t) \end{bmatrix}^{-1}. \quad (36)$$

Thus, in order to get the solution of (6) $P(t)$ which has dimensions $n \times n = (n_1 + n_2) \times (n_1 + n_2)$, we have to solve two simple algebraic equations (21) and (22) of dimensions $(2n_2 \times 2n_1)$ *and* $(2n_1 \times 2n_2)$, respectively. The efficient numerical algorithm based on the fixed point iterations and the Newton's method for solving (21) and (22) can be found in [5]. Then, two exponential forms $\exp[(T_1 - \epsilon^2 LT_3)(t - T)]$ *and* $\exp[(T_4 + \epsilon^2 T_3 L)(t - T)]$, have to be transformed in the matrix forms by using some of the well-known approaches [11]. Finally, the inversion of the matrix $N(t)$ has to be performed.

Since the matrices $M(t)$ and $N(t)$ contain unstable modes of the Hamiltonian [8], even though the product $M(t)N^{-1}(t)$ tends to a constant as $t \to \infty$ the inversion of the nonsingular matrix $N(t)$, which contains huge elements, will hurt the accuracy.

The reinitialization version of the Hamiltonian approach avoids that problem. It is considered as the most efficient numerical method for the solution of the general matrix differential Riccati equation [10]. The reinitialization technique applied to the problem under consideration will modify only terminal conditions in formulas (10), (17), and (19), respectively,

$$M(k\Delta t) = P(k\Delta t) \tag{37}$$

$$U(k\Delta t) = \begin{bmatrix} P_1(k\Delta t) \\ I \end{bmatrix}, \quad V(k\Delta t) = \begin{bmatrix} \epsilon P_2^T(k\Delta t) \\ 0 \end{bmatrix} \tag{38}$$

$$X(k\Delta t) = \begin{bmatrix} \epsilon P_2(k\Delta t) \\ 0 \end{bmatrix}, \quad Y(k\Delta t) = \begin{bmatrix} P_3(k\Delta t) \\ I \end{bmatrix} \tag{39}$$

where k represents the number of steps and Δt is an integration step.

The transformation matrix K from (20) can be easily obtained, with required accuracy, by using numerical techniques developed in [5] for solving (21)–(22). They converge with the rate of convergence of at least of $O(\epsilon^2)$. Thus, after k iterations, one gets the approximation $K^{(k)} = K + O(\epsilon^{2k})$. The use of $K^{(k)}$ in (23)–(26) instead of K, will perturb the coefficients of the corresponding systems of linear differential equations by $O(\epsilon^2)$, which implies that the approximate solutions of these differential equations are $O(\epsilon^2)$ close to the exact ones [12]. Thus, it is of interest to obtain $K^{(k)}$ with the desired accuracy, which produces the same accuracy in the sought solution.

III. The Recursive Reduced-Order Solution of an Open-Loop Optimal-Control Problem

The open-loop optimal-control problem of (1)–(4) has the solution given by

$$u(t) = -R^{-1}B^T p(t) \tag{40}$$

where $p(t) \in R^{n_1+n_2}$ is a costate variable satisfying [8]

$$\begin{bmatrix} \dot p(t) \\ \dot x(t) \end{bmatrix} = \begin{bmatrix} -A^T & -D^T D \\ -S & A \end{bmatrix} \begin{bmatrix} p(t) \\ x(t) \end{bmatrix} \tag{41}$$

with boundary conditions expressed in the standard form as

$$W\begin{bmatrix} p(t_0) \\ x(t_0) \end{bmatrix} + G\begin{bmatrix} p(T) \\ x(T) \end{bmatrix} = c \tag{42}$$

where

$$W = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad G = \begin{bmatrix} I & -F \\ 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ x(t_0) \end{bmatrix}. \tag{43}$$

Partitioning p into $p_1 \in R^{n_1}$ and $p_2 \in R^{n_2}$ *such that* $p = [p_1^T \; p_2^T]^T$ and rearranging rows in (41), we can get

$$\begin{bmatrix} \dot p_1 \\ \dot x_1 \\ \dot p_2 \\ \dot x_2 \end{bmatrix} = \begin{bmatrix} T_1 & \epsilon T_2 \\ \epsilon T_3 & T_4 \end{bmatrix} \begin{bmatrix} p_1 \\ x_1 \\ p_2 \\ x_2 \end{bmatrix} \tag{44}$$

where T_i's, $i = 1, 2, 3, 4$ are given by (14).

Introducing the notation

$$\begin{bmatrix} p_1 \\ x_1 \end{bmatrix} = w, \quad \begin{bmatrix} p_2 \\ x_2 \end{bmatrix} = \lambda \tag{45}$$

and applying the transformation (20) to (41) will produce a decoupled form

$$\dot\eta = (T_1 - \epsilon^2 LT_3)\eta \tag{46}$$

$$\dot\xi = (T_4 + \epsilon^2 T_3 L)\xi \tag{47}$$

where

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = K^{-1}\begin{bmatrix} w \\ \lambda \end{bmatrix}. \tag{48}$$

In order to be able to solve (46) and (47), we need to find their initial or terminal conditions, which can be obtained as follows. An interchange of rows for p_2 *and* x_1 in (41) will modify matrices defined in (42) and (43) as follows

$$W_1\begin{bmatrix} w(t_0) \\ \lambda(t_0) \end{bmatrix} + G_1\begin{bmatrix} w(T) \\ \lambda(T) \end{bmatrix} = c_1 \tag{49}$$

where

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix},$$

$$G_1 = \begin{bmatrix} I_{n1} & -F_1 & 0 & -\epsilon F_2 \\ 0 & 0 & 0 & 0 \\ 0 & -\epsilon F_2^T & I_{n_2} & -F_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 0 \\ x_{10} \\ 0 \\ x_{20} \end{bmatrix}. \tag{50}$$

The transformation (48) applied to (49) produces

$$W_2\begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} + G_2\begin{bmatrix} \eta(T) \\ \xi(T) \end{bmatrix} = c_1 \tag{51}$$

where

$$W_2 = W_1 K, \qquad G_2 = G_1 K. \tag{52}$$

Since solutions of (46) and (47) are given by

$$\eta(t) = e^{(T_1 - \epsilon^2 LT_3)(t - t_0)}\eta(t_0) \tag{53}$$

$$\xi(t) = e^{(T_4 + \epsilon^2 T_3 L)(t - t_0)}\xi(t_0) \tag{54}$$

we can eliminate $\eta(T)$ and $\xi(T)$ from (51); that is, we have

$$\left(W_2 + G_2\begin{bmatrix} e^{(T_1 - \epsilon^2 LT_3)(T - t_0)} & 0 \\ 0 & e^{(T_4 + \epsilon^2 T_3 L)(T - t_0)} \end{bmatrix}\right) \cdot \begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} = c_1. \tag{55}$$

It is shown in the Appendix that this system of linear algebraic equations has unique solution, assuming that a coupling parameter ϵ is sufficiently small. Namely, we have shown that if (55) is represented in the form

$$\alpha(\epsilon)\begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} = c_1 \tag{56}$$

with obvious definition for $\alpha(\epsilon)$, then $\alpha(\epsilon)$ is nonsingular.

Now we are able to find $\eta(t)$ and $\xi(t)$ from (53) and (54). Using (48), we can find $w(t)$ *and* $\lambda(t)$. Partitioning $w(t)$ *and* $\lambda(t)$ according to (45) we get values for $p_1(t)$ and $p_2(t)$, in other words, one finds the optimal reduced-order open-loop control defined by (40).

As a matter of fact, following the discussion at the bottom of Section II, we have obtained the approximate expression for the optimal control in the form

$$u^{(2k)}(t) = -R^{-1}B^Tp^{(2k)} = u^{opt}(t) + O(\epsilon^{2k}). \quad (57)$$

Apparently, as k increases, the approximate control defined in (57) converges very rapidly to the optimal solution.

Simulation results for finding the optimal closed-loop and open-loop controls in terms of the reduced-order problems can be found in [13], where a fifth-order distillation column example is solved. It is interesting to point out that the proposed method produces better accuracy for the open-loop control. This can be justified by comparing linear systems of differential equations (16)–(19) and (44). Apparently, the closed-loop solution is computationally much more involved since (16) and (18) are of the order of $2 \times (2n \times n)$, whereas (44) represents the same set of equations of order $2n \times 1$.

IV. Conclusion

The optimal finite-time closed- and open-loop control problems of weakly coupled systems are solved with any desired accuracy in terms of the reduced-order systems of linear differential equations. The proposed methods reduce considerably the size of required computations and introduce full parallelism in the problems under study.

Appendix

Let the transition matrices of (46) and (47) be denoted as $\Phi(t - t_0)$ and $\Psi(t - t_0)$, respectively, and let us partition them as follows

$$\Phi(t-t_0) = \begin{bmatrix} \Phi_{11}(t-t_0) & \Phi_{12}(t-t_0) \\ \Phi_{21}(t-t_0) & \Phi_{22}(t-t_0) \end{bmatrix}$$

$$\Psi(t-t_0) = \begin{bmatrix} \Psi_{11}(t-t_0) & \Psi_{12}(t-t_0) \\ \Psi_{21}(t-t_0) & \Psi_{22}(t-t_0) \end{bmatrix}. \quad (A.1)$$

From (55), we have

$$\alpha(\epsilon) = \left(W_2 + G_2 \begin{bmatrix} \Phi(T-t_0) & 0 \\ 0 & \Psi(T-t_0) \end{bmatrix} \right). \quad (A.2)$$

Using expressions for W_2 and G_2, defined by (52) and (20), we get

$$\alpha(\epsilon) = \begin{bmatrix} I & 0 & 0 & 0 \\ \Phi_{21} - F_1\Phi_{11} & \Phi_{22} - F_1\Phi_{12} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & \Psi_{21} - F_3\Psi_{11} & \Psi_{22} - F_3\Psi_{12} \end{bmatrix} + O(\epsilon). \quad (A.3)$$

Since matrices $\Phi_{22}(T - t_0) - F_1\Phi_{12}(T - t_0)$ and $\Psi_{22}(T - t_0) - F_3\Psi_{12}(T - t_0)$ are invertible (see [14, p. 211]), the matrix $\alpha(\epsilon)$ is invertible for sufficiently small values of ϵ.

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Vibrational Stabilization of Nonlinear Parabolic Systems with Neumann Boundary Conditions

Joseph Bentsman and Keum S. Hong

***Abstract*—This note derives the conditions for the existence of the stabilizing vibrations for a class of distributed parameter systems governed by parabolic partial differential equations with Neumann boundary conditions and gives the guidelines for the choice of the vibration parameters that ensure stabilization. Examples of vibrational stabilization of unstable systems by linear multiplicative and vector additive vibrations are given to support the theory.**

I. Introduction

The concept of vibrational control proposed in [1] is especially attractive when it is applied to distributed parameter systems (DPS), generally known as not readily amenable to sensing and actuation. Indeed, being an open-loop strategy that can ensure desired system behavior via zero-mean parametric excitations, vibrational control requires no on-line sensing and it can stabilize all system modes simultaneously. The experimental and applied theoretical results on the vibrational control of DPS include stabilization of plasma pinches

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